Nonlinear Dynamics **9:** 165–184, 1996.

Minimal Dynamic Characterization of Tree-Like Multibody Systems

P. FISETTE, B. RAUCENT, and J. C. SAMIN
Université Catholique de Louvain, Département de Mécanique, 2, Place du Levant, B-1348 Louvain-la-Neuve, Belgium

(Received: 7 September 1994; accepted: 21 October 1994)

Abstract. The dynamic model of tree-like multibody systems is linear with respect to the parameters of mass distribution for instance when barycentric parameters are used. Thus, assuming that the parameters related to the kinematics are perfectly known, these quantities can be estimated through linear regression techniques. The necessary data are obtained by measuring the joint forces and/or torques and the resulting motion given in terms of positions, velocities and accelerations. An alternative method uses measurements of the reaction forces and torques applied to the bedplate.
The linearity of the dynamic and reaction models with respect to the barycentric quantities does not however imply that the latter constitute the minimum set of parameters characterizing the mass distribution of the system. In other words, some barycentric parameters may disappear from the models or may be redundant in the sense that they appear only via linear combinations. In the first case they are not identifiable, while in the second case the linear regression technique leads to estimated values which are correct for the combinations but can be erroneous for the individual parameters.
The various options taken to derive the dynamic and reaction models by use of the ROBOTRAN programme are briefly reviewed. Then the rules leading to the minimal parametrization are presented and illustrated by means of a practical example related to a robot calibration problem.

Key words: Multibody, dynamics, identification, robot.

1. Introduction

Various problems in robotics require the computation of the dynamic model of the robot which relates the generalized control forces that are transmitted through the joints and the corresponding generalized coordinates q_i, their velocities and accelerations. For instance, high speed and high precision control can only be achieved using advanced control algorithms, such as the "computed torque" control, which require an accurate and in-depth knowledge of the robot dynamics.

The structure itself of the dynamic equations is well known. However, these equations contain parameters which correspond to the physical characteristics of the various bodies constituting the robot and therefore, a good knowledge of these parameters is absolutely necessary. In particular, measuring the parameters related to the mass distribution is not trivial and leads naturally to an identification problem [1–4].

In the second section of this paper, it is shown – often noted "*a posteriori*" as a fact [4] – that the dynamic model is linear with respect to the parameters of mass distribution if barycentric quantities are used [5]. However, it should be noted that this linearity with respect to these unknown parameters is valid only if the geometrical lengths of the system are supposed to be known, for instance by a kinematic calibration [6].

In order to emphasize these properties, the equations of motion of a tree-like multibody system have been written in a particular vectorial form derived from the Potential Power

Principle (a modified version of d'Alembert Principle). As presented in Section 3, this form is used by the programme ROBOTRAN [7] to provide automatically the dynamic model. ROBOTRAN deals with mathematical expressions by means of pointers (C programming language) and prints the resulting equations in a literal form (character strings). One purpose of this programme is to emphasize the linearity with respect to barycentric parameters, so the derived equations are not necessarily optimal as regards the number of arithmetic operations for numerical simulations.

In the fourth section of this paper, an original procedure for the estimation of the barycentric parameters of a robot is presented [8–10]. This procedure is based on the property that the relations between the robot motion and its reactions on the bedplate are completely independent of the internal joints forces. The procedure thus requires only the processing of measurements provided by an external experimental set-up [8]. The robot is placed on a sensing platform which is provided with sensors able to measure the three components of the forces and the three components of the torques between the bedplate and the first link of the robot.

Section 5 deals with the parameter combinations which are required to satisfy the identifiability conditions. A systematic way to obtain these combinations is presented. In particular, it is shown that the reaction model used for identification has the same structure as the dynamic model and that all the barycentric parameters occurring in the dynamic model also appear explicitly in the identification model. Finally, a practical example of parameter combination as well as identification numerical results are given in the sixth section.

2. Dynamic and Identification Models

2.1. Geometrical Structure and Kinematics

In this section, we will review the various options taken to describe the multibody systems under consideration.

The system is considered as a set of n rigid bodies interconnected by joints. In the present paper, it is assumed that the structure is a topological tree. The joints are numbered in ascending order, starting from the base-plate (body 0) and each body has the same index as the preceding joint. A function, INBODY, is then defined to provide for every joint the index number of the preceding body. By means of a recurrent use of this function, the whole set of indices of bodies and joints of the kinematic chain which connect an arbitrary body to the base can be obtained. The notation $i \leq j$ means that the joint (body) i belongs to this chain for body j. For later convenience, a Boolean matrix $\mathbf{T}$ is used whose elements are defined as:

$$\mathbf{T}^{ij} = \begin{cases} 1 & \text{if } i \leq j; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The characteristic geometrical dimensions of body i (see Figure 1) are given by the vectors $\underset{\sim}{\mathbf{l}}^{ij}$ and $\underset{\sim}{\mathbf{l}}^{il}$. Two indices are needed because of the tree structure. However, although such vectors should be defined only for consecutive bodies, the notation is extended to all the indices k satisfying $i < k$, defining $\underset{\sim}{\mathbf{l}}^{ik}$ as follows:

$$\underset{\sim}{\mathbf{l}}^{ik} = \underset{\sim}{\mathbf{l}}^{ij} \quad \text{for } \forall k \text{ such that } i < j \leq k. \tag{2}$$

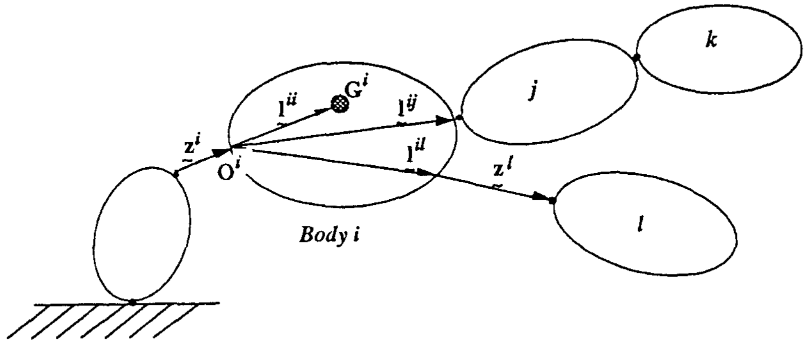

Fig. 1. Geometrical structure

Thus defined, $\underset{\sim}{\mathbf{l}}^{ik}$ provides the contribution of body i to the kinematic chain leading from the base to body k. Since the bodies are supposed to be rigid, all the vectors $\underset{\sim}{\mathbf{l}}^{ij}$ have constant components if they are expressed in a frame $\{\hat{\underset{\sim}{\mathbf{X}}}^i\}$ attached to body i.

The joints are restricted to one degree of freedom: revolute or prismatic. There is no loss of generality since any physical joint can be modelled as a succession of such elementary joints. In order to develop the kinematics, we thus consider that each joint allows:

– a relative rotation expressed by the rotation matrix $\mathbf{A}^j$ that relates the jth body frame $\{\hat{\underset{\sim}{\mathbf{X}}}^j\}$ to the previous one $\{\hat{\underset{\sim}{\mathbf{X}}}^i\}$. This matrix is constant for prismatic joints and depends on the angular coordinate q^j in the case of revolute joints.
– a relative translation represented by the vector $\underset{\sim}{\mathbf{z}}^j$. This vector is equal to zero for revolute joints and depends on the translational coordinate q^j in the case of prismatic joints.

The previous definitions are quite general and still allow us to choose the orientations of the body frames $\{\hat{\underset{\sim}{\mathbf{X}}}^i\}$ as well as the locations O^i of their origins (Figure 1) in order to minimize the number of geometric parameters such as for instance by using the Denavit and Hartenberg conventions. An equivalent minimal kinematic description (suggested in [11]) is illustrated in Figure 2 in the case of a serial link structure.

2.2. Mass Distribution Parameters and Dynamics

The dynamics of the system depends on the mass distribution of each body. The related parameters could be considered individually for each body, but some of these parameters would then combine to provide the actual parameters in the dynamic equations. These combinations could be performed manually [4] but this effort can be avoided if appropriate combinations are introduced from the outset by considering the position of the bodies in the topological structure [12].

Denoting by m^i the mass of the ith body, by $\underset{\sim}{\mathbf{l}}^{ii}$ the position vector of its centre of mass G^i with respect to O^i (Figure 3) and by $\underset{\sim}{\mathbf{I}}^i$ its inertia tensor with respect to O^i, the following barycentric parameters are introduced:

$$\bar{m}^i = \sum_k \mathbf{T}^{ik} m^k \tag{3}$$

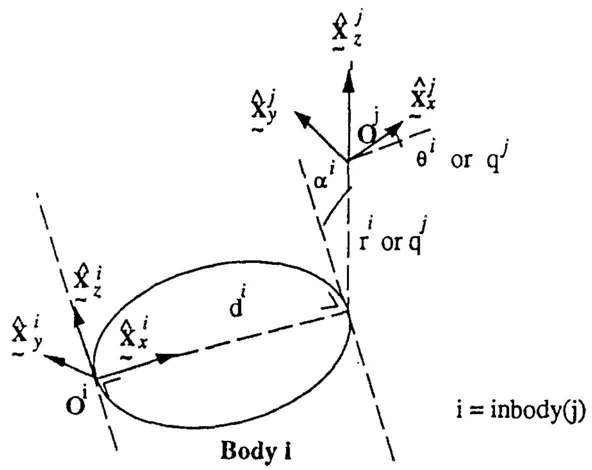

Fig. 2. Kinematic parameters [11].

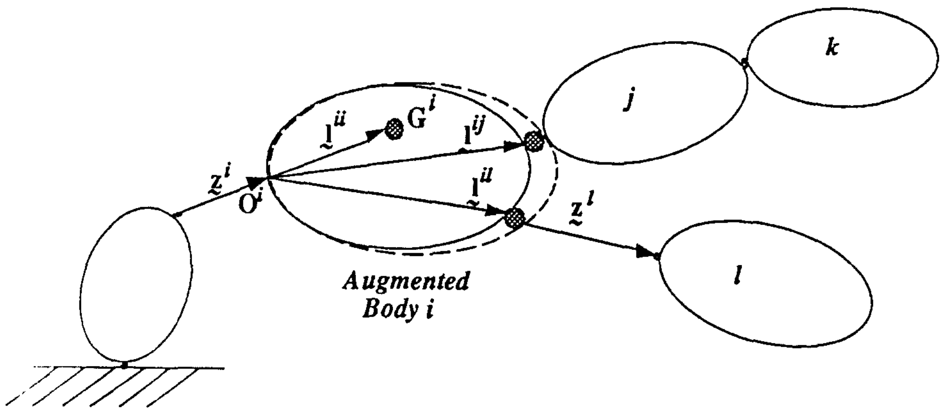

Fig. 3. Augmented body.

$$\underset{\sim}{\mathbf{mb}}^i = \sum_k \mathbf{T}^{ik} m^k \underset{\sim}{\mathbf{l}}^{ik} \tag{4}$$

and

$$\underset{\sim}{\mathbf{K}}^i = \underset{\sim}{\mathbf{I}}^i - \sum_{k \neq i} \mathbf{T}^{ik} m^k \tilde{\underset{\sim}{\mathbf{l}}}^{ik} \tilde{\underset{\sim}{\mathbf{l}}}^{ik}, \tag{5}$$

where $\tilde{\underset{\sim}{\mathbf{x}}}$ denotes the skew-symmetric tensor associated with the cross product by a vector $\underset{\sim}{\mathbf{x}}$.

These parameters are independent linear combinations of the basic mass parameters m^k, $m^k\underset{\sim}{\mathbf{I}}^{kk}$ and $\underset{\sim}{\mathbf{l}}^k$. Moreover, according to the previous definitions, $\bar{m}^i$ is constant and the moments of order 1 ($\underset{\sim}{\mathbf{mb}}^i$) and order 2 ($\underset{\sim}{\mathbf{K}}^i$) have constant components when expressed in the frame $\{\hat{\underset{\sim}{\mathbf{X}}}^i\}$. These barycentric quantities may therefore be chosen as constant parameters which fully describe the mass distribution of the multibody system. Since there are 10 such scalar quantities for each body in the system, the total set of barycentric parameters is $10n$.

From a physical point of view, the mass $\bar{m}^i$ represents the mass of an augmented body i which consists of body i and point masses $\bar{m}^j$ (with i = INBODY(j)) located by the vector $\underset{\sim}{\mathbf{l}}^{ij}$.

The vector $\underset{\sim}{\mathbf{mb}}^i$ is the moment vector locating the mass centre of this augmented body and the tensor $\underset{\sim}{\mathbf{K}}^i$ is its inertia tensor with respect to its attachment point O^i.

As shown in [5] the Principle of Potential Power for rigid body systems leads to the following vector dynamic equations

$$\begin{aligned}\underset{\sim}{\mathbf{F}}^j &= \bar{m}^j \left(\sum_{k:k<j} (\ddot{\underset{\sim}{\mathbf{z}}}^k + (\dot{\tilde{\underset{\sim}{\omega}}}^k + \tilde{\underset{\sim}{\omega}}^k \tilde{\underset{\sim}{\omega}}^k) \underset{\sim}{\mathbf{l}}^{kj}) - \underset{\sim}{\mathbf{g}} \right) \\ &+ \sum_k \mathbf{T}^{jk} (\bar{m}^k \ddot{\underset{\sim}{\mathbf{z}}}^k + (\dot{\tilde{\underset{\sim}{\omega}}}^k + \tilde{\underset{\sim}{\omega}}^k \tilde{\underset{\sim}{\omega}}^k) \underset{\sim}{\mathbf{mb}}^k)\end{aligned} \tag{6}$$

$$\begin{aligned}\underset{\sim}{\mathbf{L}}^j &= \sum_k \mathbf{T}^{jk} (\underset{\sim}{\mathbf{z}}^k \times \underset{\sim}{\mathbf{F}}^k + \underset{\sim}{\mathbf{mb}}^k \times (\ddot{\underset{\sim}{\mathbf{z}}}^k - \underset{\sim}{\mathbf{g}}) + \underset{\sim}{\mathbf{K}}^k \dot{\underset{\sim}{\omega}}^k + \underset{\sim}{\omega}^k \times \underset{\sim}{\mathbf{K}}^k \underset{\sim}{\omega}^k) \\ &+ \sum_k \mathbf{T}^{jk} \sum_{l:l<k} \underset{\sim}{\mathbf{mb}}^k \times (\ddot{\underset{\sim}{\mathbf{z}}}^l + (\dot{\tilde{\underset{\sim}{\omega}}}^l + \tilde{\underset{\sim}{\omega}}^l \tilde{\underset{\sim}{\omega}}^l) \underset{\sim}{\mathbf{l}}^{lk}) \\ &+ \sum_k \mathbf{T}^{jk} \sum_{l:k<l} \underset{\sim}{\mathbf{l}}^{kl} \times (\bar{m}^l \ddot{\underset{\sim}{\mathbf{z}}}^l + (\dot{\tilde{\underset{\sim}{\omega}}}^l + \tilde{\underset{\sim}{\omega}}^l \tilde{\underset{\sim}{\omega}}^l) \underset{\sim}{\mathbf{mb}}^l)\end{aligned} \tag{7}$$

and two important properties which are often noted "*a posteriori*" as a fact [4], arise:

(i) *the torques* $\underset{\sim}{\mathbf{L}}^j$ *and the forces* $\underset{\sim}{\mathbf{F}}^j$ *are linear functions of the barycentric parameters* $\bar{m}^k$, $\underset{\sim}{\mathbf{mb}}^k$, $\underset{\sim}{\mathbf{K}}^k$ *with* k *such that* $j \leq k$. *They become bilinear if the geometric lengths* $\underset{\sim}{\mathbf{l}}^{jk}$ $(j < k)$ *are also considered as parameters.*

(ii) the definition of $\mathbf{T}^{jk}$ given in (1) allows us to observe that $\underset{\sim}{\mathbf{L}}^i$ *and* $\underset{\sim}{\mathbf{F}}^i$, *with* i = INBODY(j), *contain all the terms included in* $\underset{\sim}{\mathbf{L}}^j$ *and* $\underset{\sim}{\mathbf{F}}^j$, *and therefore the same barycentric parameters (in addition to those related to the augmented body* i*).*

In order to obtain the scalar equations of the dynamic model, we need to project the vectors $\underset{\sim}{\mathbf{L}}^j$ (or $\underset{\sim}{\mathbf{F}}^j$) onto the axis of its corresponding revolute (or prismatic) joint. Then all the products (scalar, vector, tensor) must be performed to obtain the final form of the dynamic model in which the generalized coordinates q^i, velocities $\dot{q}^i$ and accelerations $\ddot{q}^i$ appear explicitly. The latter can be written under the following matrix form:

$$M(q,\theta)\ddot{q} + F(q,\dot{q},\theta) = Q, \tag{8}$$

where M is the $(n \times n)$ positive definite symmetric inertia matrix of the system, F is the n-vector specifying gravitation, Coriolis and centrifugal effects, Q is the n-vector of the generalized forces associated with q, and θ is the $10n$-vector containing the barycentric parameters.

Although the above operations preserve the linearity of the model with respect to the components of the barycentric parameters, the dynamic model involves a few independent combinations of those barycentric parameters. These combinations depend on the particular nature (either prismatic or revolute) of the joints and recursive methods for obtaining them

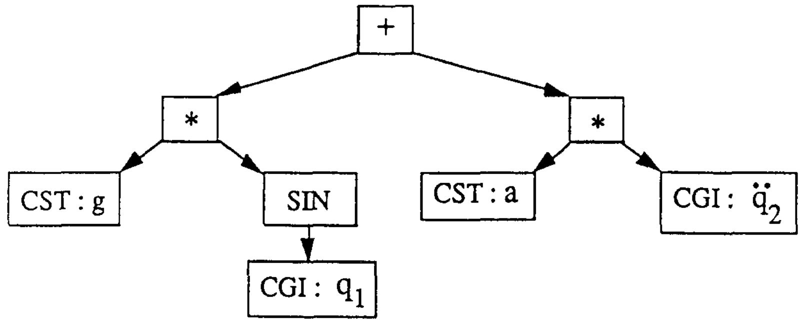

Fig. 4. Tree representation of a symbolical expression.

in a systematical way will be described in Section 5. They lead to a set of N_d combined parameters denoted δ_d:

$$\delta_d = S_d\theta, \tag{9}$$

where S_d is a full range matrix. Equation (8) can therefore be rewritten in the linear form:

$$\phi_d(q, \dot{q}, \ddot{q})\delta_d = Q, \tag{10}$$

where ϕ_d is a regressor vector depending on joint positions, velocities and accelerations.

3. Description of the ROBOTRAN Software

The main purpose of ROBOTRAN [7] is to provide a literal expression (character strings) of the dynamic model of any multibody system. By using the option dedicated to identification, the barycentric parameters appear explicitly in these expressions[1] so that the user can easily regroup their coefficients in order to obtain the regression vectors ϕ_d.

3.1. The Programme Syntax

Each mathematical expression of equation (10) contains one or several terms linked by minus or plus signs, each term being the product of several factors. A factor can be an integer, a barycentric parameter, a generalized coordinate $(q_i, \dot{q}_i, \ddot{q}_i)$, a geometrical constant, a trigonometrical function whose argument is a factor, or a mathematical expression between brackets. Thus defined, an expression can be considered as a tree whose nodes represent the nature of the expressions:

- integer whose value has to be given as a constant,
- barycentric parameter, generalized coordinate or geometrical constant which have to be identified by an appropriate string of characters,
- cosine or sine function whose argument (also an expression) must be given,
- plus, minus, times operators whose two operands must be given as expressions.

For instance, the expression $\{g * \sin(q_1) + a * \ddot{q}_2\}$ can be represented in Figure 4.

3.2. The Manipulations and Simplifications

In order to avoid expressions like: $x = a + b - a$, some order is introduced to classify the expressions. This order is given first by their *nature* and then by a lexicographical order on

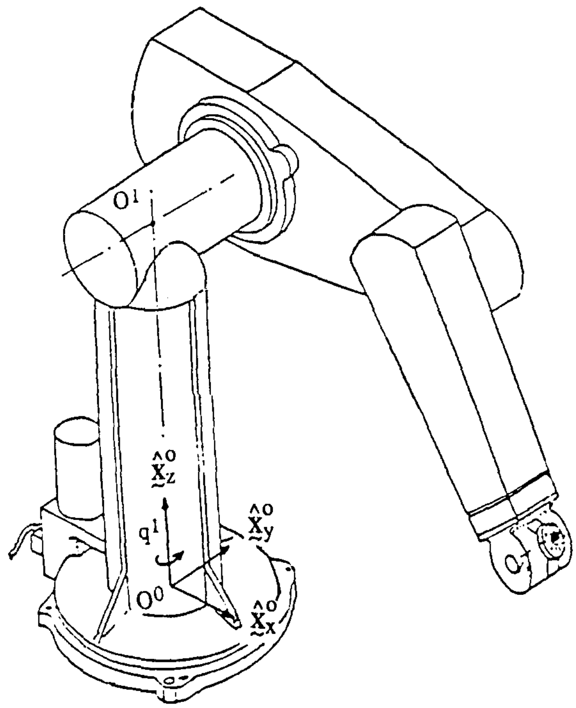

Fig. 5. The PUMA 562 (UNIMATION®).

the strings of characters (for instance, $0 < \cos(q_1) < \sin(q_1) < (q_2 + q_3)$). At each operation, the resulting expressions are rewritten according to the prescribed order and then consecutive equal terms with opposite signs are simplified. It can thus be ensured that the expressions appearing in the obtained equations cannot be further reduced without using specific formulae of trigonometry. This kind of reductions can be performed manually or by several dedicated programmes (MACSYMA, SMP, REDUCE, ...).

Finally, in order to reduce the size of the output, intermediate auxiliary variables can be used to replace each product of trigonometrical functions. These variables are produced by taking the relation order into account so it is as easy as before to find the possible combinations of barycentric parameters and the regression vectors.

3.3. Application to the PUMA Robot

The PUMA 562 (UNIMATION®) is a serial six degrees of freedom manipulator with revolute joints (Figure 5). Assuming here that the influence of the wrist is negligible, only the first three joints are considered.

Three body-fixed frames have been introduced for the dynamic modelling. In the reference configuration, when the wrist is located straight above the shoulder, all these frames are aligned with the inertial frame $\{\hat{\underset{\sim}{X}}^0\}$ attached to the bedplate. The physical characteristics (geometry and mass distribution) of each body are given in Table 1.

According to the symbols defined in Table 1, the barycentric parameters and the dynamic model are obtained automatically by the software ROBOTRAN:

Barycentric masses

$$\begin{aligned}
mm1 &= m1 + m2 + m3\\
mm2 &= m2 + m3\\
mm3 &= m3
\end{aligned}$$

Barycentric moments[2]

$$\begin{aligned}
mb1y &= l11y * m1\\
mb1z &= mm2 * l12z + l11z * m1\\
mb2y &= l22y * m2\\
mb2z &= mm3 * l23z + l22z * m2\\
mb3y &= l33y * m3\\
mb3z &= l33z * m3
\end{aligned}$$

Barycentric tensors

$$\begin{aligned}
K1xx &= J1xx + mm2 * l12z * l12z\\
&\quad - m1 * (-l11y * l11y - l11z * l11z)\\
K1yy &= J1yy + mm2 * l12z * l12z + l11z * l11z * m1\\
K1yz &= -l11y * l11z * m1\\
K1zz &= J1zz + l11y * l11y * m1\\
K2xx &= J2xx + mm3 * l23z * l23z\\
&\quad - m2 * (-l22y * l22y - l22z * l22z)\\
K2yy &= J2yy + mm3 * l23z * l23z + l22z * l22z * m2\\
K2yz &= -l22y * l22z * m2\\
K2zz &= J2zz + l22y * l22y * m2\\
K3xx &= J3xx - m3 * (-l33y * l33y - l33z * l33z)\\
K3yy &= J3yy + l33z * l33z * m3\\
K3yz &= -l33y * l33z * m3\\
K3zz &= J3zz + l33y * l33y * m3
\end{aligned}$$

Equations[3]

$$\begin{aligned}
Q3 = {} & K3yz * qpp1 * C23\\
& + K3yy * qpp2\\
& + mb3z * qpp2 * l23z * C3\\
& + K3yy * qpp3\\
& - K3xx * qp1 * qp1 * C23 * S23\\
& + K3zz * qp1 * qp1 * C23 * S23\\
& - mb3z * qp1 * qp1 * l23z * C23 * S2\\
& + mb3z * qp2 * qp2 * l23z * S3\\
& - mb3z * g * S23
\end{aligned}$$

$$\begin{aligned}
Q2 = {} & Q3\\
& + K2yz * qpp1 * C2\\
& - mb3y * qpp1 * l23z * C2\\
& + K2yy * qpp2\\
& + mb3z * qpp2 * l23z * C3\\
& + mb3z * qpp3 * l23z * C3\\
& - K2xx * qp1 * qp1 * C2 * S2\\
& + K2zz * qp1 * qp1 * C2 * S2\\
& - mb3z * qp1 * qp1 * l23z * C2 * S23\\
& - mb3z * qp2 * qp2 * l23z * S3\\
& - mb3z * qp3 * qp3 * l23z * S3\\
& - mb3z * 2 * qp2 * qp3 * l23z * S3\\
& - mb2z * g * S2\\
& - K2yz * qp2 * qp2 * S2\\
& - K3yz * qp2 * qp2 * S23\\
& + mb3y * qp2 * qp2 * l23z * S2\\
& + K2xx * qp1 * qp2 * S22\\
& - K2zz * qp1 * qp2 * S22\\
& + K3xx * qp1 * qp2 * S2233\\
& - K3zz * qp1 * qp2 * S2233\\
& + mb3z * 2 * qp1 * qp2 * l23z * S223\\
& - K3yz * qp3 * qp3 * S23\\
& + K3xx * qp1 * qp3 * S2233\\
& - K3zz * qp1 * qp3 * S2233\\
& + mb3z * 2 * qp1 * qp3 * l23z * C23 * S2\\
& - K3yz * 2 * qp2 * qp3 * S23
\end{aligned}$$

$$\begin{aligned}
Q1 = {} & K1zz * qqp1\\
& + K2xx * qpp1 * S2 * S2\\
& + K2zz * qpp1 * C2 * C2\\
& + K3xx * qpp1 * S23 * S23\\
& + K3zz * qpp1 * C23 * C23\\
& + mb3z * 2 * qpp1 * l23z * S2 * S23\\
& + K2yz * qpp2 * C2\\
& + K3yz * qpp2 * C23\\
& - mb3y * qpp2 * l23z * C2\\
& + K3yz * qpp3 * C23
\end{aligned}$$

4. Identification of Dynamic Parameters

In the classical identification approach the values of δ_d are estimated from input data (torques applied to the links) and output data (positions, velocities and accelerations of the joints) provided by *"internal" measurement* devices located inside the arms (see, e.g., [1–3]). The model relating these input and output variables is described by the linear equations (10). Therefore, in principle, parameters can be estimated through linear regression techniques. However, there is a major drawback in the practical implementation of such techniques: direct measurements of torques applied to the links are not available, so that torques have to be evaluated as sums of torques provided by the actuators and of friction torques which may be relatively large. Two problems then occur:

Table 1. PUMA description.

	Mass	Central inertia tensor	Center of mass position	Joint position
link 1	$m1$	$\begin{matrix} J^1_{xx} & 0 & 0 \\ 0 & J^1_{yy} & 0 \\ 0 & 0 & J^1_{zz} \end{matrix}$	$\begin{matrix} 0 \\ l^{11}_y \\ l^{11}_z \end{matrix}$	$\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$
link 2	$m2$	$\begin{matrix} J^2_{xx} & 0 & 0 \\ 0 & J^2_{yy} & 0 \\ 0 & 0 & J^2_{zz} \end{matrix}$	$\begin{matrix} 0 \\ l^{22}_y \\ l^{22}_z \end{matrix}$	$\begin{matrix} 0 \\ 0 \\ l^{12}_z \end{matrix}$
link 3	$m3$	$\begin{matrix} J^3_{xx} & 0 & 0 \\ 0 & J^3_{yy} & 0 \\ 0 & 0 & J^3_{zz} \end{matrix}$	$\begin{matrix} 0 \\ l^{33}_y \\ l^{33}_z \end{matrix}$	$\begin{matrix} 0 \\ 0 \\ l^{23}_z \end{matrix}$

(a) For most commercial robots, torques provided by the actuators can be obtained from *internal measurements*, but with poor accuracy. Consider for instance a permanent magnet d-c motor controlled through its armature voltage or current. The torque can be estimated from input current measurements using the torque constant available from the manufacturer's technical data, albeit with uncertainties up to 10%; furthermore the value of this constant can change over the robot lifetime.

(b) The implementation of a parameter estimation procedure requires an accurate model of friction effects and estimation of the characteristic parameters of the friction. It means that barycentric parameters and friction parameters must be estimated simultaneously, based on an adequate friction model. This coupling between barycentric and friction parameters may degrade the accuracy of the barycentric parameter estimates.

An alternative approach has been proposed in order to avoid the above drawbacks [8–10]. It is based on a reformulation of the system dynamics relating the motion of the robot to the reaction forces and torques on its bedplate. The robot is placed on a sensing platform equipped with sensors providing measurements of the three forces and the three torques components between the bedplate and the robot. The main advantage of such an experimental set-up is that it supplies the estimation algorithm with data which are far more accurate than the data which could be obtained from classical sensors located inside the robot arms. Furthermore, the reaction model is *independent from internal effects* (i.e. friction).

Analytical expressions of the reaction model can be obtained for the reaction torque and force components on the bedplate by projecting the vectors $\mathbf{L}^1$ and $\mathbf{F}^1$ (given by equations (6) and (7)) onto the axes of the inertial reference frame attached to the bedplate. As a result of conclusion (Section 2.2 (ii)), all the barycentric parameters contained in the vectorial joint equations (6) and (7) will appear in $\mathbf{L}^1$ and $\mathbf{F}^1$ and therefore in the reaction model which contains all the components of these two vectors. Moreover, the few linear combinations δ_r of barycentric parameters which will appear when projecting $\mathbf{L}^1$ and $\mathbf{F}^1$ on the axes of the bedplate frame, contain the set of linear combination δ_d which occur while deducing the dynamic model (8) from its vectorial form (6–7). In other words, the combined barycentric parameters set δ_d defined in (9) is a subset of the N_r parameters δ_r which appear independently in the reaction model:

$$\delta_d = S\delta_r \quad \text{with} \quad \delta_r = S_r\theta. \tag{11}$$

The reaction model has thus the form:

$$\phi_r(q, \dot{q}, \ddot{q})\delta_r = Q_r, \tag{12}$$

where ϕ_r is a regressor vector depending on joint positions, velocities and accelerations and where the Q_r vector contains the six components of force/torque reactions between the bedplate and the robot.

Assuming that the geometric parameters are perfectly known from a static calibration [6], the barycentric parameters can be estimated through linear regression techniques based on the dynamic or reaction model. Indeed by taking measurements of $Q(t)$, $q(t)$, $\dot{q}(t)$ and $\ddot{q}(t)$ for different values $t_1, t_2, \ldots, t_M$ of time, the following linear regression model is obtained either from (10) or (12):

$$\Phi\delta = Z, \tag{13}$$

where

$$Z = (Q^T(t_1), \ldots, Q^T(t_M))^T \quad \text{and}$$
$$\Phi = (\phi(q(t_1), \dot{q}(t_1), \ddot{q}(t_1)), \ldots, \phi(q(t_M), \dot{q}(t_M), \ddot{q}(t_M)))^T.$$

The problem of the identifiability of δ is stated as follows: does there exist only one set of values of δ satisfying the linear regression model (13) for any motion? We have the following trivial result:

"δ is identifiable iff there exists a trajectory of the robot such that Φ has full rank".

Indeed in this case:

$$\delta = (\Phi^T\Phi)^{-1}\Phi^T Z. \tag{14}$$

In order to satisfy this condition, we have to avoid redundant parametrizations and to select sufficiently rich trajectories.

(a) to avoid redundant parametrizations of the model [15–17]. A parametrization δ is said to be redundant for the model, if there exists another parametrization δ^* of dimension N^* with $N^* < N$, a full rank matrix S^* and a regressor Φ^* such that along any trajectory:

$$\delta^* = S^*\delta \quad \text{and} \quad Z = \Phi\delta = \Phi^*\delta^*. \tag{15}$$

In such a case it is clear that Φ cannot be full rank. For most mechanical structures, the full set of barycentric parameters is redundant for the dynamic and reaction models.

(b) to select trajectories which are sufficiently rich. In [13] Gauthier describes a procedure for the automatic generation of test trajectories which guarantees the identifiability rank conditions of Φ assuming a non-redundant parametrization. We have however observed in practical experiments that an empirical selection of such trajectories is fairly easy when condition (a) is satisfied [14].

5. Minimal Dynamic Parametrization

5.1. The Recursive Model

In this section a recursive method for obtaining the minimal set of parameters δ in a systematic way will be developed. Since any joint mobility is restricted to one degree of freedom (i.e.

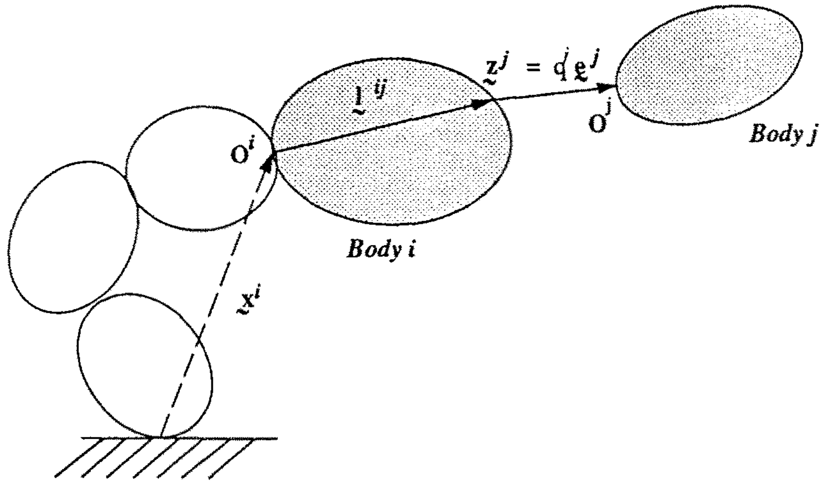

Fig. 6. Prismatic joint recursive model.

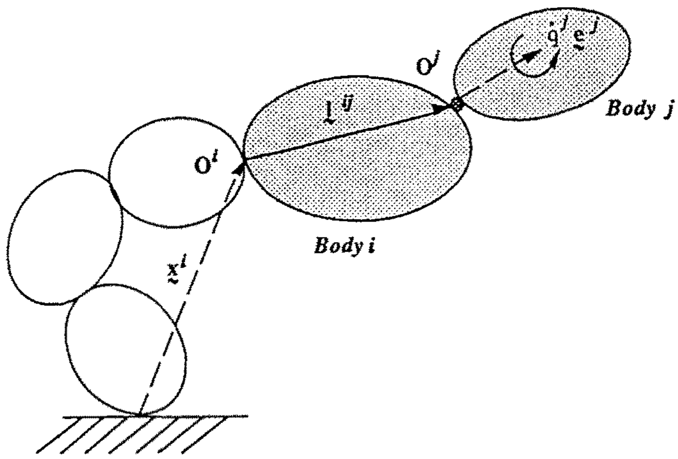

Fig. 7. Revolute joint recursive model.

revolute or prismatic), the two models depicted in Figure 6 and Figure 7 will be considered. These systems are composed of a body j and a carrying body i (with i = INBODY(j)). Depending on the kinematic chain which links body i to the base, this carrying body may have up to six degrees of freedom with respect to inertial space.

According to Section 2, frames $\{\hat{\mathbf{X}}^i\}$ and $\{\hat{\mathbf{X}}^j\}$ are attached to the bodies and the axis vector of the joint j is denoted by $\mathbf{e}^j$.

5.2. Prismatic Joint

The angular velocity of the carrying body i is represented by ω^i and the absolute position of O^i is given by the vector $\mathbf{x}^i$:

$$\mathbf{x}^i = \sum_{k:k<i} (\mathbf{z}^k + \mathbf{l}^{ki}) + \mathbf{z}^i. \tag{16}$$

Since no *a priori* assumption is made concerning the mobility of the carrying body i, the vector forms of equations (6) and (7) are retained for the body i:

$$\underline{\mathbf{F}}^i = \bar{m}^i(\ddot{\underline{\mathbf{x}}}^i - \underline{\mathbf{g}}) + (\dot{\tilde{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i\tilde{\underline{\omega}}^i)\underline{\mathbf{mb}}^i + \bar{m}^j\ddot{\underline{\mathbf{z}}}^j + (\dot{\tilde{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i\tilde{\underline{\omega}}^i)\underline{\mathbf{mb}}^j \tag{17}$$

$$\begin{aligned}\underline{\mathbf{L}}^i = {} & \underline{\mathbf{z}}^i \times \underline{\mathbf{F}}^i + \underline{\mathbf{mb}}^i \times (\ddot{\underline{\mathbf{x}}}^i - \underline{\mathbf{g}}) + \underline{\mathbf{K}}^i\dot{\underline{\omega}}^i + \underline{\omega}^i \times \underline{\mathbf{K}}^i\underline{\omega}^i \\ & + \underline{\mathbf{l}}^{ij} \times (\bar{m}^j\ddot{\underline{\mathbf{z}}}^j + (\dot{\tilde{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i\tilde{\underline{\omega}}^i)\underline{\mathbf{mb}}^j) \\ & + \underline{\mathbf{z}}^j \times \underline{\mathbf{F}}^j + \underline{\mathbf{mb}}^j \times (\ddot{\underline{\mathbf{z}}}^j - \underline{\mathbf{g}}) + \underline{\mathbf{K}}^j\dot{\underline{\omega}}^i + \underline{\omega}^i \times \underline{\mathbf{K}}^j\underline{\omega}^i \\ & + \underline{\mathbf{mb}}^j \times (\ddot{\underline{\mathbf{x}}}^i + (\dot{\tilde{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i\tilde{\underline{\omega}}^i)\underline{\mathbf{l}}^{ij})\end{aligned} \tag{18}$$

where

$$\underline{\mathbf{F}}^j = \bar{m}^j(\ddot{\underline{\mathbf{x}}}^j - \underline{\mathbf{g}}) + (\dot{\tilde{\underline{\omega}}}^j + \tilde{\underline{\omega}}^j\tilde{\underline{\omega}}^j)\underline{\mathbf{mb}}^j \tag{19}$$

and the dynamic equation of joint j is given by:

$$F^j = \underline{\mathbf{F}}^j \cdot \underline{\mathbf{e}}^j. \tag{20}$$

As a preliminary conclusion of property (Section 2.2 (ii)), the equations (17) and (18) contain the contribution of all the barycentric parameters related to the two considered bodies. For a general system in which the jth body is not a terminal one, these equations would involve additional terms corresponding to the barycentric quantities associated with the bodies located downstream from body j. Equations (17), (18) and (20) can therefore be used to detect, at each step of a recursive reasoning, the combinations which may occur in the models.

The various barycentric parameters of body j have now to be investigated in order to establish the recursive relations leading to δ^j, the minimal parametrization related to body j.

Since the frame $\{\hat{\underline{\mathbf{X}}}^j\}$ has no relative angular motion with respect to the frame $\{\hat{\underline{\mathbf{X}}}^i\}$, the components of $\underline{\mathbf{mb}}^j$ and $\underline{\mathbf{K}}^j$ are also constant in the $\{\hat{\underline{\mathbf{X}}}^i\}$ frame. Using the following properties:

$$\tilde{\underline{\mathbf{b}}}\dot{\tilde{\underline{\omega}}}\underline{\mathbf{a}} + \tilde{\underline{\mathbf{a}}}\dot{\tilde{\underline{\omega}}}\underline{\mathbf{b}} = -(\tilde{\underline{\mathbf{b}}}\tilde{\underline{\mathbf{a}}} + \tilde{\underline{\mathbf{a}}}\tilde{\underline{\mathbf{b}}})\dot{\underline{\omega}} \tag{21}$$

$$\tilde{\underline{\mathbf{b}}}\tilde{\underline{\omega}}\tilde{\underline{\omega}}\underline{\mathbf{a}} + \tilde{\underline{\mathbf{a}}}\tilde{\underline{\omega}}\tilde{\underline{\omega}}\underline{\mathbf{b}} = -\tilde{\underline{\omega}}(\tilde{\underline{\mathbf{b}}}\tilde{\underline{\mathbf{a}}} + \tilde{\underline{\mathbf{a}}}\tilde{\underline{\mathbf{b}}})\underline{\omega} \tag{22}$$

we can write

$$\begin{aligned}\underline{\mathbf{l}}^{ij} & \times (\dot{\tilde{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i\tilde{\underline{\omega}}^i)\underline{\mathbf{mb}}^j + \underline{\mathbf{mb}}^j \times (\dot{\tilde{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i\tilde{\underline{\omega}}^i)\underline{\mathbf{l}}^{ij} \\ & = -(\tilde{\underline{\mathbf{l}}}^{ij}\widetilde{\underline{\mathbf{mb}}}^j + \widetilde{\underline{\mathbf{mb}}}^j\tilde{\underline{\mathbf{l}}}^{ij})\dot{\underline{\omega}}^i - \tilde{\underline{\omega}}^i(\tilde{\underline{\mathbf{l}}}^{ij}\widetilde{\underline{\mathbf{mb}}}^j + \widetilde{\underline{\mathbf{mb}}}^j\tilde{\underline{\mathbf{l}}}^{ij})\underline{\omega}^i\end{aligned}$$

and looking at equation (18), we may suggest the following combinations:

$$
\begin{aligned}
\mathbf{mb}^{*i} &= \mathbf{mb}^{i} + \mathbf{mb}^{j} \\
\mathbf{K}^{*i} &= \mathbf{K}^{i} + \mathbf{K}^{j} - (\tilde{\mathbf{l}}^{ij}\widetilde{\mathbf{mb}}^{j} + \widetilde{\mathbf{mb}}^{j}\tilde{\mathbf{l}}^{ij})
\end{aligned} \tag{23}
$$

where it must be noted that all vectors and tensors have constant components in the $\{\hat{\mathbf{X}}^{i}\}$ frame.

Using (23), equations (17), (18) and (20) are now rewritten as:

$$\mathbf{F}^{i} = \bar{m}^{i}(\ddot{\mathbf{x}}^{i} - \mathbf{g}) + (\dot{\tilde{\omega}}^{i} + \tilde{\omega}^{i}\tilde{\omega}^{i})\mathbf{mb}^{*i} + \bar{m}^{j}\ddot{\mathbf{z}}^{j} \tag{24}$$

$$
\begin{aligned}
\mathbf{L}^{i} = {} & \mathbf{z}^{i} \times \mathbf{F}^{i} + \mathbf{mb}^{*i} \times (\ddot{\mathbf{x}}^{i} - \mathbf{g}) + \mathbf{K}^{*i}\dot{\omega}^{i} + \omega^{i} \times \mathbf{K}^{*i}\omega^{i} \\
& + \mathbf{l}^{ij} \times \bar{m}^{j}\ddot{\mathbf{z}}^{j} + \mathbf{z}^{j} \times \mathbf{F}^{j} + \mathbf{mb}^{j} \times \ddot{\mathbf{z}}^{j}
\end{aligned} \tag{25}
$$

$$F^{j} = \mathbf{e}^{j} \cdot \bar{m}^{j}(\ddot{\mathbf{x}}^{j} - \mathbf{g}) + \mathbf{e}^{j} \cdot (\dot{\tilde{\omega}}^{i} + \tilde{\omega}^{i}\tilde{\omega}^{i})\mathbf{mb}^{j}. \tag{26}$$

This new presentation clearly shows that $\mathbf{K}^{j}$ is redundant. On the contrary $\bar{m}^{j}$ can easily be identified from (26) by choosing an excitation such that $\ddot{\mathbf{x}}^{j} = \mathbf{e}^{j}\ddot{q}^{j}$. Finally, the identifiability of $\mathbf{mb}^{j}$ can be analysed by means of its individual contribution to $\mathbf{L}^{i}$ given by:

$$\tilde{\mathbf{e}}^{j}q^{j}(\dot{\tilde{\omega}}^{i} + \tilde{\omega}^{i}\tilde{\omega}^{i})\mathbf{mb}^{j} + \mathbf{mb}^{j} \times (\mathbf{e}^{j}\ddot{q}^{j} + 2\tilde{\omega}^{i}\mathbf{e}^{j}\dot{q}^{j} + (\dot{\tilde{\omega}}^{i} + \tilde{\omega}^{i}\tilde{\omega}^{i})\mathbf{e}^{j}q^{j}) \tag{27}$$

and by its contribution to F^{j} given by

$$\mathbf{e}^{j} \cdot (\dot{\tilde{\omega}}^{i} + \tilde{\omega}^{i}\tilde{\omega}^{i})\mathbf{mb}^{j}. \tag{28}$$

This analysis leads to different combinations depending on the particular values of ω^{i} [17]. In the general case, expressions (27) and (28) allow the identification of the three components of $\mathbf{mb}^{j}$ using the dynamic or the reaction model. It is therefore unnecessary to include the barycentric vector $\mathbf{mb}^{j}$ in the parameter combinations suggested in (23).

If the mobility of body i is restricted (i.e. ω^{i} is inertially fixed), the minimal set of parameters of the dynamic model depends on expressions (28) and the projection of (27) onto the fixed direction $\mathbf{w}$ of ω^{i}. In this case the component mb_{w}^{j} of $\mathbf{mb}^{j}$ along this direction is not identifiable using the dynamic model. Moreover, if $\omega^{i} = \mathbf{0}$ or if the direction of ω^{i} coincides with $\mathbf{e}^{j}$, the two normal components mb_{u}^{j} and mb_{v}^{j} also disappear from (28) and the projection of (27). In both cases the parameter combinations (23) must consequently be applied to the components of $\mathbf{mb}^{j}$ which become redundant for the dynamic model. The results may be summarized as indicated in Table 2 for the dynamic model.

Similar considerations can be applied to the reaction model in which less components are redundant because this model can retain all the components of relation (27). The results are

Table 2. Combination rules for prismatic joints in the dynamic model.

$\boldsymbol{\omega}^i$ angular velocity of body i	δ_d^j minimal set of body j parameters	θ_d^{*i} redefinition of the carrying body i barycentric parameters
general case	$(\bar{m}^j, mb_x^j, mb_y^j, mb_z^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i$ $\mathbf{K}^{*i} = \mathbf{K}^i + \mathbf{K}^j$
$\boldsymbol{\omega}^i$ inertially fixed: $\boldsymbol{\omega}^i = \omega^i \mathbf{w}$	$(\bar{m}^j, mb_u^j, mb_v^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_w^j \mathbf{w}$ $\mathbf{K}^{*i} = \mathbf{K}^i + \mathbf{K}^j - mb_w^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{w}} + \tilde{\mathbf{w}}\tilde{\mathbf{l}}^{ij})$
$\boldsymbol{\omega}^i$ inertially fixed with $\boldsymbol{\omega}^i = \omega^i \mathbf{e}^j$ or $\boldsymbol{\omega}^i = \mathbf{0}$	$(\bar{m}^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + \mathbf{mb}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i + \mathbf{K}^j - (\tilde{\mathbf{l}}^{ij}\widetilde{\mathbf{mb}}^j + \widetilde{\mathbf{mb}}^j\tilde{l}^{ij})$

Table 3. Combination rules for prismatic joints in the reaction model.

$\boldsymbol{\omega}^i$ angular velocity of body i	δ_r^j minimal set of body j parameters	θ_r^{*i} redefinition of the carrying body i barycentric parameters
general case	$(\bar{m}^j, mb_x^j, mb_y^j, mb_z^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i$ $\mathbf{K}^{*i} = \mathbf{K}^i + \mathbf{K}^j$
$\boldsymbol{\omega}^i$ inertially fixed: $\boldsymbol{\omega}^i = \omega^i \mathbf{w}$	$(\bar{m}^j, mb_u^j, mb_v^j, mb_w^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i$ $\mathbf{K}^{*i} = \mathbf{K}^i + \mathbf{K}^j$
$\boldsymbol{\omega}^i$ inertially fixed with $\boldsymbol{\omega}^i = \omega^i \mathbf{e}^j$ or $\boldsymbol{\omega}^i = \mathbf{0}$	$(\bar{m}^j, mb_x^j, mb_y^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + \mathbf{mb}_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i + \mathbf{K}^j - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij})$

summarized as indicated in Table 3 (where mb_z^j denotes the component of the vector $\mathbf{mb}^j$ along the joint axis $\mathbf{e}^j$).

For both models, the indicated rules must be applied recursively from the end(s) of the structure towards its base in order to determine the minimal sets of parameters δ^j for the whole system.

5.3. Revolute Joint

The angular velocity and acceleration of body j are:

$$\underline{\omega}^j = \underline{\omega}^i + \underline{e}^j \dot{q}^j \tag{29}$$

$$\dot{\underline{\omega}}^j = \dot{\underline{\omega}}^i + \underline{\omega}^i \times \underline{e}^j \dot{q}^j + \underline{e}^j \ddot{q}^j. \tag{30}$$

Since again no *a priori* assumption is made concerning the mobility of the carrying body i for the general case, the vector forms of equations (6) and (7) are retained:

$$\underline{\mathbf{F}}^i = \bar{m}^i(\ddot{\underline{\mathbf{x}}}^i - \underline{\mathbf{g}}) + (\tilde{\dot{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i \tilde{\underline{\omega}}^i)\underline{\mathbf{mb}}^i + (\tilde{\dot{\underline{\omega}}}^j + \tilde{\underline{\omega}}^j \tilde{\underline{\omega}}^j)\underline{\mathbf{mb}}^j \tag{31}$$

$$\begin{aligned}\underline{\mathbf{L}}^i = {} & \underline{\mathbf{z}}^i \times \underline{\mathbf{F}}^i + \underline{\mathbf{mb}}^i \times (\ddot{\underline{\mathbf{x}}}^i - \underline{\mathbf{g}}) + \underline{\mathbf{K}}^i \dot{\underline{\omega}}^i + \underline{\omega}^i \times \underline{\mathbf{K}}^i \underline{\omega}^i \\ & + \underline{\mathbf{l}}^{ij} \times (\tilde{\dot{\underline{\omega}}}^j + \tilde{\underline{\omega}}^j \tilde{\underline{\omega}}^j)\underline{\mathbf{mb}}^j + \underline{\mathbf{L}}^j\end{aligned} \tag{32}$$

where

$$\underline{\mathbf{L}}^j = \underline{\mathbf{K}}^j \dot{\underline{\omega}}^j + \underline{\omega}^j \times \underline{\mathbf{K}}^j \underline{\omega}^j + \underline{\mathbf{mb}}^j \times (\ddot{\underline{\mathbf{x}}}^j - \underline{\mathbf{g}}) \tag{33}$$

and the dynamic equation of joint j can be written:

$$L^j = \underline{\mathbf{L}}^j \cdot \underline{\mathbf{e}}^j. \tag{34}$$

The barycentric mass $\bar{m}^j$ does not appear anywhere in the equations and is therefore redundant. Using equations (29), (30) and the following relation:

$$\tilde{\dot{\underline{\omega}}}^j + \tilde{\underline{\omega}}^j \tilde{\underline{\omega}}^j = \tilde{\dot{\underline{\omega}}}^i + \tilde{\underline{\omega}}^i \tilde{\underline{\omega}}^i + 2\tilde{\underline{\omega}}^i \tilde{\underline{\mathbf{e}}}^j \dot{q}^j + \tilde{\underline{\mathbf{e}}}\ddot{q}^j + \tilde{\underline{\mathbf{e}}}^j \tilde{\underline{\mathbf{e}}}^j (\dot{q}^j)^2 \tag{35}$$

the properties (21) and (22) lead to the possible combination (23) defined in the case of a prismatic joint. However, in the present case, $\{\hat{\underline{\mathbf{X}}}^j\}$ has a relative angular motion with respect to $\{\hat{\underline{\mathbf{X}}}^i\}$. The only component of $\underline{\mathbf{mb}}^j$ that remains constant in the frame $\{\hat{\underline{\mathbf{X}}}^i\}$ is mb^j_z. In order to determine which components of $\underline{\mathbf{K}}^j$ are concerned, it is decomposed in an auxiliary frame $\{\hat{\underline{\mathbf{Y}}}^i\}$ attached to body i. Without loss of generality, this frame is chosen such that $\hat{\underline{\mathbf{Y}}}^i_z = \underline{\mathbf{e}}^j$. Due to the rotation q^j, the components of $\underline{\mathbf{K}}^j$ in the $\{\hat{\underline{\mathbf{Y}}}^i\}$ basis are given by means of the symmetric matrix:

$$\begin{pmatrix} \cos(2q^j)K^j_d - \sin(2q^j)K^j_{xy} + K^j_s & \dots & \dots \\ \cos(2q^j)K^j_{xy} + \sin(2q^j)K^j_d & -\cos(2q^j)K^j_d + \sin(2q^j)K^j_{xy} + K^j_s & \dots \\ \cos(q^j)K^j_{xz} - \sin(q^j)K^j_{yz} & \sin(q^j)K^j_{xz} + \cos(q^j)K^j_{yz} & K^j_{zz} \end{pmatrix}, \tag{36}$$

where

$$K_s^j = \frac{K_{xx}^j + K_{yy}^j}{2} \quad \text{and} \quad K_d^j = \frac{K_{xx}^j - K_{yy}^j}{2} \,. \tag{37}$$

As a preliminary result, this expression shows that the components K_{xx}^j and K_{yy}^j will only appear through the combinations K_s^j and K_d^j. The component K_{zz}^j is obviously identifiable in $\underline{L}^j \cdot \underline{e}^j$ via an excitation motion for which $\underline{\dot{\omega}}^j = \underline{e}^j \ddot{q}^j$. The following combinations:

$$\begin{aligned} \underline{\mathbf{mb}}^{*i} &= \underline{\mathbf{mb}}^i + mb_z^j \underline{e}^j \\ \underline{\mathbf{K}}^{*i} &= \underline{\mathbf{K}}^i - mb_z^j(\tilde{\underline{\mathbf{l}}}^{ij}\tilde{\underline{e}}^j + \tilde{\underline{e}}^j\tilde{\underline{\mathbf{l}}}^{ij}) - K_s^j \tilde{\underline{e}}^j \tilde{\underline{e}}^j \end{aligned} \tag{38}$$

are still applicable. Indeed it is easy to see that if we introduce these combinations into equation (31) of $\underline{F}^i$, then use (35) the only residual term that involves mb_z^j independently is:

$$(2\tilde{\underline{\omega}}^i \tilde{\underline{e}}^j \dot{q}^j + \tilde{\underline{e}}^j \ddot{q}^j + \tilde{\underline{e}}^j \tilde{\underline{e}}^j (\dot{q}^j)^2) \underline{e}^j mb_z^j \tag{39}$$

and vanishes because $\tilde{\underline{e}}^j \underline{e}^j = \underline{0}$. Introducing (38) into equation (32) of $\underline{L}^i$ leads to the following residual term:

$$(\underline{\mathbf{z}}^i + \underline{\mathbf{l}}^{ij}) \times (2\tilde{\underline{\omega}}^i \tilde{\underline{e}}^j \dot{q}^j + \tilde{\underline{e}}^j \ddot{q}^j + \tilde{\underline{e}}^j \tilde{\underline{e}}^j (\dot{q}^j)^2) \underline{e}^j mb_z^j \tag{40}$$

which vanishes for the same reason. The individual contribution of mb_z^j also disappears in (34). As a consequence, this component cannot be identified and is redundant. Similarly, using (29) and (30), the individual contribution of $\underline{\mathbf{K}}_s^j = -K_s^j \tilde{\underline{e}}^j \tilde{\underline{e}}^j$ reduces in (32) to:

$$\underline{\mathbf{K}}_s^j \underline{e}^j \ddot{q}^j + \tilde{\underline{e}}^j \dot{q}^j \underline{\mathbf{K}}_s^j \underline{e}^j \dot{q}^j + (\tilde{\underline{e}}^j \underline{\mathbf{K}}_s^j - \underline{\mathbf{K}}_s^j \tilde{\underline{e}}^j - (\underline{\mathbf{K}}_s^j \underline{e}^j)^\sim) \underline{\omega}^i \dot{q}^j \tag{41}$$

and in (34) to

$$\underline{e}^j \cdot \underline{\mathbf{K}}_s^j \underline{\dot{\omega}}^j + \underline{e}^j \cdot \tilde{\underline{\omega}}^j \underline{\mathbf{K}}_s^j \underline{\omega}^j . \tag{42}$$

It is easy to verify that both residual contributions vanish so that K_s^j is also a redundant parameter.

In the general case, i.e. when the mobility of the carrying body is not restricted, all the other components of $\underline{\mathbf{mb}}^j$ and $\underline{\mathbf{K}}^j$ are identifiable in the dynamic model as well as in the reaction model [10, 17]. As for the prismatic joint, if $\underline{\omega}^i$ is inertially fixed additional components of $\underline{\mathbf{mb}}^j$ and $\underline{\mathbf{K}}^j$ can disappear. The results are summarized in Tables 4 and 5.

6. Application to the PUMA 562 Robot

6.1. Minimal Set of Parameters

According to the recursive rules given in Table 5, the redefined (non-zero) barycentric parameters θ^* of the PUMA 562 robot are:

$$\theta_d^* = \left\{ \begin{array}{ccccccc} \bar{m}^{*3} & mb_y^{*3} & mb_z^{*3} & K_{yy}^{*3} & K_d^{*3} & K_s^{*3} & K_{yz}^{*3} \\ \bar{m}^{*2} & mb_y^{*2} & mb_z^{*2} & K_{yy}^{*2} & K_d^{*2} & K_s^{*2} & K_{yz}^{*2} \\ \bar{m}^{*1} & mb_y^{*1} & mb_z^{*1} & K_{zz}^{*1} & K_d^{*1} & K_s^{*1} & K_{yz}^{*1} \end{array} \right\}$$

Table 4. Combination rules for revolute joints in the dynamic model.

$\boldsymbol{\omega}^i$ angular velocity of the carrying body i	δ_d^j minimal set of body j parameters	θ_d^{*i} redefinition of the carrying body i barycentric parameters
general case	$(mb_x^j, mb_y^j, K_{xy}^j, K_{xz}^j, K_{yz}^j, K_d^j, K_{zz}^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij}) - K_s^j\tilde{\mathbf{e}}^j\tilde{\mathbf{e}}^j$
$\boldsymbol{\omega}^i = \omega^i \mathbf{e}^j$	$(mb_x^j, mb_y^j, K_{zz}^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij}) - K_s^j\tilde{\mathbf{e}}^j\tilde{\mathbf{e}}^j$
$\boldsymbol{\omega}^i = \omega^i \mathbf{e}^j$, $\ddot{\mathbf{x}}^j = \ddot{x}^j \mathbf{e}^j$, and $\mathbf{g} = g\mathbf{e}^j$	(K_{zz}^j)	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij}) - K_s^j\tilde{\mathbf{e}}^j\tilde{\mathbf{e}}^j$

Table 5. Combination rules for revolute joints in the reaction model.

$\boldsymbol{\omega}^i$ angular velocity of the carrying body i	δ_r^j minimal set of body j parameters	θ_r^{*i} redefinition of the carrying body i barycentric parameters
general case	$(mb_x^j, mb_y^j, K_{xy}^j, K_{xz}^j, K_{yz}^j, K_d^j, K_{zz}^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij}) - K_s^j\tilde{\mathbf{e}}^j\tilde{\mathbf{e}}^j$
$\boldsymbol{\omega}^i = \omega^i \mathbf{e}^j$	$(mb_x^j, mb_y^j, K_{xz}^j, K_{yz}^j, K_{zz}^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij}) - K_s^j\tilde{\mathbf{e}}^j\tilde{\mathbf{e}}^j$
$\boldsymbol{\omega}^i = \omega^i \mathbf{e}^j$, $\ddot{\mathbf{x}}^j = \ddot{x}^j \mathbf{e}^j$, and $\mathbf{g} = g\mathbf{e}^j$	$(mb_x^j, mb_y^j, K_{xz}^j, K_{yz}^j, K_{zz}^j)$	$\bar{m}^{*i} = \bar{m}^i$ $\mathbf{mb}^{*i} = \mathbf{mb}^i + mb_z^j \mathbf{e}^j$ $\mathbf{K}^{*i} = \mathbf{K}^i - mb_z^j(\tilde{\mathbf{l}}^{ij}\tilde{\mathbf{e}}^j + \tilde{\mathbf{e}}^j\tilde{\mathbf{l}}^{ij}) - K_s^j\tilde{\mathbf{e}}^j\tilde{\mathbf{e}}^j$

among which the minimal set of non-redundant parameters is given by:

$$\delta_d = (mb_z^3\ K_{yy}^3\ K_d^3\ K_{yz}^3\ mb_z^{*2}\ K_{yy}^{*2}\ K_d^{*2}\ K_{yz}^{*2}\ K_{zz}^{*1})^T.$$

Indeed, the barycentric masses $\bar{m}^i$ $(i = 1, \ldots, 3)$ are redundant while mb_y^{*1}, K_d^{*1} and K_{yz}^{*1} disappear because of the restricted mobility of body 1 ($\underset{\sim}{\omega}^0 = \underset{\sim}{0}$, $\ddot{\underset{\sim}{x}}^1 = \underset{\sim}{0}$, and $\underset{\sim}{g} = g\underset{\sim}{e}^1$).

For example, the equation $Q1$ of the first joint can be rewritten as follows:

Output ROBOTRAN

$Q1 = K1zz * qqp1$
$+K2xx * qpp1 * S2 * S2$
$+K2zz * qpp1 * C2 * C2$
$+K3xx * qpp1 * S23 * S23$
$+K3zz * qpp1 * C23 * C23$
$+mb3z * 2 * qpp1 * l23z * S2 * S23$
$+K2yz * qpp2 * C2$
$+K3yz * qpp2 * C23$
$-mb3y * qpp2 * l23z * C2$
$+K3yz * qpp3 * C23$
$-K2yz * qp2 * qp2 * S2$
$-K3yz * qp2 * qp2 * S23$
$+mb3y * qp2 * qp2 * l23z * S2$
$+K2xx * qp1 * qp2 * S22$
$-K2zz * qp1 * qp2 * S22$
$+K3xx * qp1 * qp2 * S2233$
$-K3zz * qp1 * qp2 * S2233$
$+mb3z * 2 * qp1 * qp2 * l23z * S223$
$-K3yz * qp3 * qp3 * S23$
$+K3xx * qp1 * qp3 * S2233$
$-K3zz * qp1 * qp3 * S2233$
$+mb3z * 2 * qp1 * qp3 * l23z * C23 * S2$
$-K3yz * 2 * qp2 * qp3 * S23$

Reduced dynamic equation

$Q1 = K * lzz * qpp1$
$+K{*}2d * qpp1 * C22$
$+K3d * qpp1 * C2233$
$+mb3z * 2 * qpp1 * l23z * S2 * S23$
$+K{*}2yz * qpp2 * C2$
$+K3yz * qpp2 * C23$
$+K3yz * qpp3 * C23$
$-K{*}2yz * qp2 * qp2 * S2$
$-K3yz * qp2 * qp2 * S23$
$-K{*}2d * qp1 * qp2 * S22$
$-K3d * qp1 * qp2 * S2233$
$+mb3z * 2 * qp1 * qp2 * l23z * S223$
$-K3yz * qp3 * qp3 * S23$
$-K3d * qp1 * qp3 * S2233$
$+mb3z * 2 * qp1 * qp3 * l23z * C23 * S2$
$-K3yz * 2 * qp2 * qp3 * S23$

6.2. Identification of δ_d: Numerical Results

Internal and external methods have been applied to estimate the PUMA 562 parameters. As mentioned above (Section 4), the internal method suffers from a lack of accuracy due to joint friction, whose characteristic parameters are unavoidably coupled with the barycentric parameters to be evaluated. In Table 6, the estimated values are compared with values calculated on the basis of blueprints [18] and by pendulum determination [19].[4] This table clearly shows that the estimated values (columns 2 and 3) are of the same order of magnitude as those deduced from the literature although there are some significant differences.

7. Conclusions

A recursive method for determining the minimum set of dynamic parameters of a tree-like multibody system has been presented. Using simple rules, the number of independent dynamic parameters can be determined *a priori*. Depending on the mobility of the first joints of the kinematic chain, this number is equal to or less than $4n_{\text{prismatic}} + 7n_{\text{revolute}}$ while the total number of inertial parameters which is commonly used is given by $10(n_{\text{prismatic}} + n_{\text{revolute}})$. The reduction of independent parameters due to restricted mobility is also obtained from these rules in a straightforward manner. For instance, the first revolute joint of the PUMA arm being vertical, it can be seen from Table 4 that 9 parameters disappear from the dynamic model.

Table 6. Estimation of inertial parameters of a PUMA 562.

Parameters		External identification $\delta_d \pm \sigma$	Internal identification $\delta_d \pm \sigma$	Ref. [18]	Ref. [19]
mb_z^3	kgm	0.802±0.027	0.815±0.027	1.061	0.864
K_{yy}^3	kgm^2	0.545±0.044	0.566±0.045	0.547	0.336
$2 * K_d^3$	kgm^2	0.567±0.262	0.408±0.264	0.533	0.300
K_{yz}^3	kgm^2	−0.103±0.029	−0.073±0.030	−0.150	−0.142
mb_z^{*2}	kgm	3.212±0.039	3.211±0.040	3.702	3.790
K_{yy}^{*2}	kgm^2	2.234±0.100	2.230±0.264	2.786	2.174
$2 * K_d^{*2}$	kgm^2	2.888±0.250	3.168±0.250	2.384	2.829
K_{yz}^{*2}	kgm^2	−0.598±0.049	−0.507±0.095	−0.558	−0.605
K_{zz}^{*1}	kgm^2	1.665±0.138	1.221±0.124	1.920	1.357

Notes

[1]These expressions are thus not necessarily optimal as regards the number of arithmetic operations for online applications.
[2]The particular values given in Table 1 lead to a zero value for $mbix$, $Kixy$ and $Kixz$ (with $i = 1, 2$ and 3).
[3]Where for instance, $qp1$ stands for $\dot{q}(1)$, $qpp1$ for $\ddot{q}(1)$, $S2$ for $\sin(q(2))$, $C2$ for $\cos(q2))$, $S223$ for $\sin(q(2) + q(2) + q(3))$, ...
[4]In fact, the results presented in [18] and [19] are related to the PUMA 560 robot which differs from the PUMA 562 by the presence of a counterweight on the second segment. These results were easily adapted to the PUMA 562 robot considered here.

References

1. Mayeda, H., Osuka, K., and Kangawa, A., 'A new identification method for serial manipulator arms', in *IFAC 9th World Congress*, Budapest, Hungary, July 2–6, 1984, pp. 74–79.
2. Gautier, M., 'Identification of robot dynamics', in *Proceedings IFAC Symposium on Theory of Robots*, Vienna, Austria, December 1986, pp. 351–367.
3. Atkeson, C. G., An, C. H., and Hollerbach, J. M., 'Estimation of inertial parameters of manipulator loops and links', *International Journal of Robotics Research* **5**(3), 1986, 101–119.
4. Ferreira, E. P., 'Contribution à l'identification des paramètres et à la commande dynamique adaptive des robots manipulateurs', Thesis, Université Paul Sabatier de Toulouse, 6 July 1984.
5. Maes, P., Samin, J. C., and Willems, P. Y., 'Linearity of multibody systems with respect to barycentric parameters: Dynamics and identification models obtained by symbolic generation', *Mechanics of Structures and Machines* **17**(2), 1989, 219–237.
6. Duelen, J. and Schroër, K., 'Robot calibration: Method and results', *Robotic and Computer Integrated Manufacturing* **8**, 1991, 223–231.
7. Maes, P., Samin, J. C., and Willems, P. Y., 'AUTODYN and ROBOTRAN: Computer programmes', in *Multibody Systems Handbook*, W. Schiehlen (ed.), Springer-Verlag, Berlin, 1990, pp. 225–264.
8. Raucent, B., Samin, J. C., and Willems, P. Y., 'Identification of dynamic parameters of robot manipulators from external measurements', *European Journal of Mechanical Engineering* **34**, 1989, 21–26.
9. Raucent, B., Campion, G., Bastin, G., Samin, J. C., and Willems, P. Y., 'Identification of the barycentric parameters of robot manipulators from external measurements', *Automatica* **28**(5), 1992, 1011–1016.
10. Raucent, B. and Samin, J. C., 'Modeling and identification of dynamic parameters of robot manipulators', in *Robot Calibration*, R. Bernhardt and S. Albright (eds.), Chapman & Hall, London, 1993, pp. 197–232.
11. Khalil, W. and Kleinfinger, J. F., 'A new geometric notation for open and closed-loop robots', in *Proceedings IEEE Conference on Robotics and Automation*, San Francisco, CA, April 1986, pp. 1174–1180.
12. Wittenburg, J., *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart, Germany, 1977.
13. Gautier, M., 'Contribution à la modélisation et à l'identification des robots', Thesis, Université de Nantes, 1990.

14. Raucent, B., 'Identification des paramètres dynamiques des robots manipulateurs', Thesis, Université Catholique de Louvain, 1990.
15. Gautier, M. and Khalil, W., 'Direct calculation of a minimum set of inertial parameters of serial robot', *IEEE Transaction on Robotics and Automation* **5**(3), 1990, 368–372.
16. Lin, S. K., 'An identification method for estimating the inertial parameters of a manipulator', *Journal of Robotics Systems* **9**(4), 1992, 505–528.
17. Raucent, B. and Samin, J. C., 'Minimal parametrization of robot dynamic models', *Mechanics of Structures and Machines* **22**(3), 1994, 371–396.
18. Tarn, T. J., Bejczy, A. K., Shuotiao, H., and Xiaoping, Y., 'Inertial parameters of Puma 560 robot arm', Rapport 63130, Department of Systems Sciences and Mathematics, Washington University, September 1985.
19. Armstrong, B., Khatlib, O., and Burdick, J., 'The explicit dynamic model and inertial parameters of the PUMA 560 arm', in *Proceedings IEEE Conference on Robotics and Automation*, 1986, pp. 510–518.