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Research Papers

Additive selections of superadditive set-valued functions

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Dedicated to Professor Otto Haupt on his 100th birthday

Summary. A set-valued function F from a cone C with a cone-basis of a topological vector space X into the family of all non-empty compact convex subsets of a locally convex space Y is called superadditive provided that $F(x) + F(y) \subset F(x + y)$, for all $x, y \in C$. We show that every superadditive set-valued function admits an additive selection.

Let $(C, +)$ and $(Y, +)$ be two semigroups. A set-valued function F from C into the family of all non-empty subsets of Y is said to be superadditive provided that

$$F(x) + F(y) \subset F(x + y)$$

for all $x, y \in C$. Properties of superadditive set-valued functions were investigated by W. Smajdor in [13]. In this paper the existence of their additive selections is considered, i.e. homomorphisms $f\colon C \to Y$ such that $f(x) \in F(x)$ for all $x \in C$. Such a problem for additive set-valued functions was studied by H. Rådström [11], K. Nikodem [7], [8] and K. Przesławski [10]. Some existence theorems for additive selections of subadditive set-valued functions are given by P. Kranz [5], Z. Gajda and R. Ger [3] and by W. Smajdor [12].

EXAMPLE. Consider $C = Y = (0, \infty)$ and a function $g\colon C \to Y$. The set-valued function $F(x) = [g(x), \infty)$ is superadditive if and only if g is subadditive, i.e. $g(x + y) \leqslant g(x) + g(y)$, $x, y \in C$. Assume that F admits an additive selection f. The theorem of Bernstein and Doetsch implies the continuity of f and the equality $f(x) = cx$ holds for $x \in (0, \infty)$ with a certain $c \in \mathbb{R}$. Thus the function $x^{-1}g(x)$ is bounded above by c. Conversely, if the function $x^{-1}g(x)$ is bounded above by a constant $c > 0$, then the function $x \mapsto cx$ is an additive selection of F. The

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superadditive set-valued function $F(x) = [\sqrt{x}, \infty)$ does not admit any additive selection.

Now, let X and Y be two topological vector spaces and let C be a convex cone in X. Throughout this paper we shall assume that topological vector spaces satisfy the T_0 separation axiom, and that vector spaces are over the real numbers $\mathbb{R}$. In this paper $\mathrm{cc}(Y)$ denotes the family of all non-empty compact and convex subsets of Y.

A set-valued function $F: C \to \mathrm{cc}(Y)$ such that $F(tx) = tF(x)$ for every $x \in C$ and for every positive rational (positive) t is said to be Q_+-homogeneous (positively homogeneous).

A function $f: C \to \mathbb{R}$ is said to be Jensen-convex or Jensen or Jensen-concave if

$$f\left[\frac{1}{2}(x+y)\right] \leqslant \frac{1}{2}[f(x)+f(y)] \quad \text{or } f\left[\frac{1}{2}(x+y)\right] = \frac{1}{2}[f(x)+f(y)]$$

$$\text{or } f\left[\frac{1}{2}(x+y)\right] \geqslant \frac{1}{2}[f(x)+f(y)] \quad \text{for all } x, y \in C,$$

respectively.

First we shall prove the following theorem.

THEOREM 1. *Let X be a vector space and Y be a real topological vector space and let C be a convex cone in X. If $F: C \to \mathrm{cc}(Y)$ is a superadditive set-valued function, then there exists a minimal superadditive set-valued function $F_0: C \to \mathrm{cc}(Y)$ contained in F. It is $\mathbb{Q}_+$-homogeneous.*

Proof. Let $\mathscr{F}$ denote the family of all superadditive set-valued functions $G: C \to \mathrm{cc}(Y)$ contained in F. The family $\mathscr{F}$ is partially ordered by the relation $F \subset G$ iff $F(x) \subset G(x)$ for all $x \in C$. Let $\mathscr{T}$ be a chain in $\mathscr{F}$. Sets $G(x)$ $(x \in C, G \in \mathscr{T})$ are non-empty, compact and convex. Consequently the sets $G_0(x) := \cap\{G(x): G \in \mathscr{T}\}$ $(x \in C)$ are non-empty, compact and convex. Since $G_0(x) + G_0(y) \subset G(x) + G(y) \subset G(x+y)$ for $x, y \in C$ and $G \in \mathscr{T}$, therefore the function G_0 is superadditive. In virtue of the Kuratowski–Zorn lemma there exists a minimal element F_0 of the family $\mathscr{F}$. We have $nF_0(n^{-1}x) \subset F_0(x)$ for every $x \in C$ and for every positive integer n since F_0 is superadditive. The function $x \mapsto nF_0(n^{-1}x)$ belongs to $\mathscr{F}$ and by the minimality of F_0 we have $nF_0(n^{-1}x) = F_0(x)$ for every $x \in C$ and for every positive integer n. This forces F_0 to be $\mathbb{Q}_+$-homogeneous. This concludes the proof.

Observe that the set-valued function F_0 appearing in Theorem 1 is a Jensen-convex map, i.e.

$$\frac{1}{2}[F_0(x) + F_0(y)] \subset F_0\left[\frac{1}{2}(x+y)\right].$$

The following three theorems will be needed.

THEOREM A (Ger–Kominek [4]). *Let D be a non-empty open and convex subset of a topological vector space and let $f\colon D \to \mathbb{R}$ be a Jensen-convex functional. Then, for every $x \in D$, there exists a Jensen function $g_x\colon D \to \mathbb{R}$ such that $g_x(x) = f(x)$ and $g_x(y) \leqslant f(y)$ for all $y \in D$.*

THEOREM B (Ger–Kominek [4]). *Let D be a non-empty open and convex subset of a topological vector space X. If $g\colon D \to \mathbb{R}$ is a Jensen functional, then there exists an additive functional $a\colon X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $g(x) = a(x) + c$ for all $x \in D$.*

THEOREM C (Nikodem [9]). *Let D be an open convex subset of $\mathbb{R}^n$. If $g\colon D \to \mathbb{R}$ is a Jensen-convex function and $h\colon D \to \mathbb{R}$ is a Jensen-concave function and the inequality $g(x) \leqslant h(x)$ holds for $x \in D$, then there exist an additive function $a\colon \mathbb{R}^n \to \mathbb{R}$, a continuous and convex function $g_1\colon D \to \mathbb{R}$ and a continuous and concave function $h_1\colon D \to \mathbb{R}$ such that $g(x) = g_1(x) + a(x)$, $h(x) = h_1(x) + a(x)$ for $x \in D$.*

LEMMA 1. *Let X be a topological vector space and let $C \subset X$ be a cone with finite cone-basis E. If $F\colon C \to \mathrm{cc}([0, \infty))$ is a superadditive and $\mathbb{Q}_+$-homogeneous set-valued function, then it is continuous and positively homogeneous in the set $\mathrm{int}_L C$, where L is a subspace of X spanned by E.*

Proof. There exist a Jensen-convex function $g\colon C \to [0, \infty)$ and a Jensen-concave function $h\colon C \to [0, \infty)$ such that $F(x) = [g(x), h(x)]$ for all $x \in C$.

Let $E = \{e_1, e_2, \ldots, e_n\}$. The set

$$C = \{x_1 e_1 + \cdots + x_n e_n\colon x_i \geqslant 0, i = 1, \ldots, n\}$$

has non-empty interior in L, namely,

$$\mathrm{int}_L C = \{x_1 e_1 + \cdots + x_n e_n\colon x_i > 0, i = 1, \ldots, n\}$$

(cf. [2] Ch. 1, §2). In virtue of Theorem C, there exist an additive function $a\colon L \to \mathbb{R}$ and a continuous convex function $g_1\colon \mathrm{int}_L C \to \mathbb{R}$ and a continuous concave function $h_1\colon \mathrm{int}_L\, C \to \mathbb{R}$ such that

$$g_1(x) \leqslant h_1(x), \qquad g(x) = a(x) + g_1(x), \qquad h(x) = a(x) + h_1(x). \tag{1}$$

Let $x_0 \in \mathrm{int}_L\, C$. In a neighbourhood of x_0 the function $-a$ is bounded above, thus, by the theorem of Mehdi [6], it is continuous. Consequently g and h are continuous in $\mathrm{int}_L C$. Since g and h are $\mathbb{Q}_+$-homogeneous, they have to be positively homogeneous in $\mathrm{int}_L\, C$. This completes the proof.

LEMMA 2. *Assume that X is a topological vector space and C is a convex cone in X such that $0 \in C$. If $F\colon C \to \mathrm{cc}([0, \infty))$ is a superadditive and $\mathbb{Q}_+$-homogeneous set-valued function, then it is positively homogeneous.*

Proof. Since $F(0)$ is a bounded set, then $F(0) = \{0\}$ (see [13], Remark 1). Consequently

$$F(tx) = tF(x) \tag{2}$$

for $x = 0$ and $t > 0$. Suppose that $x \in C\backslash\{0\}$. Applying Lemma 1 to the set $E = \{x\}$ we get (2) for $t > 0$.

LEMMA 3. *Assume that X is a topological vector space, $C \subset X$ is a cone with a cone-basis E and $L = \mathrm{lin}\, E$. If $y \in C$, $x_0 \in \mathrm{int}_L\, C$ and $\lambda \in (0, 1]$, then $(1 - \lambda)y + \lambda x_0 \in \mathrm{int}_L\, C$.*

The easy proof is omitted.

Now, we shall show a selection theorem for superadditive set-valued functions with values contained in $\mathbb{R}$.

THEOREM 2. *Let X be a topological vector space and let $C \subset X$ be a cone with a finite cone-basis. If $F\colon C \to \mathrm{cc}(\mathbb{R})$ is a superadditive set-valued function, then there exists an additive selection of F.*

Proof. By Theorem 1 there exists a minimal superadditive set-valued function $F_0\colon C \to \mathrm{cc}(\mathbb{R})$ contained in F. It is $\mathbb{Q}_+$-homogeneous. There exist a Jensen-convex function $g\colon C \to \mathbb{R}$ and a Jensen-concave function $h\colon C \to \mathbb{R}$ such that

$$F_0(x) = [g(x), h(x)]$$

for all $x \in C$. Take $x_0 \in \mathrm{int}_L\, C$, where $L := \mathrm{lin}\, E$, where $E = \{e_1, \ldots, e_n\}$ is a basis for the cone C. According to Theorems A and B there exist an additive function $a\colon L \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $g(x) \geqslant a(x) + c$ for all $x \in \operatorname{int}_L C$ and $g(x_0) = a(x_0) + c$. Let $\lambda \in (0, 1) \cap \mathbb{Q}$. For every $y \in C$ we have

$$(1-\lambda)g(y) + \lambda g(x_0) \geqslant g((1-\lambda)y + \lambda x_0). \tag{3}$$

Applying Lemma 3 we have $(1-\lambda)y + \lambda x_0 \in \operatorname{int}_L C$. Consequently

$$\begin{aligned} g((1-\lambda)y + \lambda x_0) &\geqslant a((1-\lambda)y + \lambda x_0) + c = (1-\lambda)a(y) + \lambda a(x_0) + c \\ &= (1-\lambda)[a(y) + c] + \lambda g(x_0). \end{aligned} \tag{4}$$

Hence, by (3) and (4) we get

$$g(y) \geqslant a(y) + c \tag{5}$$

for every $y \in C$. Since the functions g and a are $\mathbb{Q}_+$-homogeneous, we have, in view of (5), for every $x \in C$ and for every positive integer n

$$g(x) \geqslant a(x) + n^{-1}c.$$

The last relation yields

$$g(x) \geqslant a(x) \tag{6}$$

for every $x \in C$.

Let us consider the set-valued function

$$F_a(x) := F_0(x) - a(x)$$

for $x \in C$. The function F_a is superadditive, $\mathbb{Q}_+$-homogeneous and its values are convex and compact subsets of $[0, \infty)$. By Lemma 2, F_a is positively homogeneous. For $x = x_1e_1 + \cdots + x_ne_n \in C$ we define

$$F_1(x) := x_1F_a(e_1) + \cdots + x_nF_a(e_n).$$

This set-valued function has compact and convex values. Since F_a is superadditive and positively homogeneous the inclusion

$$F_1(x) \subset F_a(x)$$

holds for all $x \in C$. Taking $x = x_1 e_1 + \cdots + x_n e_n \in C$ and $y = y_1 e_1 + \cdots + y_n e_n \in C$ we get $F_1(x) + F_1(y) = F_1(x + y)$. $F_1 + a$ is an additive set-valued function contained in F_0 with compact convex values. By the minimality of F_0 we have

$$F_0 = F_1 + a.$$

Thus $F_0(x_1 e_1 + \cdots + x_n e_n) = x_1 F_0(e_1) + \cdots + x_n F_0(e_n)$. Each function $x_1 e_1 + \cdots + x_n e_n \mapsto x_1 c_1 + \cdots + x_n c_n$, with $c_i \in F_0(e_i)$, $i = 1, \ldots, n$, is additive and contained in F_0. Consequently F_0 has to be single-valued and the proof is complete.

Now, we shall give the main results.

THEOREM 3. *Assume that X is a topological vector space and $C \subset X$ is a cone with a finite cone-basis. If Y is a locally convex space, then every superadditive set-valued function $F: C \to \mathrm{cc}(Y)$ has an additive selection.*

Proof. According to Theorem 1 there exists a minimal superadditive set-valued function $F_0: C \to \mathrm{cc}(Y)$ contained in F. Suppose that there exist $x_0 \in C$ and $u, v \in F_0(x_0)$ with $u \neq v$. By the Hahn–Banach theorem there exists an $l \in Y^*$ such that $l(u) \neq l(v)$. The set-valued function

$$(l \circ F_0)(x) := l[F_0(x)]$$

from C into $\mathbb{R}$ fulfils the hypotheses of Theorem 2. Therefore $l \circ F_0$ admits an additive selection $f: C \to \mathbb{R}$. Write

$$F_1(x) := \{y \in F_0(x) : l(y) = f(x)\}$$

for $x \in C$. If $y_1 \in F_1(x_1)$, $y_2 \in F_1(x_2)$, then $y_1 + y_2 \in F_0(x_1 + x_2)$ as F_0 is superadditive. Moreover, since f is additive we have $l(y_1 + y_2) = l(y_1) + l(y_2) = f(x_1) + f(x_2) = f(x_1 + x_2)$. Consequently F_1 is superadditive. The set $F_1(x)$, for all $x \in C$, is a closed subset of the compact set $F_0(x)$. The sets $F_1(x)$ are also convex in virtue of the linearity of l. It has been shown that the set-valued function $F_1 \subset F_0$ is superadditive with compact and convex values. F_1 is also not equal to F_0, since $\{u, v\} \not\subset F_1(x_0)$. But F_0 is a minimal set-valued function, a contradiction. Consequently the function F_0 is single-valued and the function a such that $\{a(x)\} = F_0(x)$ is an additive selection of F. The proof is finished.

The assumption that a cone-basis of a cone C is finite may be omitted.

THEOREM 4. *Assume that X is a topological vector space, $C \subset X$ is a cone with a cone-basis. If Y is a locally convex space, then every superadditive set-valued function $F: C \to \mathrm{cc}(Y)$ admits an additive selection.*

Proof. Let E be a cone-basis for C. In virtue of Theorem 1 there exists a minimal superadditive set-valued function $F_0: C \to \mathrm{cc}(Y)$ contained in F. Fix an x in C. We have $x = \sum_{i=1}^{n} \alpha_i e_i$, for some $\alpha_i \geqslant 0$, $e_i \in E$, $i = 1, 2, \ldots, n$. Put

$$C_0 := \{\beta_1 e_1 + \cdots + \beta_n e_n : \beta_1, \ldots, \beta_n \geqslant 0\}.$$

According to Theorem 3 there exists an additive selection $f: C_0 \to Y$ of the restriction $F_0|_{C_0}$ of function F_0. Write

$$F_1(y) := \begin{cases} \{f(y)\} & \text{for } y \in C_0 \\ F_0(y) & \text{for } y \in C \setminus C_0. \end{cases}$$

In the case $y, z \in C_0$, we have

$$F_1(y) + F_1(z) = \{f(y)\} + \{f(z)\} = \{f(y+z)\} = F_1(y+z).$$

On the other hand, for $y \in C$, $z \in C \setminus C_0$, we get

$$y = \sum_{i=1}^{m} \beta_i e_i, \qquad z = \sum_{i=1}^{m} \gamma_i e_i,$$

where $m > n$, $\beta_i, \gamma_i \geqslant 0$. Moreover there exists $i_0 > n$ such that $\gamma_{i_0} > 0$. Consequently

$$y + z = \sum_{i=1}^{m} (\beta_i + \gamma_i) e_i \in C \setminus C_0 \qquad \text{since } \beta_{i_0} + \gamma_{i_0} > 0.$$

In this case

$$F_i(y) + F_1(z) \subset F_0(y) + F_0(z) \subset F_0(y+z) = F_1(y+z).$$

Thus F_1 is a superadditive set-valued function such that $F_1 \subset F_0$. By the minimality of F_0 we obtain

$$F_1 = F_0$$

and $F_0(x)$ is a singleton. But x was fixed arbitrarily. Let $a(x)$ be the only element of the set $F_0(x)$. Since $F_0(x)$ is superadditive this function has to be additive. The proof has been finished.

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