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Impact Dynamics of Multibody Systems with Frictional Contact Using Joint Coordinates and Canonical Equations of Motion

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Abstract. This paper presents a methodology in computational dynamics for the analysis of mechanical systems that undergo intermittent motion. A canonical form of the equations of motion is derived with a minimal set of coordinates. These equations are used in a procedure for balancing the momenta of the system over the period of impact, calculating the jump in the body momentum, velocity discontinuities and rebounds. The effect of dry friction is discussed and a contact law is proposed. The present formulation is extended to open and closed-loop mechanical systems where the jumps in the constraints' momenta are also solved. The application of this methodology is illustrated with the study of impact of open-loop and closed-loop examples.

Key words: Impact dynamics, contact, friction, multibody systems, joint coordinates.

1. Introduction

Impact occurs in many mechanical systems, such as crushing and feeding machinery. Such multibody systems undergo intermittent motion essentially characterized by discontinuities they encounter when two bodies within the system collide.

The use of the canonical form of equations of motion, in impact analysis involving impulse and momentum variables, has been presented in several text books [1–3], where solutions are obtained for velocity changes during impact of particles and simple examples of rigid body collisions. The problem is more complex for kinetically constrained mechanical systems, once the change in momenta and velocities are not solely due to the impulsive forces of the colliding bodies, but also involves the change in the reactions of the kinematic constraints.

Piece-wise analysis formulations have been developed for the solutions of the rebounds in the intermittent motion of the mechanical systems [4–6]. During the period of the impact, the integration of the equations of motion is halted and a momentum balance analysis is performed to calculate the velocity jumps. Impulsive forces are a byproduct of these calculations. In these methods it is assumed that no significant change in the system configuration occur during the collision time which is considered small compared to a typical time scale of the motion before and after the impact.

Based on Newton's impact law and the Poisson's impact hypothesis, Stronge [18] proposed an energetically consistent theory for dynamics of partly elastic 2D collisions. Also a theory of the impact or collision of two rigid bodies tacking account of friction was presented by Keller

[19]. The main difficulty in using his theory arises in calculating the direction of tangential impulse when the direction of sliding is not constant during the collisions.

Lankarani and Nikravesh [7] solved the direct central frictionless impact of a mechanical system using a canonical form of the equations of motion expressed in terms of a large set of Cartesian variables. The work presented here starts from that Cartesian canonical form of the equations of motion, then the equations are converted to a minimal set of equations in terms of a set of joint coordinates and associated generalized momenta.

The resulting formulation together with a proposed contact law corresponding to the general oblique impact problem between two bodies of a multibody system is applied to several examples. The role of the coefficient of restitution and the coefficient of friction is discussed and a methodology is suggested to establish the conditions whether or not sliding or stiction can take place.

2. Equations of Motion

The equations of motion can be described in terms of different sets of coordinates. If the number of generalized coordinates is greater than the number of system's degrees of freedom, then algebraic equations are required to show the dependency of the coordinates. One such set of coordinates which leads to defining algebraic constraints for the kinematic joints is the so-called absolute Cartesian coordinates [8].

Another set of generalized coordinates which can provide a minimal set of equations is known as the joint coordinates. In the following sections, the equations of motion in terms of the joint coordinates are discussed.

2.1. Standard Form

In order to specify the position of a rigid body in a global non-moving x–y–z coordinate system, it is sufficient to specify the spatial location of the origin (center of mass) and the angular orientation of a body fixed ξ–η–ζ coordinate system. For the ith body in a multibody system $\mathbf{q}_i$ denotes a vector of coordinates which contains a vector of translational coordinates $\mathbf{r}_i$ and a set of rotational coordinates. Matrix $\mathbf{A}_i$ represents the rotational transformation of the ξ_i–η_i–ζ_i axes relative to the x–y–z axes. A vector of velocities for body i is defined as $\mathbf{v}_i$, which contains a three vector of translational velocities $\dot{\mathbf{r}}_i$ and a three vector of angular velocities ω_i. A vector of accelerations of this body is denoted by $\dot{\mathbf{v}}_i$ which contains $\ddot{\mathbf{r}}_i$ and $\dot{\omega}_i$. For a multibody system containing b bodies, the vector of coordinates, velocities and accelerations are $\mathbf{q}$, $\mathbf{v}$, and $\dot{\mathbf{v}}_i$, respectively, for body $i = 1, \ldots b$. Also a generalized mass matrix is denoted by $\mathbf{M}$ and a vector of generalized forces $\mathbf{g}$ is defined for the multibody system [8].

The relative configurations of two adjacent bodies can be defined by one or more so-called joint coordinates equal in number to the number of relative degrees of freedom between these bodies. The vector of coordinates for an open-loop system is denoted by θ containing all of the joint coordinates and the absolute coordinates of a base body if the base body is not the ground. Therefore, vector θ has a dimension k, equal in number to the number of the degrees of freedom in the system. The vector of joint velocities is defined as $\dot{\theta}$. It can be shown that there is a linear transformation between $\dot{\theta}$ and $\mathbf{v}$ as [9–11]

$$\mathbf{v} = \mathbf{B}\dot{\theta}, \tag{1}$$

where **B** is a $n \times k$ matrix. When the number of selected coordinates is equal to the number of degrees of freedom, the generalized equations of motion for an open-loop multibody system can be written as

$$M\ddot{\theta} = f \tag{2}$$

where

$$M = \mathbf{B}^T\mathbf{M}\mathbf{B} \tag{3}$$

$$f = \mathbf{B}^T(\mathbf{g} - \mathbf{M}\dot{\mathbf{B}}\dot{\theta}). \tag{4}$$

In order to solve these equations, a set of initial conditions must also be defined as

$$\theta(0) = \theta^0, \qquad \dot{\theta}(0) = \dot{\theta}^0. \tag{5}$$

Assume that there are one or more closed kinematic loops in a multibody system. To derive the equations of motion for such a system, each closed-loop is cut at one of the kinematic joints in order to obtain an open-loop system without defining any joint coordinates for the cut joints. Therefore, vector of joint coordinates θ has a dimension k greater than the number of degrees of freedom of the closed-loop system. If the cut joints are reassembled, the joint coordinates are no longer independent. Therefore, there exist algebraic constraints between the joint coordinates as [12]

$$\Psi(\theta) = \mathbf{0}. \tag{6}$$

The first and second time derivatives of the constraints are

$$\dot{\Psi} \equiv \mathbf{C}\dot{\theta} = \mathbf{0} \tag{7}$$

$$\ddot{\Psi} \equiv \mathbf{C}\ddot{\theta} + \dot{\mathbf{C}}\dot{\theta} = \mathbf{0}, \tag{8}$$

where **C** is the Jacobian matrix of the constraints. Then, the differential equations of motion of equation (2) are modified as

$$M\ddot{\theta} - \mathbf{C}^T\nu = f, \tag{9}$$

where ν is a vector of Lagrange multipliers.

Equations (6)–(9) represent a set of differential-algebraic equations for a closed-loop system. A set of initial conditions, such as the set given by equation (5), but consistent with the constraints of equations (6) and (7), must also be defined. These equations can further be reduced to a minimal set of second-order differential equations, equal in number to the number of the degrees of freedom of the system [12].

2.2. Canonical Form

The equations of motion for a multibody system can also be derived in terms of the total momenta of the system. The process of converting the equations of motion described in terms of a large set of absolute accelerations to a canonical form has been shown in [7]. In order to transform the open-loop equations of motion of equation (2) to the canonical form, a vector of joint momenta **p** is defined as [13]

$$M\dot{\theta} = \mathbf{p}, \tag{10}$$

where

$$\dot{\mathbf{p}} = f + \dot{M}\dot{\theta}. \tag{11}$$

Equations (10) and (11) form a simultaneous system of $2 \times k$ differential equations of the first-order which can be considered as the solution of the Lagrangian problem assuming that θ and $\mathbf{p}$ are varied independently [14]. For these equations, an appropriate set of initial conditions must be defined as

$$\theta(0) = \theta^0, \qquad \mathbf{p}(0) = \mathbf{p}^0. \tag{12}$$

For closed-loop systems, a number of prescribed scleronomic conditions may be given by equation (6) and a mixed representation corresponding to the general canonical transformation is defined as [13]

$$M\dot{\theta} - \mathbf{C}^T\sigma = \mathbf{p}, \tag{13}$$

where vector of Lagrange multipliers σ is defined as $\dot{\sigma} = \nu$. The time derivative of equation (13) yields

$$\dot{\mathbf{p}} = f + \dot{M}\dot{\theta} - \dot{\mathbf{C}}^T\sigma. \tag{14}$$

Equations (13) and (14), in conjunction with equations (6) and (7), provide the constrained equations of motion in canonical form. A proper set of initial conditions such as the set given by equation (12) and consistent with the constraints of equation (6) are also required.

Numerical solution of the canonical equations of motion, for either an open- or a closed-loop system, can be obtained by introducing integration arrays as

$$\mathbf{y} = \begin{bmatrix} \theta \\ \mathbf{p} \end{bmatrix}, \qquad \dot{\mathbf{y}} = \begin{bmatrix} \dot{\theta} \\ \dot{\mathbf{p}} \end{bmatrix}. \tag{15}$$

At every integration time step, $\dot{\mathbf{y}}$ array is integrated to obtain $\mathbf{y}$. For open-loop systems, vector θ is used to determine the absolute coordinates $\mathbf{q}$, vector $\mathbf{p}$ is used to obtain $\dot{\theta}$, and then $\mathbf{v}$ is found from equation (1). Equation (11) yields $\dot{\mathbf{p}}$, which in addition to $\dot{\theta}$ provides all the elements of $\dot{\mathbf{y}}$ in order to continue with the integration.

For closed-loop systems, vector $\dot{\theta}$ is found from the solution of equations (7) and (13), where σ is also found at the same time, then equation (14) is used to find $\dot{\mathbf{p}}$. The rest of the process is the same as that of the open-loop systems. A possible problem with this procedure is that due to the accumulation of numerical errors during integration, the position constraints of equation (6) may become violated. Different techniques for eliminating the possibility of constraint violation can be found in [8].

In a piece-wise impact analysis, it is assumed that the contact between points P_i and P_j lasts for a short period from $t^{(-)}$ to $t^{(+)}$, during which the configuration of the system does not change. As a result, the dynamics of the open-loop and closed-loop systems are governed under this smoothness assumption by equations (10) and (11), or equations (13) and (14) respectively. A contact-impact law involving the relative velocity $\mathbf{v}_r$ and the generalized forces f must also be provided.

3. Points of Contact and Relative Velocities

Assume that an impact occurs between bodies i and j of a multibody system. The relative velocity between the points of contact P_i and P_j is written in terms of the absolute velocities

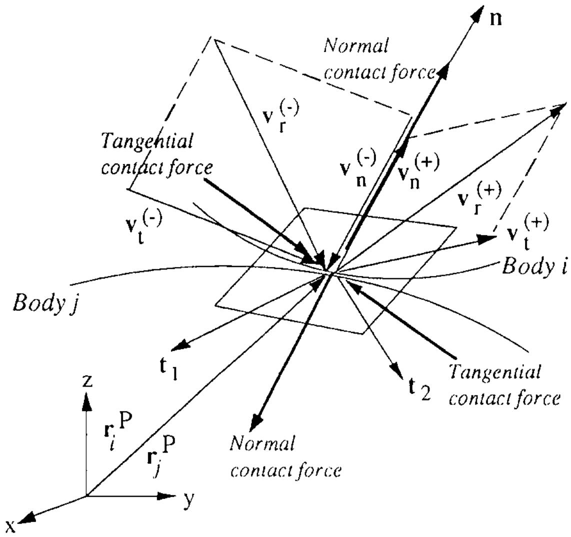

Fig. 1. Impact between two bodies of a multibody system.

of these points as

$$\begin{aligned} \mathbf{v}_r &= \dot{\mathbf{r}}_i^P - \dot{\mathbf{r}}_j^P \\ &= \mathbf{D}_i \mathbf{v}_i - \mathbf{D}_j \mathbf{v}_j \\ &= \mathbf{D}_{i,j} \mathbf{v}, \end{aligned} \tag{16}$$

where $\mathbf{D}_{i,j}$ is a $3 \times n$ incidence matrix and it is a function of the coordinates of bodies i and j only. Substituting equation (1) into equation (16) yields

$$\mathbf{v}_r = \mathbf{R}\dot{\theta}, \tag{17}$$

where $\mathbf{R} = \mathbf{D}_{i,j}\mathbf{B}$ is a $3 \times k$ matrix describing a θ-dependent linear mapping of the generalized velocities.

The relative velocity vector $\mathbf{v}_r$ is represented by its components relative to some chosen orthonormal base $\mathbf{n}$–$\mathbf{t}_1$–$\mathbf{t}_2$. We shall assume that the unit vector $\mathbf{n}$ is defined in the normal direction to the contact surfaces and directed toward body i. A schematic representation of the contact between bodies i and j is shown in Figure 1.

The component of vector $\mathbf{v}_r$ along the normal direction is

$$\begin{aligned} \mathbf{v}_n &= (\mathbf{n}^T \mathbf{v}_r)\mathbf{n} \\ &= (\mathbf{n}^T \mathbf{R}\dot{\theta})\mathbf{n} \\ &= (\mathbf{c}_n^T \dot{\theta})\mathbf{n}, \end{aligned} \tag{18}$$

where

$$\mathbf{c}_n^T = \mathbf{n}^T \mathbf{R} \tag{19}$$

is a k-vector. The tangential component of $\mathbf{v}_r$ corresponding to the slip velocity is denoted as $\mathbf{v}_t$ and lies in the tangential plane $\mathbf{t}_1$–$\mathbf{t}_2$, perpendicular to $\mathbf{n}$, such that

$$\mathbf{v}_r = \mathbf{v}_t + \mathbf{v}_n. \tag{20}$$

At $t^{(-)}$ and $t^{(+)}$ the relative velocity vectors are denoted by $\mathbf{v}_r^{(-)}$ and $\mathbf{v}_r^{(+)}$ respectively. The normal components of these vectors, $\mathbf{v}_n^{(-)}$ and $\mathbf{v}_n^{(+)}$, are collinear along $\mathbf{n}$. However, their tangential components, $\mathbf{v}_t^{(-)}$ and $\mathbf{v}_t^{(+)}$, are not necessarily collinear. Velocity jumps are defined as $\Delta\mathbf{v}_r = \mathbf{v}_r^{(+)} - \mathbf{v}_r^{(-)}$, $\Delta\mathbf{v}_n = \mathbf{v}_n^{(+)} - \mathbf{v}_n^{(-)}$, and $\Delta\mathbf{v}_t = \mathbf{v}_t^{(+)} - \mathbf{v}_t^{(-)}$.

A coefficient of restitution in the normal direction is introduced as the ratio between the relative separation velocity and the relative approach velocity as

$$e = -\frac{v_n^{(+)}}{v_n^{(-)}} = -\frac{\mathbf{c}_n^T\dot{\theta}^{(+)}}{\mathbf{c}_n^T\dot{\theta}^{(-)}}\,. \tag{21}$$

The introduction of the intrinsically non-negative values of e has been traditionally associated with the loss of kinetic energy in impacts in which the tangential component of impulse is absent. This occurs when the contacting surfaces are perfectly smooth or in direct/oblique and central/eccentric impacts. The explicit use of e as a kinematic constraint is extended for the analysis of more general impact situations, playing a role in the material constant based on the common assumption that local inelastic material behavior is solely characterized by that coefficient.

From equation (20) the relative velocity jump during impact is given by

$$\Delta\mathbf{v}_r = \Delta\mathbf{v}_t + \Delta\mathbf{v}_n. \tag{22}$$

Substituting equations (17) and (21) into equation (22) yields

$$\mathbf{R}\Delta\dot{\theta} = \Delta\mathbf{v}_t - (1+e)\mathbf{c}_n^T\dot{\theta}^{(-)}\mathbf{n}. \tag{23}$$

This equation, which involves the coefficient of restitution and the incoming velocities, yields the jumps in the generalized velocities and slip velocity.

4. Impulsive Forces and Impulses

During impact, a pair of impulsive forces, $\mathbf{f}^{(i)}$, act at the point of contact between bodies i and j. Superscript (i) denotes the *impulsive* nature of these forces. The corresponding power of the pair of contact forces can be written as

$$\begin{aligned} P &= \mathbf{v}_r^T\mathbf{f}^{(i)} \\ &= \dot{\theta}^T\mathbf{R}^T\mathbf{f}^{(i)} \\ &= \dot{\theta}^T f^{(i)}, \end{aligned} \tag{24}$$

where

$$f^{(i)} = \mathbf{R}^T\mathbf{f}^{(i)} \tag{25}$$

is the k-dimensional generalized force vector associated with the impulsive force $\mathbf{f}^{(i)}$.

The impulsive force $\mathbf{f}^{(i)}$ can be decomposed along the normal direction $\mathbf{n}$, and along a tangential direction $\mathbf{t}$ in the plane $\mathbf{t}_1, \mathbf{t}_2$ (the direction of $\mathbf{t}$ will be discussed later on). Thus

$$\begin{aligned} \mathbf{f}^{(i)} &= \mathbf{f}_n^{(i)} + \mathbf{f}_t^{(i)} \\ &= f_n^{(i)}\mathbf{n} + f_t^{(i)}\mathbf{t} \end{aligned} \tag{26}$$

and

$$\begin{aligned} f^{(i)} &= \mathbf{R}^T(\mathbf{f}_n^{(i)}\mathbf{n} + \mathbf{f}_t^{(i)}\mathbf{t}) \\ &= \mathbf{f}_n^{(i)}\mathbf{c}_n + \mathbf{f}_t^{(i)}\mathbf{c}_t, \end{aligned} \tag{27}$$

where $\mathbf{c}_t$ is defined similar to $\mathbf{c}_n$ in equation (19) and using $\mathbf{t}$ instead of $\mathbf{n}$.

The generalized impulse vector π due to the generalized impulsive forces $f^{(i)}$ is

$$\begin{aligned} \pi &= \int_{t^{(-)}}^{t^{(+)}} f^{(i)}\,\mathrm{d}t \\ &= \mathbf{c}_n \int_{t^{(-)}}^{t^{(+)}} \mathbf{f}_n^{(i)}\,\mathrm{d}t + \mathbf{c}_t \int_{t^{(-)}}^{t^{(+)}} \mathbf{f}_t^{(i)}\,\mathrm{d}t \\ &= \pi_n\mathbf{c}_n + \pi_t\mathbf{c}_t, \end{aligned} \tag{28}$$

where π_n and π_t are the normal and the tangential impulses due to the impulsive forces $\mathbf{f}_n^{(i)}$ and $\mathbf{f}_t^{(i)}$, respectively.

The method of obtaining the generalized impulsive forces $f^{(i)}$ and the generalized impulse π through the linear mappings $\mathbf{c}_n$ and $\mathbf{c}_t$ suggests the usual concept of virtual motions at fixed time, expressed as a family of configurations $\theta(\varepsilon)$, depending on a real variable ε, in a differentiable manner with subsequent calculations made at constant t.

5. Open-Loop Systems

From equation (10) the generalized velocity jumps can be obtained as

$$\Delta\dot{\theta} = M^{-1}\Delta\mathbf{p}. \tag{29}$$

Integrating the canonical equations of motion described by equation (11) for the period of contact, the generalized impulse vector $\Delta\mathbf{p}$ is given as

$$\Delta\mathbf{p} = \int_{t^{(-)}}^{t^{(+)}} f\,\mathrm{d}t + \int_{t^{(-)}}^{t^{(+)}} \dot{M}\dot{\theta}\,\mathrm{d}t. \tag{30}$$

The generalized force vector f can be described as the sum of two generalized impulsive and non-impulsive forces

$$f = f^{(i)} + f^{(ni)}. \tag{31}$$

Since the period of contact is assumed to be very short, i.e., almost zero, only the impulsive forces have non-zero impulse – all other forces are finite including the gyroscopic and Coriolis forces. In fact it can be shown that integrating by parts, the last term in equation (30) vanishes; i.e.,

$$\begin{aligned} \int_{t^{(-)}}^{t^{(+)}} \dot{M}\dot{\theta}\,\mathrm{d}t &= M\dot{\theta}\Big|_{t^{(-)}}^{t^{(+)}} - M\int_{t^{(-)}}^{t^{(+)}} \ddot{\theta}\,\mathrm{d}t \\ &= M\dot{\theta}\Big|_{t^{(-)}}^{t^{(+)}} - M\dot{\theta}\Big|_{t^{(-)}}^{t^{(+)}} = \mathbf{0}. \end{aligned} \tag{32}$$

Therefore equation (30) becomes

$$\Delta \mathbf{p} = \int_{t^{(-)}}^{t^{(+)}} f^{(i)} \, dt = \pi. \tag{33}$$

5.1. The Sliding Problem for Open-Loop Systems

Assuming that during contact the process of tangential force generation is the result of dry friction, then the impulsive force $\mathbf{f}^{(i)}$ can be written as

$$\mathbf{f}^{(i)} = \mathbf{f}_n^{(i)}(\mathbf{n} + \mu \mathbf{t}), \tag{34}$$

where μ is the coefficient of friction. The generalized impulse vector, for this case, yields

$$\pi = \pi_n(\mathbf{c}_n + \mu \mathbf{c}_t). \tag{35}$$

Substituting equations (35), (33) and (29) into equation (23) yields

$$\pi_n \mathbf{R} M^{-1}(\mathbf{c}_n + \mu \mathbf{c}_t) = \Delta \mathbf{v}_t - (1 + e)\mathbf{c}_n^T \dot{\theta}^{(-)} \mathbf{n}. \tag{36}$$

Premultiplying both sides by $\mathbf{n}^T$ and realizing that by definition $\Delta \mathbf{v}_t$ is perpendicular to $\mathbf{n}$, $\mathbf{n}^T\mathbf{n} = 1$, and from equation (18) the scalar value of $\mathbf{v}_n$ is given by $v_n^{(-)} = \mathbf{c}_n^T \dot{\theta}^{(-)}$, then

$$\pi_n \mathbf{n}^T \mathbf{R} M^{-1}(\mathbf{c}_n + \mu \mathbf{c}_t) = -(1 + e)\mathbf{v}_n^{(-)}. \tag{37}$$

This scalar equation can be solved for the normal impulse

$$\pi_n = -\frac{(1 + e)\mathbf{v}_n^{(-)}}{\mathbf{c}_n^T M^{-1}(\mathbf{c}_n + \mu \mathbf{c}_t)} \, . \tag{38}$$

5.2. The Sticking Problem for Open-Loop Systems

When sticking occurs as the result of an impact, in addition to equation (28) an extra kinematic condition must be considered in order for the outgoing relative tangential velocity to vanish; i.e., $\mathbf{v}_t^{(+)} = \mathbf{0}$ and $\Delta \mathbf{v}_t = -\mathbf{v}_t^{(-)}$. The slip velocity of equation (20), for $t = t^{(-)}$, can be expressed in the more convenient form $\mathbf{v}_t^{(-)} = \mathbf{v}_r^{(-)} - \mathbf{v}_n^{(-)}$. Using equations (17) and (18), we have

$$\mathbf{v}_t^{(-)} = (\mathbf{R} - \mathbf{n}\mathbf{c}_n^T)\dot{\theta}^{(-)}, \tag{39}$$

which allows us to write equation (23) in the form

$$\mathbf{R}\Delta\dot{\theta} = -(\mathbf{R} + e\mathbf{n}\mathbf{c}_n^T)\dot{\theta}^{(-)}. \tag{40}$$

In order to write an impulsive equation, substitute equations (28), (33) and (29) in equation (40):

$$\mathbf{R} M^{-1}(\mathbf{c}_n \pi_n + \mathbf{c}_t \pi_t) = -(\mathbf{R} + e\mathbf{n}\mathbf{c}_n^T)\dot{\theta}^{(-)}. \tag{41}$$

Finally, premultiplying the preceding equation by $\mathbf{n}^T$ and by $\mathbf{t}^T$, a set of two momentum balance impulse equations can be written in matrix form as

$$\begin{bmatrix} \mathbf{c}_n^T M^{-1}\mathbf{c}_n & \mathbf{c}_n^T M^{-1}\mathbf{c}_t \\ \mathbf{c}_t^T M^{-1}\mathbf{c}_n & \mathbf{c}_t^T M^{-1}\mathbf{c}_t \end{bmatrix} \begin{bmatrix} \pi_n \\ \pi_t \end{bmatrix} = \begin{bmatrix} -(1+e)\mathbf{v}_n^{(-)} \\ -\mathbf{v}_t^{(-)} \end{bmatrix}. \tag{42}$$

Equation (42) is a set of two linear equations in terms of the normal and tangential components of the impulse.

6. Closed-Loop Systems

The change in the generalized joint velocities during impact must satisfy the constraints of equation (7); i.e.,

$$\mathbf{C}\Delta\dot{\theta} = \mathbf{0}. \tag{43}$$

From equation (13) we have

$$\Delta\dot{\theta} = M^{-1}(\Delta\mathbf{p} + \mathbf{C}^T\Delta\sigma). \tag{44}$$

Then,

$$\mathbf{C}M^{-1}(\Delta\mathbf{p} + \mathbf{C}^T\Delta\sigma) = \mathbf{0}. \tag{45}$$

Integrating the canonical equations of motion described by equation (14) for the period of contact yields

$$\Delta\mathbf{p} = \int_{t^{(-)}}^{t^{(+)}} \boldsymbol{f}\,\mathrm{d}t + \int_{t^{(-)}}^{t^{(+)}} \dot{M}\dot{\theta}\,\mathrm{d}t - \int_{t^{(-)}}^{t^{(+)}} \dot{\mathbf{C}}^T\sigma\,\mathrm{d}t. \tag{46}$$

All the forces in the right hand side of equation (46) are bounded except for the impulsive forces. The integral of the bounded forces, including the term containing σ are zero. This can be shown easily by writing

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{C}^T\sigma) = \dot{\mathbf{C}}^T\sigma + \mathbf{C}^T\dot{\sigma}. \tag{47}$$

Then,

$$\begin{aligned} \int_{t^{(-)}}^{t^{(+)}} \dot{\mathbf{C}}^T\sigma\,\mathrm{d}t &= \mathbf{C}^T\sigma\Big|_{t^{(-)}}^{t^{(+)}} - \mathbf{C}^T\int_{t^{(-)}}^{t^{(+)}} \dot{\sigma}\,\mathrm{d}t \\ &= \mathbf{C}^T\Delta\sigma - \mathbf{C}^T\Delta\sigma = 0 \end{aligned} \tag{48}$$

which allows us to conclude that the change in the total momenta for closed-loop systems is the result of the impulsive forces as described by equation (33).

6.1. The Sliding Problem for Closed-Loop Systems

Using the same procedure as in Section 5.1 and taking into consideration equation (44) instead of equation (29), an equivalent of equation (37) is obtained for the closed-loop systems as

$$\mathbf{c}_n^T M^{-1}[\pi_n(\mathbf{c}_n + \mu\mathbf{c}_t) + \mathbf{C}^T\Delta\sigma] = -(1+e)\mathbf{c}_n^T\dot{\theta}^{(-)}. \tag{49}$$

Equation (45), after substituting $\Delta\mathbf{p}$ from equations (33) and (35), finds the form

$$\pi_n \mathbf{C}M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t) + \mathbf{C}M^{-1}\mathbf{C}^T\Delta\sigma = \mathbf{0}. \tag{50}$$

Equations (49) and (50) can then be written in matrix form as

$$\begin{bmatrix} \mathbf{C}M^{-1}\mathbf{C}^T & \mathbf{C}M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t) \\ \mathbf{c}_n^T M^{-1}\mathbf{C}^T & \mathbf{c}_n^T M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t) \end{bmatrix} \begin{bmatrix} \Delta\sigma \\ \pi_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -(1+e)\mathbf{v}_n^{(-)} \end{bmatrix}, \tag{51}$$

which can be solved for the internal impulses $\Delta\sigma$ and the normal component of the impulse π acting at the point of contact.

6.2. The Sticking Problem for Closed-Loop Systems

Using the same procedure as in Section 5.2 and taking into consideration equation (45) instead of equation (29), an equivalent of equation (42) is then obtained for closed-loop systems as

$$\begin{bmatrix} \mathbf{C}M^{-1}\mathbf{C}^T & \mathbf{C}M^{-1}\mathbf{c}_n & \mathbf{C}M^{-1}\mathbf{c}_t \\ \mathbf{c}_n^T M^{-1}\mathbf{C}^T & \mathbf{c}_n^T M^{-1}\mathbf{c}_n & \mathbf{c}_n^T M^{-1}\mathbf{c}_t \\ \mathbf{c}_t^T M^{-1}\mathbf{C}^T & \mathbf{c}_t^T M^{-1}\mathbf{c}_n & \mathbf{c}_t^T M^{-1}\mathbf{c}_t \end{bmatrix} \begin{bmatrix} \Delta\sigma \\ \pi_n \\ \pi_t \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -(1+e)\mathbf{v}_n^{(-)} \\ -\mathbf{v}_t^{(-)} \end{bmatrix}, \tag{52}$$

which can be solved for the internal impulses $\Delta\sigma$ and the normal and tangential components of the impulse π acting at the point of contact.

7. Contact Law

A contact law is now suggested for the traditional isotropic Coulomb law of friction:

– $\mathbf{v}_t^{(+)}$ is zero when the magnitude of the tangential impulse is less than μ times the magnitude of the normal impulse:

$$\text{If } \pi_t \geq \mu\pi_n, \text{ then } \mathbf{v}_t^{(+)} = \mathbf{0}. \tag{53}$$

– $\mathbf{v}_r^{(+)}$ has a non-zero tangential component, i.e., $\mathbf{v}_t^{(+)} \neq \mathbf{0}$, if the tangential impulse has a magnitude μ times the magnitude of the normal impulse:

$$\text{If } \pi_t = \mu\pi_n, \text{ then } \mathbf{v}_t^{(+)} \neq \mathbf{0}. \tag{54}$$

For planar oblique impacts, the contact law involves the preceding conditions and the assumption that the tangential impulse acts along the opposite direction of $\mathbf{v}_t^{(-)}$. This means

$$\mathbf{t} = -\frac{1}{v_t^{(-)}}\,\mathbf{v}_t^{(-)}. \tag{55}$$

This is a plausible assumption which will be shown to correspond to a maximum energy loss in planar impacts. However, in spatial oblique impacts, the tangential velocity generally undergoes a change in direction which enables a direct determination of the direction of the tangential impulse. Formulations using standard Convex Analysis [15, 16] have shown that the Coulomb's law of friction is exactly similar to the *law of perfect plasticity* and it can be derived from a "principle" of maximal dissipation. As a result, the direction of the tangential impulse, $\mathbf{t}$, should be such that the energy loss during impact is maximized. The impact law for spatial cases is better illustrated in the example described in Section 9.3.

8. Velocity Jump and Rebounds

Procedures for updating velocities after impact are now obtained. Without any loss of generality, consider the sliding problem for open-loop systems described in Section 5.1. Taking into consideration equation (33) and substituting equation (38) in equation (29), the jump in the generalized velocities can be obtained as

$$\Delta\dot{\theta} = -M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t)\frac{(1+e)v_n^{(-)}}{\mathbf{c}_n^T M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t)} \,. \tag{56}$$

From equations (23) and (21) the jump in the tangential velocity can be described as

$$\Delta\mathbf{v}_t = -(\mathbf{R} - \mathbf{n}\mathbf{c}_n^T)M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t)\frac{(1+e)v_n^{(-)}}{\mathbf{c}_n^T M^{-1}(\mathbf{c}_n + \mu\mathbf{c}_t)} \,. \tag{57}$$

A procedure for updating velocities in a general oblique impact can be stated for closed-(open-)loop systems as:

ALGORITHM:
Knowing the positions and velocities at $t^{(-)}$:

1. Find π_n and π_t from equation (52) (42):
   if $\pi_t \leq \mu\pi_n$ go to 2.
   if $\pi_t > \mu\pi_n$ go to 3.
2. We have stiction with $\mathbf{v}_t^{(+)} = \mathbf{0}$:
   2.1. From equation (28) we evaluate $\Delta\mathbf{p} = \mathbf{c}_n\pi_n + \mathbf{c}_t\pi_t$.
   2.2. Go to 4.
3. We have sliding:
   Find π_n from equation (51) (38).
4. Evaluate $\Delta\dot{\theta}$ from (44) (29).
5. Update joint velocities at $\dot{\theta}^{(+)} = \dot{\theta}^{(-)} + \Delta\dot{\theta}$.
6. Resume with integration.

9. Examples and Numerical Results

The purpose of this section is to apply the present piece-wise analysis to typical examples of mechanical systems and to show the validity of the method by examining the impact responses.

9.1. Slider-Crank Mechanism

This first example is taken from [7] where it was solved using a piece-wise formulation with equations of motion described in terms of a large set of absolute Cartesian coordinates. This is a closed-loop multibody system impacting a sliding body. A schematic representation of the system is shown in Figure 2. A set of relative joint coordinates θ_1, θ_2, θ_3 and θ_4 are used

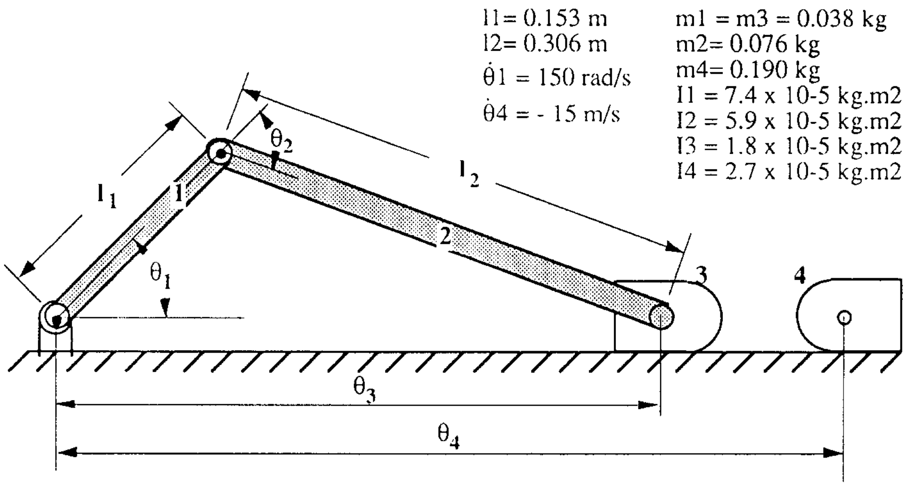

Fig. 2. Impact between a slider-crank mechanism and another slider.

Table 1. Comparison of results.

	Present results (m/s)	Reference [7] (m/s)
$\Delta\dot{\theta}_4$	41.58	41.8
$\Delta\dot{\theta}_3$	−68.285	−68.3

to describe the configuration of the system. Due to the closed-loop, θ_1, θ_2, and θ_3 must satisfy equation (6), and their first time derivatives must satisfy equation (7) (or equation (43)).

The slider crank is driven by a restoring torque such that the crank maintains almost a constant angular velocity. At some instant, the slider (body 3) impacts the free slider (body 4) which is driven inertially to the left at a constant speed. A coefficient of restitution $e = 0.83$ between the blocks is considered. This case corresponds to a central impact problem, therefore, no tangential relative velocities are observed and no tangential impulsive forces are developed between bodies 3 and 4 during the impact. The results from this analysis and those of [7] are summarized in Table 1.

The implied loss of kinetic energy induced by the impact is given by

$$T_L = T^{(-)} - T^{(+)} = \frac{1}{2}\,\dot{\theta}^{(-)T}\boldsymbol{M}\dot{\theta}^{(-)} - \frac{1}{2}\,\dot{\theta}^{(+)T}\boldsymbol{M}\dot{\theta}^{(+)}. \tag{58}$$

For central impacts, equation (58) can be written in the form

$$T_L = \frac{1}{2\mathbf{c}_n^T\boldsymbol{M}^{-1}\mathbf{c}_n}\,(1 - e^2)\mathbf{v}_n^{(-)^2}. \tag{59}$$

For general oblique impacts, the induced energy loss can be shown to be

$$T_L = -\frac{1}{2}\,\pi_n^2(m_{nm} + 2\mu m_{nt} + \mu^2 m_{tt}) - \pi_n(v_n^{(-)} + \mu v_t^{(-)}), \tag{60}$$

where

$$m_{nn} = \mathbf{c}_n^T\boldsymbol{M}^{-1}\mathbf{c}_n; \qquad m_{nt} = \mathbf{c}_n^T\boldsymbol{M}^{-1}\mathbf{c}_t; \qquad m_{tt} = \mathbf{c}_t^T\boldsymbol{M}^{-1}\mathbf{c}_t. \tag{61}$$

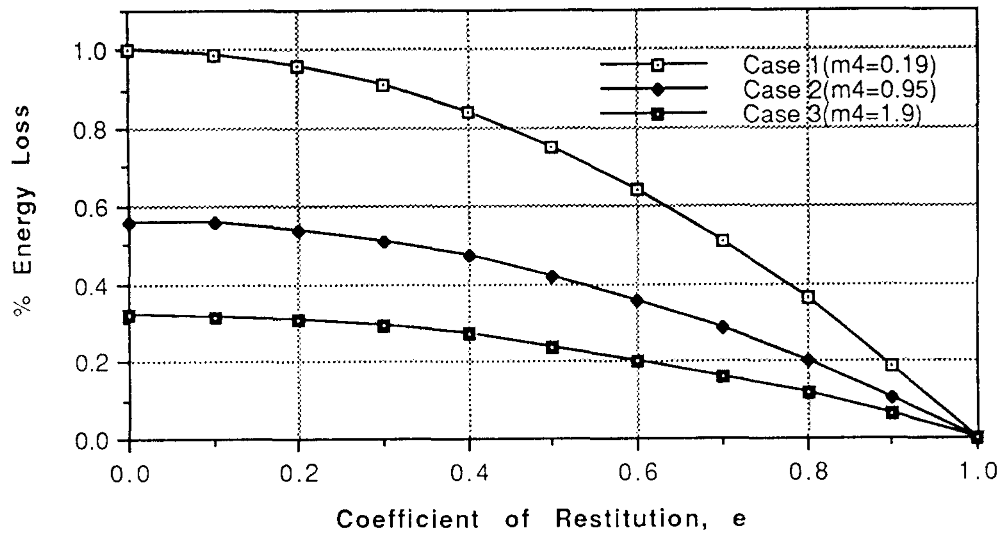

Fig. 3. Energy loss versus the coefficient of restitution.

Figure 3 shows a plot of the non-dimensionalized energy loss of the system, $\Delta T/^{(-)}$, for the range of the coefficient of restitution $0 \le e \le 1.0$, and for three different masses of the inertially driven slider (case 1: $m_4 = 0.19$; case 2: $m_4 = 0.95$ and case 3: $m_4 = 1.9$).

For case 1, with $e = 0$, the energy loss is almost equal to the initial kinetic energy of the system before impact, which means that the system came to a halt. For cases 2 and 3, where the mass of body 4 was increased, only 54% and 32%, respectively, of the initial kinetic energies were dissipated during perfectly plastic collisions ($e = 0$). Case 1 is a particular case where the generalized mass of the crank and slider is equivalent to the inertially driven mass. Case 1 can be reasoned to be "equivalent" to the perfectly plastic central impact of two particles that is known to come to a halt when the masses are equal, thus releasing all of the kinetic energies.

9.2. Double Pendulum-Planar Cases

9.2.1. *Case 1*

Figure 4a shows two slender rods connected together and to a fixed support with revolute joints. The system is only allowed to move in the x–y plane. The rods are identical and the approaching joint velocities are $\dot{\theta}_1 = -1$ and $\dot{\theta}_2 = -1$ (rad/s). This configuration implies an incoming velocity $\mathbf{v}_r^{(-)} = (-1.309\mathbf{n} - 2.683\mathbf{t})$.

Figure 5 shows the tangential velocity, $\mathbf{v}_t^{(+)}$, for $0 \le e \le 1.0$ and $-0.5 \le \mu \le 0.5$. Positive values of μ correspond to tangential impulses in the t-direction whereas negative values of μ correspond to tangential impulses along the positive x-direction. A plot of the energy loss, as defined in equation (60), is shown in Figure 6.

It can be observed that the negative values of μ corresponding to smaller energy losses or even energy gains. On the other hand, sticking is not always possible. For example, for values of $e < 0.3$, even for larger values of μ, there is always a positive $\mathbf{v}_t^{(+)}$. For larger values of e, sticking can only occur, eventually with energy gains which is not plausible.

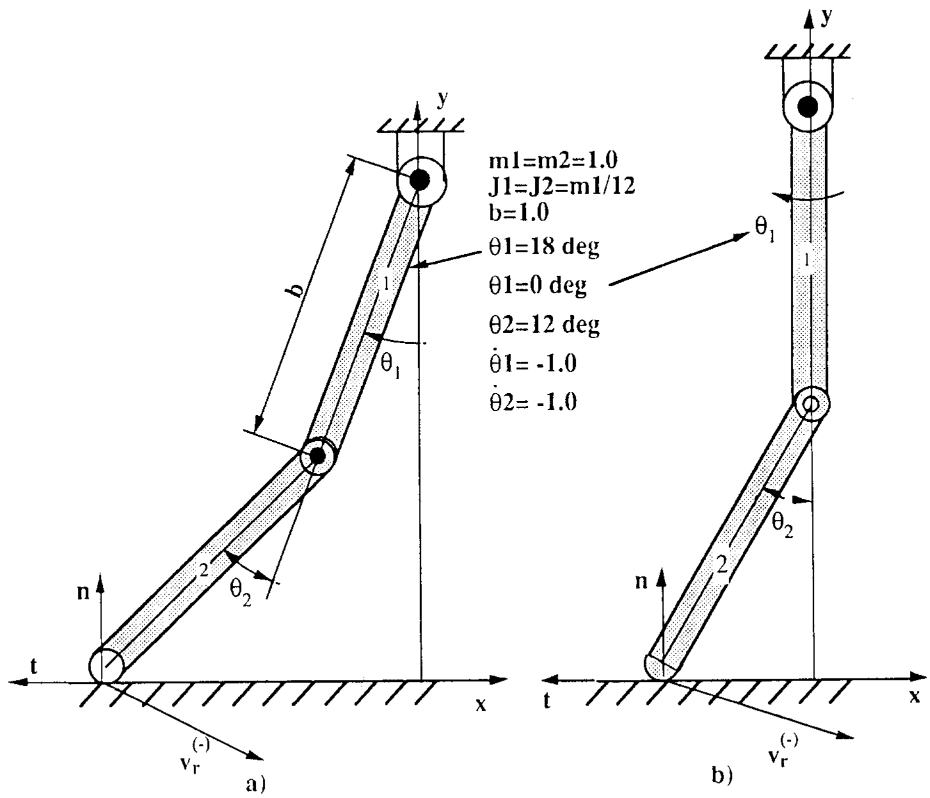

Fig. 4. Double pendulum examples.

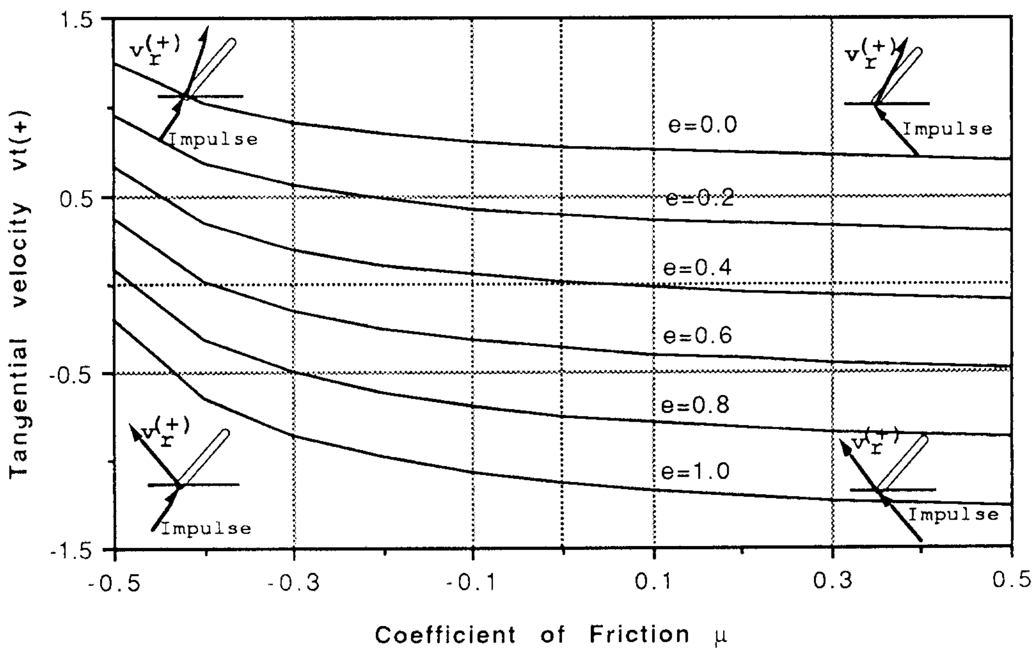

Fig. 5. Tangential rebound predictions for the pendulum (a).

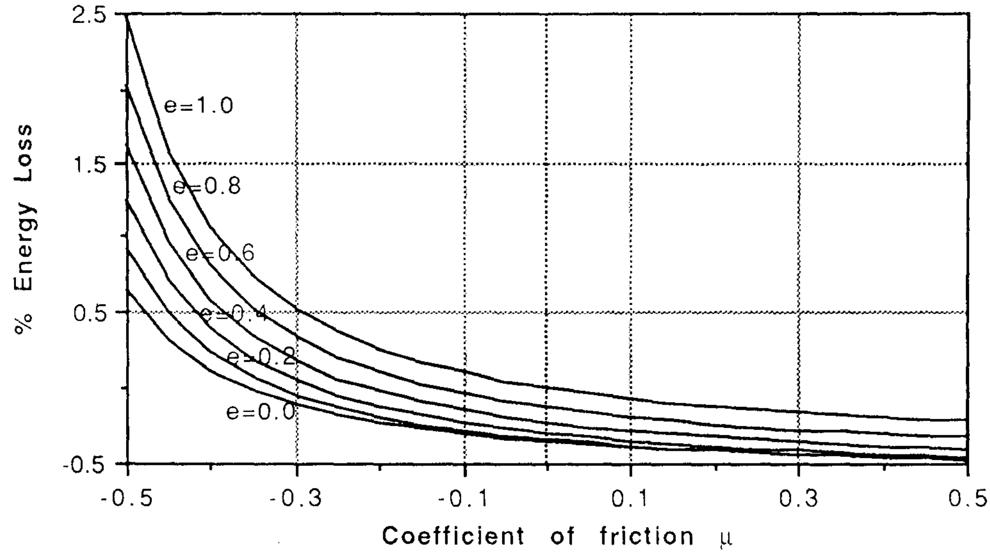

Fig. 6. Energy loss for the double pendulum (a).

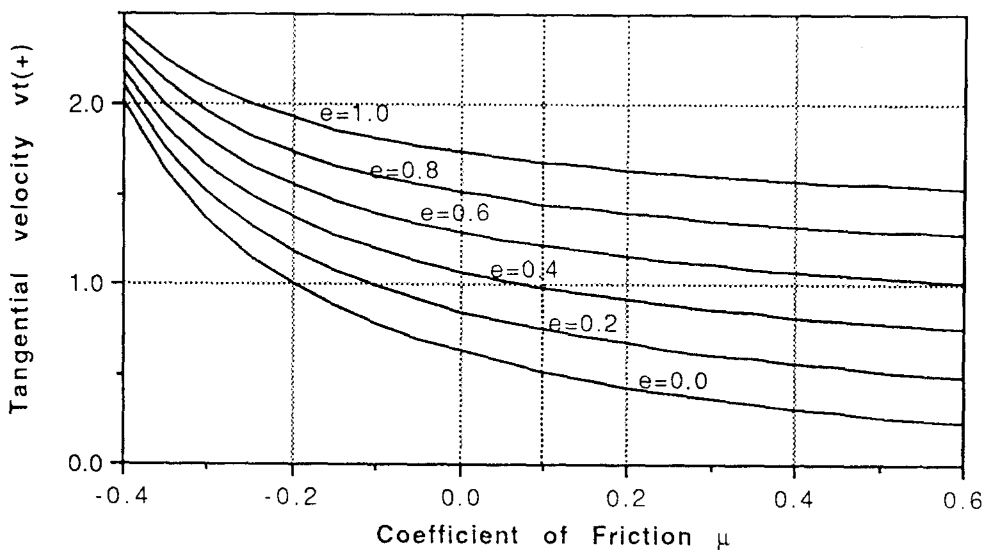

Fig. 7. Tangential rebound predictions for the pendulum (b).

9.2.2. *Case 2*

The double pendulum illustrated in Figure 4b is also analyzed, where the first rod is in vertical position. For this configuration, and assuming the same angular velocities as in case (a), the relative incoming velocity is $\mathbf{v}_r^{(-)} = (-0.416\mathbf{n} - 2.956\mathbf{t})$.

Figure 7 shows the predicted outgoing tangential velocities for $0 \leq e \leq 1$ and $-0.4 \leq \mu \leq 0.6$. The results show that this pendulum never sticks, even for large values of μ, the outgoing tangential velocity is always positive.

These results clearly show that the impact conditions are strongly dependent on the mechanical system and its configuration at the time of impact. A new method is now proposed to verify analytically whether or not stiction can occur.

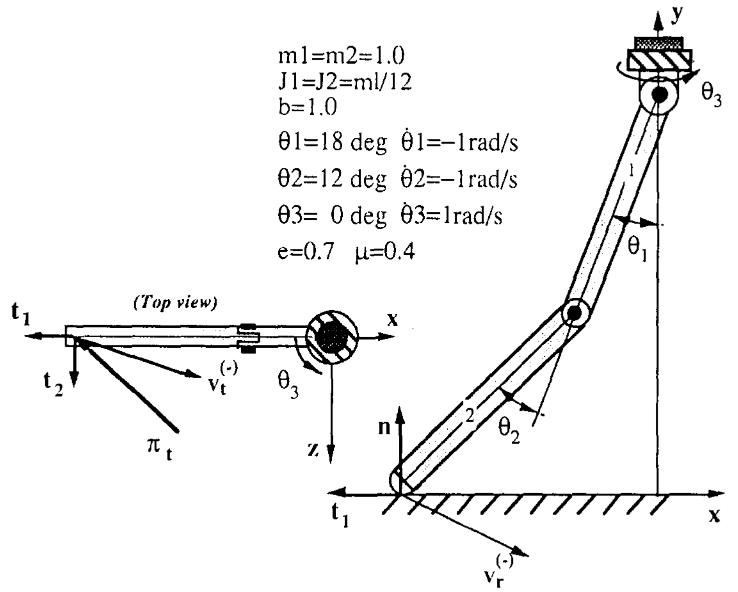

Fig. 8. 3D double pendulum.

Without any loss of generality, consider the open-loop case under sticking conditions. As the normal $\mathbf{n}$ was defined from body j to body i, and the tangential vector $\mathbf{t}$ was considered along $\Delta \mathbf{v}_t$, or, in the opposite sense of $\mathbf{v}_t^{(-)}$, the solution of equation 942) should always yield positive normal and tangential impulses.

According to Crammer's rule, $\pi_n = D_1/D$ and $\pi_t = D_2/D$ where

$$D = \begin{vmatrix} \mathbf{c}_n^T \boldsymbol{M}^{-1}\mathbf{c}_n & \mathbf{c}_n^T \boldsymbol{M}^{-1}\mathbf{c}_t \\ \mathbf{c}_t^T \boldsymbol{M}^{-1}\mathbf{c}_n & \mathbf{c}_t^T \boldsymbol{M}^{-1}\mathbf{c}_t \end{vmatrix}; \qquad D_1 = \begin{vmatrix} -1(1+e)v_n^{(-)} & \mathbf{c}_n^T M^{-1}\mathbf{c}_t \\ -v_t^{(-)} & \mathbf{c}_t^T M^{-1}\mathbf{c}_t \end{vmatrix}$$

$$D_2 = \begin{vmatrix} \mathbf{c}_n^T \boldsymbol{M}^{-1}\mathbf{c}_n & -(1+e)v_n^{(-)} \\ \mathbf{c}_t^T \boldsymbol{M}^{-1}\mathbf{c}_n & -v_t^{(-)} \end{vmatrix}.$$

Therefore, if $D > 0$, then $D_1 > 0$ and $D_2 > 0$, or if $D < 0$, then $D_1 < 0$ and $D_2 < 0$. Since $D > 0$, then the necessary conditions for stiction to occur can be written as

$$\mathbf{c}_t^T \boldsymbol{M}^{-1}\mathbf{c}_t v_t^{(-)} - (1+e)\mathbf{c}_t^T \boldsymbol{M}^{-1}\mathbf{c}_t v_n^{(-)} > 0 \tag{62a}$$

$$(1+e)\mathbf{c}_t^T \boldsymbol{M}^{-1}\mathbf{c}_n v_n^{(-)} - \mathbf{c}_n^T \boldsymbol{M}^{-1}\mathbf{c}_n v_t^{(-)} > 0, \tag{62b}$$

which can be used to find bounds on the coefficient of restitution for stiction to be possible.

9.3. Three-Dimensional Double Pendulum

Consider the double pendulum is also allowed to rotate around the y-axis with an angular velocity $\dot{\theta}_3$. As shown in Figure 8, vectors $\mathbf{t}_1$ (opposite to the x-axis) and $\mathbf{t}_2$ (along the z-axis) define the contact plane. At the time of impact, the joint velocities are $\dot{\theta}_1 = -1$, $\dot{\theta}_2 = -1$, and $\dot{\theta}_3 = 1$ (rad/s), resulting in the approach velocity $\mathbf{v}_r^{(-)} = (-1.309\mathbf{n} - 2.683\mathbf{t}_1 + 0.809\mathbf{t}_2)$.

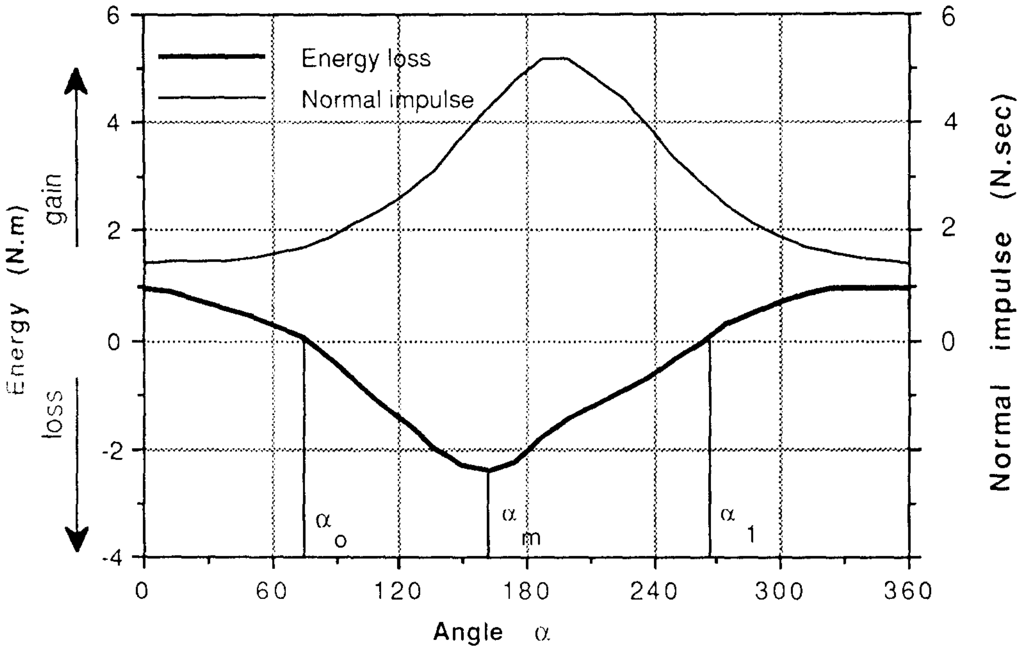

Fig. 9. Energy loss and normal impulse versus the direction of the tangential impulse.

The algorithm described in Section 8 was used and it can be shown that sliding conditions prevail for this case, therefore, equation (38) must be used to obtain the normal component of the impulse. However, according to the proposed contact law, the direction **t** of the tangential impulse must be such that the energy dissipation during the impact is maximized. Equation (60) clearly shows the dependency of T_L on **t**. A plot of the variation of T_L for the present 3D pendulum, when **t** is considered to rotate an angle α counter-clockwise from the direction of $\mathbf{v}_t^{(-)}$, is shown in Figure 9. The direction of **t** corresponding to a maximum energy dissipation, is denoted by the angle α_m. It should be observed that for this direction a value of $\pi_n =$ 4.02 N.sec is predicted which is 78% of the maximum normal impulse (for $\alpha = 196.25$). The angle values α_0 and α_1 define a sector corresponding to the directions of the tangential impulse that imply energy gains, therefore are not acceptable.

Figure 10 illustrates the incoming tangential velocity, $\mathbf{v}_t^{(-)}$, the tangential velocity jump, $\Delta\mathbf{v}_t$, the outgoing tangential velocity, $\mathbf{v}_t^{(+)}$, and the tangential impulse, π_t, corresponding to the maximum energy dissipation. Another vector **t** for the direction of the tangential impulse is also represented which was obtained from [17] where an averaging process of the tangential components of the approach and separation velocities was used. This figure clearly shows that the tangential impulse, in general is not collinear with $\mathbf{v}_t^{(-)}$ and $\mathbf{v}_t^{(+)}$. The present calculation predicts a rebound velocity and an impulse at the point of contact as

$$\mathbf{v}_r^{(+)} = 0.916\mathbf{n} + 0.675\mathbf{t}_1 + 0.41\mathbf{t}_2$$

$$\Delta\mathbf{p} = 4.026\mathbf{n} + 1.26\mathbf{t}_1 - 0.99\mathbf{t}_2.$$

A modification in the algorithm of Section 8, step 3, must then be introduced where π_n must be found from equation (51) (38) for closed-(open-)loop systems, using a direction **t** that maximizes T_L. This process is highly non-linear and can be carried out with the use of any standard optimization algorithms.

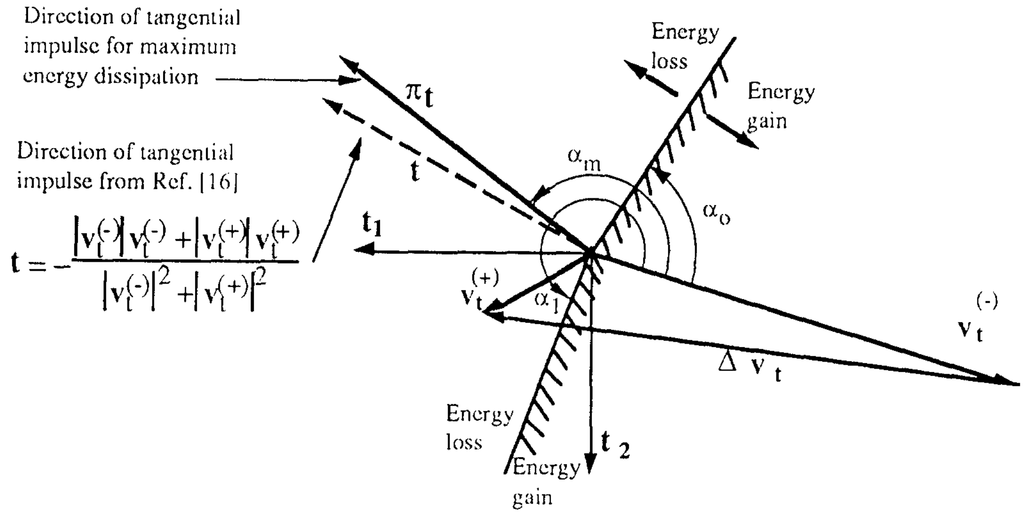

Fig. 10. Impulse and rebound predictions for the 3D pendulum.

10. Conclusions

In this paper, a method was presented for the piece-wise analysis of the intermittent motion of mechanical systems. A canonical form of the impulse-momentum equations for multibody systems can be established in terms of a minimum set of joint coordinates and can be solved for the jump in the joint velocities during the impact. This solution also finds the impulsive forces exerted on the contact surfaces of the colliding bodies and provides a means by which necessary conditions are introduced for the occurrence of stiction or sliding. The analytical set is completed by a system of contact laws involving the introduction of a coefficient of restitution traditionally used for the description of two body collisions and a coefficient of friction associated with the Coulomb's dry friction, assuming that friction is developed in such a way that a maximum dissipation of energy is observed. Since the impulse-momentum represents a first integral of the motion, velocity jumps can be calculated in a straightforward manner allowing an immediate assessment of energy changes.

Acknowledgement

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