Some Relations Between Additive Functions - II

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The question raised in our paper 'Some relations between additive functions - I' (see [1]) regarding additive functions, can be formulated as follows:

Let $\mathfrak{A}$ be the set of all additive functions $f: R \to R$ (where R is the reals), that is, f's satisfying

$$\text{(c)} \qquad f(x+y) = f(x) + f(y), \quad \text{for all} \quad x, y \in R.$$

Then $\mathfrak{A}$ is a vector space over R. A function $f: R \to R$ satisfying (c) and

$$\text{(d)} \qquad f(xy) = xf(y) + f(x)\,y, \qquad x, y \in R$$

is called a derivative on R. Let $\mathfrak{B}$ be the subspace of $\mathfrak{A}$ spanned by $x \to f(x) = f(1)\,x$, continuous functions, and by all derivations on R.

PROBLEM. Let the u_i's be rational functions in x, the p_i's be continuous functions on R except at the singular points of u_i and the f_i's be additive functions. When does a condition of the form

$$\sum_{i=1}^{n} p_i(x)\, f_i(u_i(x)) = 0$$

imply that $f_i \in \mathfrak{B}$ $(i = 1, 2, \dots, n)$ or that the f_i are linearly dependent relative to $\mathfrak{B}$.

In the sequel we shall often use, when f is additive, that

$$\text{(c}'\text{)} \qquad f(rx) = rf(x), \qquad \text{where } r \text{ is any rational},$$

which is a consequence of (c), and, when f is a derivative, that

$$\text{(d}'\text{)} \qquad f(x^2) = 2xf(x),$$

which is easily obtainable from (d). It is well known that the general continuous solutions of (c) are $f(x) = cx$, where c is an arbitrary constant, and that $f \equiv 0$ is the only common, continuous solution of (c) and (d).

Here we prove the following theorem.

THEOREM. *If $f, g \in \mathfrak{A}$ and if there exist a number α, two continuous functions p_1 $(\not\equiv 0)$, p_2 $(\not\equiv 0)$, two constants A, B such that*

$$f\left(\frac{1}{x-\alpha}\right) + f\left(\frac{A}{(x-\alpha)^2}\right) = p_1(x)\, g\left(\frac{1}{x-\alpha}\right) + p_2(x)\, g\left(\frac{B}{(x-\alpha)^2}\right) \tag{1}$$

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for all $x \neq \alpha$, *then* f *and* g *are linearly dependent modulo the subspace* $\mathfrak{B}$, *that is, there are constants* a, b, c *such that* $f(x) = ag(x) + bH(x) + cx$, *where* H *is a derivative on* R.

Further, (a) *if* $A \neq 0$, $B = 0$, *then either* f *and* g *are continuous or* f *and* g *are derivatives or* $f, g \in \mathfrak{B}$ *and* $p_1(x) = (\varepsilon x + \delta)/(x - \alpha)$, *where* ε, δ *are constants or* $f = 0$ *and* $g = 0$.

(b) *if* $A = 0$, $B \neq 0$, *then either* $f = 0$, $g \in \mathfrak{B}$ *or both* f *and* $g \in \mathfrak{B}$ *and* $p_1(x) = c_1 + c_2 p_2(x)/(x - \alpha)$, *where the* c_i*'s are constants.* (c) *if* $A \neq 0$, $B \neq 0$, *then either* f *and* g *are continuous or* f *is continuous*, $g \in \mathfrak{B}$ *or* f *is a derivative and* $g \in \mathfrak{B}$ *or both* f *and* $g \in \mathfrak{B}$ *and* $p_1(x) = d_1 + d_2/(x - \alpha) + d_3\, p_2(x)/(x - \alpha)$, *where the* d_i*'s are constants,*

or $f = 0$, $g \in \mathfrak{B}$, $p_1(x) = a/(x - \alpha)$, $p_2(x) = b$

or $f, g \in \mathfrak{B}$, $p_1(x) = a/(x - \alpha)$, $p_2(x) = e_1 + e_2(x - \alpha)$, *where the* e_i*'s are constants,*

or $f, g \in \mathfrak{B}$, $p_1(x) = e_1 + e_2/(x - \alpha)$, $p_2(x) = b$, *where the* e_i*'s are constants*

or $f(x) = ag(x)$, $p_1(x) = a$, $p_2(x) = b$.

Proof. In (1), replacing x by $(1/x) + \alpha$, we have

$$f(x) + f(Ax^2) = p_1\left(\frac{1}{x} + \alpha\right) g(x) + p_2\left(\frac{1}{x} + \alpha\right) g(Bx^2). \tag{2}$$

Replacing x by rx in (2), where r is a rational, using (c′) and then letting $r \to 1/x$, we obtain

$$xf(x) + f(Ax^2) = axg(x) + bg(Bx^2), \tag{3}$$

where $a = p_1(1 + \alpha)$ and $b = p_2(1 + \alpha)$.

Putting $x + r$ for x in (3), where r is a rational, and using (3) and then letting $r \to x$, we get

$$xf(x) + x^2 f(1) + 2xf(Ax) = axg(x) + ax^2 g(1) + 2bxg(Bx). \tag{4}$$

Setting

$$k(x) = f(Ax) - bg(Bx), \tag{5}$$

we obtain from (3), (4), and (5) that

$$k(x^2) = 2xk(x) + \gamma x^2, \quad \text{where} \quad \gamma = f(1) - ag(1). \tag{6}$$

Now, define

$$H(x) = k(x) + \gamma x. \tag{7}$$

Then $H(x^2) = k(x^2) + \gamma x^2 = 2xk(x) + 2\gamma x^2 = 2xH(x)$. Thus H is a derivative. Now, (3), (5), and (7) yield

$$f(x) - ag(x) = -2H(x) + \gamma x, \tag{8}$$

and

$$f(Ax) - bg(Bx) = H(x) - \gamma x. \tag{9}$$

From (8) we see that f and g are linearly dependent relative to $\mathfrak{B}$.

We consider the following cases and subcases:

Case 1. $A \neq 0, B=0$. Subcases $a=0$; $a \neq 0$.

Case 2. $A=0, B \neq 0$. Subcases $a=0, b=0$; $a \neq 0, b=0$; $b \neq 0$.

Case 3. $A \neq 0, B \neq 0$. Subcases $a=0, b=0$; $a=0, b \neq 0$; $a \neq 0, b \neq 0$.

Case 1. Let $A \neq 0$ and $B=0$.

Subcase. Let $a=0$.

From (8) and (9) we get

$$f[(1+A)x] = -H(x). \tag{10}$$

If $A=-1$ we see from (10) that $H=0$, and from (8) we get $f(x)=\gamma x$, that is, f is continuous. This in (2) gives

$$\gamma x(1-x) = p_1\left(\frac{1}{x}+\alpha\right)g(x). \tag{11}$$

From (11) it is clear that g is continuous in some points and hence we can conclude that g is continuous everywhere. Thus from (11) we have

$$\left.\begin{aligned} p_1(x) &= c\left(1-\frac{1}{x-\alpha}\right), &&\text{where } c \text{ is a constant} \\ &= \frac{\varepsilon x+\delta}{x-\alpha}, &&\text{where } \varepsilon, \delta \text{ are constants}. \end{aligned}\right\} \tag{12}$$

Now suppose that $A \neq -1$. Then from (10), we have

$$\left.\begin{aligned} f(x) &= -H\left(\frac{x}{1+A}\right) \\ &= -\frac{1}{1+A}H(x) + \frac{x}{(1+A)^2}H(A). \end{aligned}\right\} \tag{13}$$

From (13) and (8) (with $a=0$) we get

$$(2A+1)f(x) = \frac{2H(A)}{1+A}x - \gamma x. \tag{14}$$

If further $A \neq -\frac{1}{2}$ we see from (14) that f is continuous.

As before, from (2) we can conclude that g is also continuous and that

$$\begin{aligned} p_1(x) &= c_1\left(1-\frac{A}{x-\alpha}\right), &&\text{where } c_1 \text{ is a constant} \\ &= \frac{\varepsilon x+\delta}{x-\alpha}, &&\text{where } \varepsilon, \delta \text{ are constants}, \end{aligned}$$

and that is (12). If $A=-\frac{1}{2}$ (with $A \neq -1$), we have from (13) that $f(x)=-2H(x)$.

Hence f is a derivative. Now (2) and (3) give

$$(1-x)f(x)=p_1\left(\frac{1}{x}+\alpha\right)g(x). \tag{15}$$

In (15), replacing x by rx, where r is a rational, and then letting

$$r\to\frac{1}{x(x_0-\alpha)},\quad \text{where } x_0 \text{ is an arbitrary point},$$

we get

$$\left(1-\frac{1}{x_0-\alpha}\right)f(x)=p_1(x_0)\,g(x),$$

which implies that $g(x)=kf(x)$, where k is a constant, $\neq 0$. If $k=0$, there is nothing to prove. Thus g is also a derivative. Now (15) and $g(x)=kf(x)$ imply that

$$p_1(x)=\frac{x-\alpha-1}{k(x-\alpha)}=\frac{\varepsilon x+\delta}{x-\alpha},\quad \text{where } \varepsilon, \delta \text{ are constants}. \tag{12}$$

Thus we have in this subcase either f and g are continuous or f and g are derivatives and p_1 is given by (12).

Subcase. Let $a\neq 0$. Then (8) and (9) yield

$$\left.\begin{aligned} f(x)&=\frac{1}{A}H(x)+f(1)\,x,\\ g(x)&=\left(\frac{2}{a}+\frac{1}{aA}\right)H(x)+g(1)\,x.\end{aligned}\right\} \tag{16}$$

Now (2) and (3) imply

$$(1-x)f(x)=\left(p_1\left(\frac{1}{x}+\alpha\right)-ax\right)g(x). \tag{17}$$

From (16) and (17), we get

$$\left[\frac{1}{A}H(x)+f(1)\,x\right](1-x)=\left[p_1\left(\frac{1}{x}+\alpha\right)-ax\right]\left[\left(\frac{2}{a}+\frac{1}{aA}\right)H(x)+g(1)\,x\right]. \tag{18}$$

In (18), putting $x=r$, where r is a rational, using the fact that H is a derivative and then allowing $r\to x$, we obtain

$$\begin{aligned} p_1(x)&=\frac{a}{x-\alpha}+\frac{f(1)}{g(1)}\left(1-\frac{1}{x-\alpha}\right)\\ &=\frac{\varepsilon x+\delta}{x-\alpha},\quad \text{where } \varepsilon, \delta \text{ are constants},\end{aligned}$$

provided $g(1)\neq 0$.

If $g(1)=0$, then from (16) we see that g is a derivative. Choose x_0 such that $g(x_0)\neq 0$. [$g=0$ in (2) gives $f=0$.] Then $H(x_0)\neq 0$. Replacing x by rx_0, where r is a rational, using $H(x_0)\neq 0$, $g(1)=0$ and then taking the limit

$$r\to\frac{1}{x(x_0-\alpha)}\quad (x\text{ any real}),$$

we have from (18),

$$\left.\begin{aligned} p_1(x) &= \frac{a}{x-\alpha}+\frac{\left[\frac{1}{A}H(x_0)+f(1)x_0\right]}{\left(\frac{2}{a}+\frac{1}{aA}\right)H(x_0)}\left(1-\frac{1}{x-\alpha}\right)\\ &= \frac{\varepsilon x+\delta}{x-\alpha},\quad \text{where } \varepsilon,\ \delta \text{ are constants,}\end{aligned}\right\}\tag{19}$$

which is (12), provided that $A\neq -\frac{1}{2}$. But $A=-\frac{1}{2}$ in (16) gives $g=0$. Then from (2), we get $f=0$. Thus in this subcase we have, $f, g\in\mathfrak{B}$, or $f\in\mathfrak{B}$, g is a derivative and p_1 is given by (12) or $f=0$, $g=0$.

Case 2. Let $A=0$, $B\neq 0$.

Subcase. Let $a=0$, $b=0$.

From (3), we get $f(x)=0$. Since $p_2(x)\not\equiv 0$, let x_0 be such that $p_2(x_0)\neq 0$.

Now in (2), replacing x by rx, where r is a rational and then taking the limit as

$$r\to\frac{1}{x(x_0-\alpha)},$$

we get

$$g(Bx^2)=kxg(x),\qquad \text{where } k \text{ is a constant, } \neq 0,\tag{20}$$

and

$$p_1(x)=-\frac{k}{x-\alpha}\,p_2(x).\tag{21}$$

In (20), replacing x by $x+r$, where r is a rational, we obtain

$$2g(Bx)=kg(1)\,x+kg(x).\tag{22}$$

Hence (20) and (22) yield

$$g(x^2)=2xg(x)-g(1)\,x\ .\tag{23}$$

Now set

$$D(x)=g(x)-g(1)\,x.\tag{24}$$

Now from (23) and (24) we see that D is a derivative. Of course g can be obtained from (24). Hence we have in this subcase $f=0$, $g\in\mathfrak{B}$ and p_1 and p_2 are given by (21)

Subcase. Let $b=0$, $a\neq 0$.

Then (3) gives $f(x)=ag(x)$.

Now (2) becomes

$$g(x)\left[a - p_1\left(\frac{1}{x} + \alpha\right)\right] = p_2\left(\frac{1}{x} + \alpha\right) g(Bx^2). \tag{25}$$

Replacing x by rx in (25), where r is a rational, and letting

$$r \to \frac{1}{x(x_0 - \alpha)}, \quad \text{where } x_0 \text{ is an arbitrary point},$$

$$g(Bx^2) = kxg(x), \quad \text{where } k \text{ is a constant}, \neq 0\,. \tag{20}$$

If $k=0$ there is nothing to prove.

As before g can be obtained from (24) and f from $f(x)=ag(x)$. From (20) and (25) we get

$$p_1(x) = a - \frac{k}{x - \alpha}\, p_2(x). \tag{26}$$

So we have $f, g \in \mathfrak{B}$ and p_1 and p_2 are given by (26).

Subcase. Let $b \neq 0$.

Now (8) and (9) give

$$\left.\begin{aligned} f(x) &= a_1 H(x) + f(1)\,x, \\ g(x) &= a_2 H(x) + g(1)\,x, \end{aligned}\right\} \tag{27}$$

where $a_2 = -(1/bB) \neq 0$, $a_1 = aa_2 - 2$.

From (2) and (27), using H as a derivative, we have

$$\left.\begin{aligned} a_1 H(x) + f(1)\,x &= p_1\left(\frac{1}{x} + \alpha\right)[a_2 H(x) + g(1)\,x] \\ &+ p_2\left(\frac{1}{x} + \alpha\right)[a_2 x^2 H(B) + 2a_2 BxH(x) + g(1)\,Bx^2]. \end{aligned}\right\} \tag{28}$$

In (28), put $x=r$, where r is a rational, make use of the fact that H is a derivative and then allow $r \to x$, where x is any real number; we have

$$\begin{aligned} p_1(x) &= \frac{f(1)}{g(1)} - \frac{[a_2 H(B) + g(1)\,B]}{g(1)}\,\frac{p_2(x)}{x - \alpha} \\ &= c_1 + \frac{c_2}{x - \alpha}\, p_2(x), \quad \text{where } c_1, c_2 \text{ are constants}, \end{aligned}$$

provided $g(1) \neq 0$. If $g(1)=0$ (27) gives that g is a derivative. Choose y_0 such that $g(y_0) \neq 0$ ($g=0$ in (2) gives $f=0$). Hence $H(y_0) \neq 0$.

Replacing x by ry_0, r a rational, using $H(y_0) \neq 0$, $g(1)=0$ and then taking the limit as

$$r \to \frac{1}{x(y_0 - \alpha)}, \quad \text{where } x \text{ is any real number},$$

we have from (28)

$$\left.\begin{aligned} p_1(x) &= \frac{a_1 H(y_0) + f(1)\, y_0}{a_2 H(y_0)} - \frac{[a_2 H(B)\, y_0 + 2a_2 B H(y_0)]}{a_2 H(y_0)} \frac{p_2(x)}{x-\alpha} \\ &= c_1 + \frac{c_2}{x-\alpha}\, p_2(x), \text{ where } c_1, c_2 \text{ are constants}. \end{aligned}\right\} \tag{29}$$

Hence we have either $f, g \in \mathfrak{B}$ or $f \in \mathfrak{B}$, g is a derivative and p_1 and p_2 are given by (29).

Case 3. Let $A \neq 0$, $B \neq 0$.

Subcase. Let $a=0$, $b=0$.

From (3), we get

$$f(Ax^2) = -\, x f(x). \tag{30}$$

Replacing x by $x+r$ in (30), where r is a rational, and using (30), we have

$$2f(Ax) = -\, f(1)\, x - f(x). \tag{31}$$

Now (30) and (31) yield

$$f(x^2) = 2x f(x) - f(1)\, x^2 \tag{32}$$

similar to (23). Defining

$$L(x) = f(x) - f(1)\, x, \tag{33}$$

we see that L is a derivative. Now f can be obtained from (33).

From (30) and (33) we have

$$\begin{aligned} L(Ax^2) &= -\, x f(x) - f(1)\, A x^2 \\ \text{also} \quad &= x^2 [f(A) - f(1)\, A] + 2A x f(x) - 2A f(1)\, x^2 . \end{aligned}$$

Thus we have

$$(2A+1) f(x) = [2A f(1) - f(A)]\, x . \tag{34}$$

From (34) we have two complementary sub-subcases, that is, either $A = -\frac{1}{2}$ or f is continuous.

First let us consider the sub-subcase that $A = -\frac{1}{2}$. Now (30) shows that f is a derivative.

Choose y_0 such that $g(y_0) \neq 0$. Replacing x by $r y_0$ in (2), where r is a rational, and then allowing

$$r \to \frac{1}{x(y_0 - \alpha)}, \quad \text{where } x \text{ is a real number},$$

we obtain

$$\left.\begin{aligned} p_1(x) &= \frac{f(y_0)}{g(y_0)} \left(1 - \frac{1}{x-\alpha}\right) - \frac{g(B y_0^2)}{g(y_0)\, y_0} \frac{p_2(x)}{x-\alpha} \\ &= d_1 + \frac{d_2}{x-\alpha} + \frac{d_3}{x-\alpha}\, p_2(x), \quad \text{where } d_1, d_2, d_3 \text{ are constants}. \end{aligned}\right\} \tag{35}$$

Using (2) and (35) we get

$$(1-x)f(x) = p_2\left(\frac{1}{x}+\alpha\right)g(Bx^2) + \left[d_1 + d_2 x + d_3 x p_2\left(\frac{1}{x}+\alpha\right)\right]g(x). \qquad (36)$$

Replacing x by rx in (36), where r is a rational, and allowing

$$r \to \frac{1}{x(x_0-\alpha)}$$

(where $p_2(x_0) \neq 0$), we get

$$k_1 x f(x) = k_2 x g(x) + g(Bx^2), \quad \text{where } k_1, k_2 \text{ are constants}. \qquad (37)$$

Now $k_2 = 0$ gives

$$g(Bx^2) = k_1 x f(x), \qquad (38)$$

which in turn gives, using the fact that f is a derivative, that

$$2g(Bx) = k_1 f(x).$$

Hence

$$g(x) = \frac{k_1}{2B} f(x) + \frac{k_1}{2} f\left(\frac{1}{B}\right) x. \qquad (39)$$

If $k_2 \neq 0$, replacing x by $x+r$ in (37), where r is a rational, using the fact that f is a derivative and (37), we get

$$k_1 f(x) = k_2 g(1)\, x + k_2 g(x) + 2g(Bx). \qquad (40)$$

From (37), (40) and using the fact that f is a derivative, we have

$$g(x^2) = 2x g(x) - g(1)\, x^2. \qquad (23)$$

Hence g can be obtained from (24). Thus we have f as a derivative $g \in \mathfrak{B}$ and p_1 and p_2 are given by (35).

Now we take up the other sub-subcase, that f is continuous, say $f(x) = cx$, where c is a constant. Using (30), (2) can be rewritten as

$$(1-x)\,cx = p_1\left(\frac{1}{x}+\alpha\right)g(x) + p_2\left(\frac{1}{x}+\alpha\right)g(Bx^2). \qquad (41)$$

Replacing x by rx in (41), where r is a rational, and allowing

$$r \to \frac{1}{x(x_0-\alpha)}$$

(where $p_2(x_0) \neq 0$), we get

$$k_1 x^2 = k_2 x g(x) + g(Bx^2), \quad \text{where } k_1, k_2 \text{ are constants}. \qquad (42)$$

Now $k_2=0$ implies that g is continuous, say $g(x)=dx$, where d is a constant. From (41), we now obtain

$$p_1(x)=\frac{c}{d}\left(1-\frac{1}{x-\alpha}\right)-\frac{B}{x-\alpha}p_2(x)$$
$$=d_1+\frac{d_2}{x-\alpha}+\frac{d_3}{x-\alpha}p_2(x), \quad \text{where the } d_i\text{'s are constants}.$$

If $k_2\neq 0$, from (42)

$$2k_1x=k_2xg(1)+k_2g(x)+2g(Bx)$$

can be obtained, which with (42) gives

$$g(x^2)=2xg(x)-g(1)\,x. \tag{23}$$

Hence g can be obtained from (24).

Now (2) can be rewritten using (24) as

$$\left.\begin{aligned}(1-x)\,cx&=p_1\left(\frac{1}{x}+\alpha\right)[D(x)+g(1)\,x]\\&\quad+p_2\left(\frac{1}{x}+\alpha\right)[x^2D(B)+2BxD(x)+g(1)\,Bx^2].\end{aligned}\right\} \tag{43}$$

Putting $x=r$ in (43), where r is a rational, using D as a derivative and then taking the limit as $r\to x$, for any real number x, we get

$$p_1(x)=\frac{c}{g(1)}\left[1-\frac{1}{x-\alpha}\right]-\frac{D(B)+g(1)\,B}{g(1)}\frac{p_2(x)}{x-\alpha}$$
$$=d_1+\frac{d_2}{x-\alpha}+\frac{d_3}{x-\alpha}p_2(x), \quad \text{where the } d_i\text{'s are constants},$$

provided $g(1)\neq 0$. If $g(1)=0$, choose y_0 such that $g(y_0)\neq 0$. Now from (24) g is a derivative and $D(y_0)\neq 0$.

Replacing x by ry_0 in (43), where r is a rational, and making

$$r\to\frac{1}{x(y_0-\alpha)} \quad \text{where } x \text{ is a real number},$$

we get

$$\left.\begin{aligned}p_1(x)&=\frac{cy_0}{D(y_0)}\left(1-\frac{1}{x-\alpha}\right)-\frac{D(B)\,y_0+2BD(y_0)}{D(y_0)}\frac{p_2(x)}{x-\alpha}\\&=d_1+\frac{d_2}{x-\alpha}+\frac{d_3}{x-\alpha}p_2(x), \quad \text{where the } d_i\text{'s are constants}.\end{aligned}\right\} \tag{35}$$

Thus we have in the subcase $a=0$, $b=0$ either f and g are continuous or f is continuous and $g\in\mathfrak{B}$ and p_1 and p_2 are given by (35).

Subcase. Let $a=0, b\neq 0$.

From (8) and (9) we have

$$\left.\begin{aligned} f(x) &= -2H(x)+\gamma x \\ g(x) &= vH(x)+\mu x, \qquad \text{where } v=\frac{1+2A}{bB}, \quad \mu=g(1). \end{aligned}\right\} \tag{44}$$

Using (44) and the fact that H is a derivative, (2) can be rewritten as

$$\left.\begin{aligned} &-2H(x)+\gamma x-4AxH(x)-2x^2H(A)+\gamma Ax^2 = p_1\left(\frac{1}{x}+\alpha\right) \\ &\quad \times [\mu x+vH(x)] + p_2\left(\frac{1}{x}+\alpha\right)[\mu Bx^2+vH(B)x^2+2vBxH(x)]. \end{aligned}\right\} \tag{45}$$

In (45), taking $x=r$, where r is a rational, using the fact that H is a derivative and then taking the limit as $r\to x$ (for any real number x), we get

$$\left.\begin{aligned} p_1(x) &= \frac{\gamma}{\mu} - \frac{2H(A)-\gamma A}{(x-\alpha)} - \frac{\mu B+vH(B)}{\mu}\frac{p_2(x)}{x-\alpha} \\ &= d_1+\frac{d_2}{x-\alpha}+\frac{d_3}{x-\alpha}\,p_2(x), \quad \text{where the } d_i\text{'s are constants}, \end{aligned}\right\} \tag{35}$$

provided $g(1)\neq 0$.

If $g(1)=0$, from (44) we have g as a derivative. Choose y_0 such that $g(y_0)\neq 0$. Then $H(y_0)\neq 0$. [$g=0$ in (2) gives $f=0$.] In (45) replacing x by ry_0, where r is a rational, using $H(y_0)\neq 0$, $g(1)=0$ and then letting

$$r\to\frac{1}{x(y_0-\alpha)}, \qquad \text{where } x \text{ is any real number},$$

we get

$$p_1(x)=d_1+\frac{d_2}{x-\alpha}+\frac{d_3}{x-\alpha}\,p_2(x), \qquad \text{where the } d_i\text{'s are constants}. \tag{35}$$

So, we have either $f, g\in\mathfrak{B}$ or $f\in\mathfrak{B}$ and g is a derivative and p_1 and p_2 are given by (35).

The subcase $a\neq 0$, $b=0$ can be discussed similarly. To finish the proof let us consider the subcase $a\neq 0, b\neq 0$.

Subcase. Let $a\neq 0, b\neq 0$.

Now (2) and (3) imply

$$(1-x)f(x)=\left[p_1\left(\frac{1}{x}+\alpha\right)-ax\right]g(x)+\left[p_2\left(\frac{1}{x}+\alpha\right)-b\right]g(Bx^2). \tag{46}$$

Changing x into rx in (46), where r is a rational, and then allowing

$$r\to\frac{1}{x(y-\alpha)},$$

where y is any real number, we have

$$\left(1-\frac{1}{y-\alpha}\right)f(x)=\left[p_1(y)-\frac{a}{y-\alpha}\right]g(x)+\frac{[p_2(y)-b]\,g(Bx^2)}{x(y-\alpha)}. \tag{47}$$

With the equation (47), we discuss the following cases:

(i) $p_1(x)=\dfrac{a}{x-\alpha}$, $p_2(x)=b$ for all x.

(ii) $p_1(x)=\dfrac{a}{x-\alpha}$ for all x, there is an x such that $p_2(x)\neq b$.

(iii) there is an x for which $p_1(x)\neq\dfrac{a}{x-\alpha}$ and $p_2(x)=b$ for all x.

(iv) neither of (i) nor (ii) nor (iii) holds.

Let us consider (i), that is

$$p_1(x)=\frac{a}{x-\alpha} \quad\text{and}\quad p_2(x)=b \qquad \text{for all } x.$$

Then from (47) results $f=0$, and from (8) results

$$g(x)=\frac{2}{a}H(x)-\frac{\gamma}{a}x.$$

(ii) Let us take up case (ii). From (47) we obtain

$$g(Bx^2)=kxf(x), \qquad \text{where } k\neq 0 \text{ is a constant} \tag{48}$$

($k=0$ gives $g=0, f=0$). From (48) results

$$2g(Bx)=kf(1)\,x+kf(x). \tag{49}$$

Thus (48) and (49) give

$$f(x^2)=2xf(x)-f(1)\,x^2. \tag{32}$$

Hence f can be determined from (33) and g from (49),

$$g(x)=\frac{k}{2B}L(x)+g(1)\,x. \tag{50}$$

Utilizing (2), (33), (48), (50), and (ii) we get

$$\left.\begin{aligned}&L(x)+f(1)\,x+x^2L(A)+2AxL(x)+f(1)\,Ax^2\\&\quad=ax\left[\frac{k}{2B}L(x)+g(1)\,x\right]+p_2\left(\frac{1}{x}+\alpha\right)[kxL(x)+f(1)\,kx^2].\end{aligned}\right\} \tag{51}$$

Putting $x=r$ in (51), where r is a rational, using the fact that L is a derivative and allowing $r \to x$, we have

$$\left.\begin{aligned} p_2(x) &= \frac{L(A) + f(1)\,A - ag(1)}{kf(1)} + \frac{1}{k}(x-\alpha) \\ &= e_1 + e_2(x-\alpha), \quad \text{where the } e_i\text{'s are constants}, \end{aligned}\right\} \tag{52}$$

provided $f(1) \neq 0$.

If $f(1)=0$, then from (33) it follows that f is a derivative. Choose z_0 such that $f(z_0) \neq 0$. Then $H(z_0) \neq 0$.

In (51), using $f(1)=0$, putting $x = rz_0$, where r is a rational and allowing

$$r \to \frac{1}{x(z_0 - \alpha)}, \quad \text{where } x \text{ is any real number},$$

we get

$$\left.\begin{aligned} p_2(x) &= \frac{L(A)\,z_0 + 2AL(z_0) - \frac{ka}{2B}L(z_0) - g(1)\,az_0}{kL(z_0)} + \frac{1}{k}(x-\alpha) \\ &= e_1 + e_2(x-\alpha), \quad \text{where the } e_i\text{'s are constants}. \end{aligned}\right\} \tag{52}$$

(iii) Let us now consider case (iii) $p_2(x) = b$, for all x. Then (47) gives $f(x) = kg(x)$, where k is a constant ($\neq 0$). Hence f and g can be determined from (8), unless $k = a$. Further, from (46) and $f(x) = kg(x)$, we get

$$p_1(x) = k + \frac{a-k}{x-\alpha}. \tag{53}$$

For $k = a$, $f(x) = ag(x)$ in (46) gives $p_1(x) = a$.

(iv) The last case to be considered is (iv). From (47), we have

$$k_1 x f(x) = k_2 x g(x) + g(Bx^2), \quad \text{where the } k_i\text{'s are constants} \quad (k_1 \neq 0, k_2 \neq 0). \tag{54}$$

From (54) results

$$2g(Bx) = k_1 f(1)\,x + k_1 f(x) - k_2 g(1)\,x - k_2 g(x). \tag{55}$$

From (55) and (54), we have

$$k_1\left[f(x^2) - 2xf(x) + f(1)\,x^2\right] = k_2\left[g(x^2) - 2xg(x) + g(1)\,x^2\right]. \tag{56}$$

Now set

$$R(x) = k_1 f(x) - k_2 g(x) \tag{57}$$

Then, from (56) and (57) we have

$$R(x^2) = 2xR(x) - R(1)\,x^2. \tag{58}$$

Now define

$$S(x) = R(x) + R(1)\,x. \tag{59}$$

Then S is a derivative.

From (55), (57) and (59) result

$$\left.\begin{aligned} g(x) &= \frac{1}{2B} S(x) + g(1)\,x \\ f(x) &= c_1 S(x) + f(1)\,x, \quad \text{where } c_1 = \frac{1}{k_1}\left(1 - \frac{k_2}{2B}\right). \end{aligned}\right\} \tag{60}$$

From (2) and (60), we get

$$\left.\begin{aligned} c_1 S(x) + f(1)\,x + c_1 x^2 S(A) + 2c_1 A x S(x) + f(1)\,A x^2 \\ = p_1\left(\frac{1}{x} + \alpha\right)\left[\frac{S(x)}{2B} + g(1)\,x\right] \\ + p_2\left(\frac{1}{x} + \alpha\right)\left[xS(x) + \frac{x^2 S(B)}{2B} + g(1)\,Bx^2\right]. \end{aligned}\right\} \tag{61}$$

Putting $x = r$ in (61), where r is a rational, using the fact that S is a derivative and then taking the limit as $r \to x$ (for any real number x), we get

$$p_1(x) = d_1 + \frac{d_2}{x - \alpha} + \frac{d_3}{x - \alpha}\, p_2(x), \quad \text{where the } d_i\text{'s are constants},$$

that is (35), provided $g(1) \neq 0$. If on the other hand $g(1) = 0$, (60) shows g is a derivative. Choose y_0 such that $g(y_0) \neq 0$. Then $S(y_0) \neq 0$.

In (61), replacing x by ry_0, r, a rational, using $g(1) = 0$, $H(y_0) \neq 0$ and allowing

$$r \to \frac{1}{x(y_0 - \alpha)}, \quad \text{where } x \text{ is any real number},$$

we get

$$p_1(x) = d_1 + \frac{d_2}{x - \alpha} + \frac{d_3}{x - \alpha}\, p_2(x), \quad \text{where the } d_i\text{'s are constants},$$

that is (35). Thus the proof of the theorem is complete.

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