A Probabilistic Interpretation of Complete Monotonicity

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1. Introduction

If $\{X_n\}$ is a sequence of *independent* random variables, then the joint distribution function $F_{i_1 \ldots i_m}$ of the m-element subset $\{X_{i_1}, \ldots, X_{i_m}\}$ of the sequence $\{X_n\}$ is given by the expression

$$F_{i_1 \ldots i_m}(x_1, \ldots, x_m) = F_{i_1}(x_1) \ldots F_{i_m}(x_m), \tag{1}$$

where F_{i_1} is the distribution function of X_{i_1}, etc. Now (1) can be immediately rewritten in the form:

$$F_{i_1 \ldots i_m}(x_1, \ldots, x_m) = \exp(-[-\log F_{i_1}(x_1) - \cdots - \log F_{i_m}(x_m)]). \tag{2}$$

It is the purpose of this paper to investigate the extent to which the well-known and essentially trivial result (2) can be extended in a non-trivial manner to sequences of *dependent* random variables.

To this end, let f be a function defined, continuous, and strictly decreasing on the extended half-line $[0, \infty]$, with $f(0)=1$ and $f(\infty) \geqslant 0$. Denote the inverse of f by f^{-1}. Then, if $\{X_n\}$ is a sequence of (not necessarily independent) random variables, with corresponding respective distribution functions $\{F_n\}$, we shall call $\{X_n\}$ *admissible* (or *exchangeable*) *under* f if the joint distribution function $F_{i_1 \ldots i_m}$ of any m-element subset $\{X_{i_1}, \ldots, X_{i_m}\}$ of the sequence is given by the expression

$$F_{i_1 \ldots i_m}(x_1, \ldots, x_m) = f(f^{-1}[F_{i_1}(x_1)] + \cdots + f^{-1}[F_{i_m}(x_m)]). \tag{3}$$

In the other direction, let $\{F_n\}$ be a sequence of 1-dimensional distribution functions. Then we shall call f *admissible over* $\{F_n\}$ if there exists a probability space $(\Omega, \mathscr{A}, P)$ and a sequence $\{X_n\}$ of random variables defined on that space such that: (a) F_n is the distribution function of X_n for each $n \geqslant 1$; (b) $\{X_n\}$ is exchangeable under f.

The principal results of this paper are the following:

THEOREM 1. *Suppose f is a strictly decreasing function from* $[0, \infty]$ *into* $[0, 1]$, *that $f(0)=1$ and $f(\infty) \geqslant 0$, and that $\{F_n\}$ is a sequence of continuous distribution functions over which f is admissible. Then f is completely monotone on* $[0,\infty)$. (*That is, f is*

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continuous on $[0, \infty)$ *and has derivatives of all orders on* $(0, \infty)$ *which alternate successively in sign:* $(-1)^n f^{(n)} \geqslant 0, n=0, 1, 2, \ldots$. See [14, p. 145].)

THEOREM 2. *Suppose* $\{F_n\}$ *is a sequence of distribution functions. Suppose f is a function from* $[0, \infty]$ *onto* $[0, 1]$ *which is completely monotone on* $[0, \infty)$. *Then f is admissible over* $\{F_n\}$.

THEOREM 3. *Suppose f is a completely monotone function from* $[0, \infty)$ *into* $(0, 1]$. *Then*

$$f(x)\, f(y) \leqslant f(x+y) \leqslant [f(x+my)]^{1/m} [f(x)]^{1/n}$$

for all $x, y \geqslant 0$ *and all positive real m and n satisfying* $1/m + 1/n = 1$.

THEOREM 4. *Suppose* $\{X_n\}$ *is a sequence of random variables with corresponding continuous distribution functions* $\{F_n\}$. *Suppose f is a strictly decreasing function from* $[0, \infty]$ *into* $[0, 1]$ *under which* $\{X_n\}$ *is exchangeable. Then there exists g, strictly decreasing from* $[0, \infty]$ *into* $[0, 1]$, *under which* $\{-X_n\}$ *is exchangeable, if and only if* $f(x) = r^{-x}$ *for some* $r > 0$ *and all x in* $[0, \infty]$.

THEOREM 5. *Suppose* $\{X_n\}$ *is a sequence of random variables and that f is a strictly decreasing function from* $[0, \infty]$ *into* $[0, 1]$ *under which* $\{X_n\}$ *is admissible. Let* $\{F_n\}$ *be the sequence of distribution functions corresponding to* $\{X_n\}$. *Then*

i) *if* $\sum_{n=1}^{\infty} [1 - F_n(x_n)] < \infty$, *then* $\sum_{n=1}^{\infty} f^{-1}[F_n(x_n)] < \infty$;

ii) $P[X_n > x_n \text{ inf. oft.}] > 0$ *if and only if* $\sum_{n=1}^{\infty} f^{-1}[F_n(x_n)] = \infty$;

iii) *if* $f(x) \geqslant e^{-nx}$ *for some n and all x in* $(0, \infty)$ *and* $\sum_{n=1}^{\infty}[1 - F_n(x_n)] = \infty$, *then* $\sum_{n=1}^{\infty} f^{-1}[F_n(x_n)] = \infty$.

There is a counterpart of Theorems 1 and 2 for *events* rather than random variables, as follows: *Given a sequence* $\{r_n\}$ *of real numbers with* $r_0 = 1$ *and* $0 \leqslant r_1 \leqslant 1$, *there exists a probability space* $(\Omega, \mathscr{A}, P)$ *and events* $\{E_m\}$ *over that space satisfying*

$$P\left(\bigcap_{j=0}^{n} E_{m_j}\right) = r_n$$

for all n and $m_1 < m_2 < \cdots < m_n$, *if and only if* $\{r_n\}$ *is a completely monotone sequence.*

The 'if' part of the above statement is a corollary of Theorem 2 in virtue of the following connection between completely monotone functions and completely monotone sequences [14, p. 164]: For completely monotone f on $[0, \infty)$, the sequence $\{f(n)\}$ is completely monotone; for given $\{r_n\}$ with r_0 the least number for which $\{r_n\}$ is completely monotone, the sequence $\{r_n\}$ has an extension to a completely monotone function f on $[0, \infty)$ which satisfies $f(n) = r_n$ for $n = 0, 1, 2, \ldots$.

The 'only if' part of the statement is not a corollary of Theorem 1 because the distribution functions involved are discontinuous. It is, instead, a restatement of a result of de Finetti [5]; for an exposition of the proof, see [4, p. 225].

By a *strict t-norm* T, we shall mean a function defined on the unit square $[0, 1] \times [0, 1]$ which can be represented there by

$$T(x_1, x_2) = f[f^{-1}(x_1) + f^{-1}(x_2)] \tag{4}$$

for some function f which is strictly decreasing from $[0, \infty]$ onto $[0, 1]$. We shall call f a *generator* of T and note that f_1 and f_2 generate the same T if and only if $f_2(x) = f_1(ux)$ for some positive constant u (see [10], p. 171).

Suppose T is a strict t-norm. For any positive integer m and m numbers $x_1, x_2, \ldots, x_m$ in $[0, 1]$, we define

$$\begin{aligned} T(x_1) &= T(x_1, 1) = x_1, \\ T(x_1, x_2, x_3) &= T[T(x_1, x_2), x_3] \\ &= f[f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)], \\ &\vdots \\ T(x_1, \ldots, x_m) &= T[T(x_1, \ldots, x_{m-1}), x_m] \\ &= f[f^{-1}(x_1) + \cdots + f^{-1}(x_m)]. \end{aligned}$$

Now we shall be able, in the sequel, to denote the right side of (3) more simply $T[F_1(x_1), \ldots, F_m(x_m)]$.

Suppose $\{F_n\}$ is a sequence of distribution functions and T is a strict t-norm with generator f. We shall call T *admissible over* $\{F_n\}$ if f is admissible over $\{F_n\}$.

A historical note on t-norms may be in order. The name is an abbreviation of *triangle norm*, as introduced by Menger [8] in connection with statistical metric spaces. Literature on t-norms and related semigroups includes [6], [7], [9], [10], [11], and [12].

2. Proofs

First, let us adopt the notations E^m and I^m to denote, respectively, m-dimensional Euclidean space, and the m-dimensional closed unit cube in E^m.

To prove Theorem 1, we shall use the following lemma adapted from Widder ([14], p. 147): f is completely monotone over $[0, \infty)$ if and only if

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(y - kh) \geqslant 0$$

for all $n \geqslant 0$ and all y and h satisfying

$$0 \leqslant y - nh < \cdots < y - h < y < \infty.$$

To start the proof, let such n, y, and h be given. Determine α and β by $y=nf^{-1}(\beta)$ and $h=f^{-1}(\beta)-f^{-1}(\alpha)$. For $1\leqslant j\leqslant n$, determine a_j and b_j by $F_j(a_j)=\alpha$ and $F_j(b_j)=\beta$, and write A_j for $f^{-1}[F_j(a_j)]$ and B_j for $f^{-1}[F_j(b_j)]$. Now we have $f(y-kh)=(n-k)B_j+kA_j$ for $j=1,\ldots,n$. Consequently,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(y-kh)=\sum_{k=0}^{n}(-1)^k\sum_{\delta'\in\Delta'_{k,n}}f[\sigma(\delta')], \tag{5}$$

where δ' ranges through the set $\Delta'_{k,n}$ of $\binom{n}{k}$ vertices of the cell

$$((A_1,\ldots,A_n),(B_1,\ldots,B_n)]$$

which consist of k A_j's and $n-k$ B_j's, and $\sigma(\delta')$ is the sum of the components of δ'. Since $(n-k)B_j+kA_j=(n-k)f^{-1}(\alpha)+f^{-1}(\beta)$ for $j=1,\ldots,n$, the right side of (5) becomes

$$\sum_{k=0}^{n}(-1)^k\sum_{\delta\in\Delta_{k,n}}F_{1,\ldots,n}(\delta), \tag{6}$$

where δ ranges through the set $\Delta_{k,n}$ of $\binom{n}{k}$ vertices of the cell

$$(a,b]=((a_1,\ldots,a_n),(b_1,\ldots,b_n)]$$

which consist of k a_i's and $n-k$ b_i's. As the Stieltjes measure of $(a,b]$ with respect to the joint distribution function $F_{1,\ldots,n}$, (6) is nonnegative. Now the lemma applies, and we conclude that f is completely monotone on $[0,\infty)$.

DEFINITION 1. A strict t-norm T is m-monotone if for every cell $(a,b]=((a_1,\ldots,a_m),(b_1,\ldots,b_m)]$ in I^m,

$$\sum_{k=0}^{m}(-1)^k\sum_{\delta\in\Delta_{k,m}}T(\delta)\geqslant 0, \tag{7}$$

where δ ranges through the set $\Delta_{k,m}$ of those $\binom{m}{k}$ vertices of $(a,b]$ which consist of k a_i's and $m-k$ b_i's. If T is m-monotone for every $m\geqslant 1$, then we shall call T *completely monotone.*

COROLLARY TO THEOREM 1. *If a strict t-norm T is admissible over a sequence $\{F_n\}$ of continuous distribution functions, then T and its generator(s) are completely monotone.*

LEMMA 2a. *Let T be a completely monotone t-norm and let $\{G_n\}$ be a sequence of distribution functions. Then T is admissible over $\{G_n\}$.*

Proof. Given a completely monotone t-norm T and sequence $\{G_n\}$, define, for

$m=1, 2, \dots$ and each m-element set $\{n_1, \dots, n_m\}$ of positive integers, a function $F_{n_1,\dots,n_m}$ on E^m by

$$F_{n_1,\dots,n_m}(x_1, \dots, x_m) = T\left[G_1(x_1), \dots, G_m(x_m)\right].$$

Then the collection

$$\Gamma = \{F_{n_1,\dots,n_m} : n_1, \dots, n_m \text{ are distinct positive integers}\}$$

clearly satisfies items a, b, c, e, and f of the hypothesis of the Kolmogorov Theorem as found in Tucker [13], p. 30. It remains to be seen that item d is also satisfied.

Let m be any positive integer and let

$$(a, b] = ((a_1, \dots, a_m), (b_1, \dots, b_m)]$$

be an arbitrary m-dimensional cell in E^m. Then for given $n_1 < \cdots < n_m$,

$$((F_{n_1}(a_1), \dots, F_{n_m}(a_m)), (F_{n_1}(b_1), \dots, F_{n_m}(b_m))]$$

is a cell in I^m and, by (7),

$$\sum_{k=0}^{m} (-1)^k \sum_{\delta \in \Delta_{k,m}} T(\delta) \geqslant 0,$$

where δ ranges through the set $\Delta_{k,m}$ of those $\binom{m}{k}$ vertices of the cell

$$((F_{n_1}(a_1), \dots, F_{n_m}(a_m)), (F_{n_1}(b_1), \dots, F_{n_m}(b_m))]$$

which consist of k $F_{n_i}(a_i)$'s and $m-k$ $F_{n_i}(b_i)$'s. But this means

$$\sum_{k=0}^{m} (-1)^k \sum_{\delta \in \Delta_{k,m}} F_{n_1,\dots,n_m}(\delta) \geqslant 0,$$

where, *in this expression*, δ ranges through the set $\Delta_{k,m}$ of those $\binom{m}{k}$ vertices of the cell $(a, b]$ which consist of k a_i's and $m-k$ b_i's.

Thus, the Kolmogorov Theorem applies to the collection Γ and there exist a probability space $(\Omega, \mathscr{A}, P)$ and random variables X_n over Ω whose distribution functions and joint distribution functions are, with corresponding indices, those in Γ. Therefore T is admissible over $\{G_n\}$.

DEFINITION 2. Suppose $0 \leqslant a \leqslant b \leqslant 1$. In the class of functions $T(x_1, \dots, x_m)$, define

$$\Delta_k(a, b)\, T(x_1, \dots, x_m) = T(x_1, \dots, x_{k-1}, b, x_{k+1}, \dots, x_m) - T(x_1, \dots, x_{k-1}, a, x_{k+1}, \dots, x_m).$$

We shall write $\Delta a_k b_k$ for $\Delta_k(a_k, b_k)$ and note that these operators commute:

$$\Delta a_k b_k \Delta a_j b_j T(x_1, \dots, x_m) = \Delta a_j b_j \Delta a_k b_k T(x_1, \dots, x_m).$$

LEMMA 2b. *Let* $m \geqslant 1$. *Suppose* T *is any function which carries* I^m *into* E^1 *and suppose the cell* $((a_1, \dots, a_m), (b_1, \dots, b_m)]$ *lies in* I^m. *Then*

$$\sum_{k=0}^{m} (-1)^k \sum_{\delta \in \Delta_{k,m}} T(\delta) = \Delta a_m b_m \Delta a_{m-1} b_{m-1} \dots \Delta a_1 b_1 T(x_1, \dots, x_m), \tag{8}$$

where δ *ranges as in* (7).

Proof. If $m=1$, clearly (8) holds. Assume for arbitrary q that (8) holds for all functions carrying I^{q-1} into E^1. Let T be any function from I^q into E^1 and let $((a_1, \dots, a_q), (b_1, \dots, b_q)]$ be a cell in I^q. Then T_b and T_a, given respectively by

$$T(x_1, \dots, x_{q-1}, b_q) \quad \text{and} \quad T(x_1, \dots, x_{q-1}, a_q),$$

are $q-1$ place functions to which the induction hypothesis applies:

$$\begin{aligned} \sum_{k=0}^{q} (-1)^k \sum_{\delta \in \Delta_{k,q}} T(\delta) &= \sum_{k=0}^{q-1} (-1)^k \sum_{\delta \in \Delta_{k,q-1}} T_b(\delta) - \sum_{k=0}^{q-1} (-1)^k \sum_{\delta \in \Delta_{k,q-1}} T_a(\delta) \\ &= \Delta a_{q-1} b_{q-1} \dots \Delta a_1 b_1 T_b(x_1, \dots, x_{q-1}) - \Delta a_{q-1} b_{q-1} \dots \Delta a_1 b_1 T_a(x_1, \dots, x_{q-1}) \\ &= \Delta a_{q-1} b_{q-1} \dots \Delta a_1 b_1 T(x_1, \dots, x_{q-1}, b_q) - \Delta a_{q-1} b_{q-1} \dots \Delta a_1 b_1 T(x_1, \dots, x_{q-1}, a_q) \\ &= \Delta a_q b_q \Delta a_{q-1} b_{q-1} \dots \Delta a_1 b_1 T(x_1, \dots, x_{q-1}, x_q). \end{aligned}$$

LEMMA 2c. *Let* $m \geqslant 1$. *If* f *is completely monotone from* $[0, \infty]$ *onto* $[0, 1]$, *then*

$$\frac{\partial}{\partial x_k} \cdots \frac{\partial}{\partial x_1} f\left[\sum_{j=1}^{m} f^{-1}(x_j)\right] = \frac{f^{(k)}\left[\sum_{j=1}^{m} f^{-1}(x_j)\right]}{f'[f^{-1}(x_1)] \dots f'[f^{-1}(x_k)]} \geqslant 0 \tag{9}$$

for all $(x_1, \dots, x_m) \in I^m$ and $1 \leqslant k \leqslant m$. Moreover, if

$((a_1, \dots, a_m), (b_1, \dots, b_m)]$ is a cell in I^m, then

$$\Delta a_m b_m \dots \Delta a_1 b_1 f\left[\sum_{j=1}^{m} f^{-1}(x_j)\right] \geqslant 0.$$

Proof. We shall write simply Ξ for $\sum_{j=1}^{m} f^{-1}(x_j)$. The first assertion obviously holds for $k=1$. Suppose $1 \leqslant q \leqslant m-1$ and (9) holds for $k=q-1$. Then

$$\frac{\partial}{\partial x_q}\left(\frac{\partial}{\partial x_{q-1}} \cdots \frac{\partial}{\partial x_1} f(\Xi)\right) = \frac{\partial}{\partial x_q} \frac{f^{(q-1)}(\Xi)}{f'[f^{-1}(x_1)] \dots f'[f^{-1}(x_{q-1})]},$$

which by the chain rule is the desired

$$\frac{f^{(q)}(\Xi)}{f'[f^{-1}(x_1)] \dots f'[f^{-1}(x_q)]}.$$

It follows for odd q that both numerator and denominator are nonpositive, and for even q, both nonnegative. Thus (9) holds.

To prove the second assertion, first note

$$\frac{\partial}{\partial x_{m-1}} \cdots \frac{\partial}{\partial x_1} \Delta a_m b_m f(\Xi) \geqslant 0,$$

since by (9), the function

$$\frac{\partial}{\partial x_{m-1}} \cdots \frac{\partial}{\partial x_1} f(\Xi)$$

is nondecreasing in x_m. Suppose now for $2 \leqslant k \leqslant m$ that

$$\frac{\partial}{\partial x_{k-1}} \cdots \frac{\partial}{\partial x_1} \Delta a_m b_m \ldots \Delta a_k b_k f(\Xi) \geqslant 0.$$

Then the function

$$\frac{\partial}{\partial x_{k-2}} \cdots \frac{\partial}{\partial x_1} \Delta a_m b_m \ldots \Delta a_k b_k f(\Xi)$$

is nondecreasing in x_{k-1}, so

$$\frac{\partial}{\partial x_{k-2}} \cdots \frac{\partial}{\partial x_1} \Delta a_{k-1} b_{k-1} \Delta a_m b_m \ldots \Delta a_k b_k f(\Xi) \geqslant 0,$$

whence

$$\frac{\partial}{\partial x_{k-2}} \cdots \frac{\partial}{\partial x_1} \Delta a_m b_m \ldots \Delta a_k b_k \Delta a_{k-1} b_{k-1} f(\Xi) \geqslant 0.$$

Interpreting $\partial/\partial x_0$ as the identity operator, we have $\Delta a_m b_m \ldots \Delta a_1 b_1 f(\Xi) \geqslant 0$.

We now rephrase Theorem 2 as follows:

THEOREM 2'. *Suppose $\{F_n\}$ is a sequence of distribution functions. If T is a strict t-norm with completely monotone generator f, then T is admissible over $\{F_n\}$ and f is admissible over $\{F_n\}$.*

Proof. We intend to show that T is completely monotone. Then, by Lemma 2a, T is admissible over $\{F_n\}$. Consequently, f is admissible over $\{F_n\}$.

Let $(a, b] = ((a_1, \ldots, a_m), (b_1, \ldots, b_m)]$ be a cell in I^m. By Lemma 2b,

$$\sum_{k=0}^{m} (-1)^k \sum_{\delta \in \Delta_{k,m}} T(\delta) = \Delta a_m b_m \Delta a_{m-1} b_{m-1} \ldots \Delta a_1 b_1 T(x_1, \ldots, x_m).$$

By Lemma 2c, the right side is nonnegative, since

$$T(x_1, \ldots, x_m) = f\left[\sum_{j=1}^{m} f^{-1}(x_j)\right]$$

for all $(x_1, \ldots, x_m)$ in I^m.

LEMMA 3a. ([2, p. 245]) *Let* $-\infty \leqslant a \leqslant b \leqslant \infty$. *Suppose* f, g, *and* α *are nonnegative over* $[a, b]$ *and* $\alpha(b) - \alpha(a) \leqslant 1$. *If* α *is nondecreasing on* $[a, b]$ *and both* f *and* g *are nonincreasing on* $[a, b]$, *then*

$$\int_a^b f(t)\, g(t)\, d\alpha(t) \geqslant \int_a^b f(t)\, d\alpha(t) \int_a^b g(t)\, d\alpha(t).$$

LEMMA 3b. ([14], p. 160) *A function* f *is completely monotone on* $[0, \infty)$ *if and only if*

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t), \tag{10}$$

where $\alpha(t)$ *is bounded and nondecreasing and the integral converges for* $0 \leqslant x < \infty$.

THEOREM 3a. *Suppose* f *is a completely monotone function from* $[0, \infty)$ *into* $(0, 1]$. *Then* $f(x+y) \geqslant f(x) f(y)$. *If* T *is a strict t-norm generated by* f, *then* $T \geqslant$ *Product.*

Proof. For such a function f, we have $f(0) \leqslant 1$, so that in Lemma 3b, we have $\alpha(\infty) - \alpha(0) \leqslant 1$. Thus Lemma 3a applies with $f(t) = e^{-xt}$ and $g(t) = e^{-yt}$ and we conclude that $f(x+y) \geqslant f(x) f(y)$.

Then, for $a, b \in I$, we set $x = f^{-1}(a)$ and $y = f^{-1}(b)$ to get

$$T(a, b) = f[f^{-1}(a) + f^{-1}(b)] \geqslant ab.$$

THEOREM 3b. *Suppose* f *is a completely monotone function from* $[0, \infty)$ *into* $[0, \infty)$. *Then*

$$f(x + y) \leqslant [f(x + my)]^{1/m} [f(x)]^{1/n}$$

for all $x, y \geqslant 0$ *and positive real* m *and* n *satisfying* $1/m + 1/n = 1$.

Proof. For fixed $x \geqslant 0$ and variable $y \geqslant 0$, the function $f(x+y)$ is completely monotone and therefore can be represented

$$f(x + y) = \int_0^1 t^y\, d\alpha_x(t),$$

where α_x is bounded and nondecreasing on $[0, 1]$. (This integral arises from (10) by a simple change of variable.)

We shall apply the following form of Hölder's Inequality:

$$\int_0^1 |f(t)\, g(t)|\, d\alpha_x(t) \leqslant \left[\int_0^1 |f(t)|^m\, d\alpha_x(t)\right]^{1/m} \left[\int_0^1 |g(t)|^n\, d\alpha_x(t)\right]^{1/n},$$

where $f \in L^n$, $g \in L^m$, and $1/m+1/n=1$. For $f(t)=t^y$ and $g(t)=1$,

$$\int_0^1 t^y \, d\alpha_x(t) \leqslant \left[\int_0^1 t^{my} \, d\alpha_x(t)\right]^{1/m} \left[\int_0^1 d\alpha_x(t)\right]^{1/n},$$

i.e.,

$$f(x+y) \leqslant [f(x+my)]^{1/m} [f(x)]^{1/n}.$$

Taking Theorems 3a and 3b together, we now have Theorem 3.

DEFINITION 3. Let $\{r_n\}$ be a moment sequence which satisfies $r_n = \int_0^1 t^n \, d\alpha(t)$, $n=0, 1, 2, \ldots$, for some integrator $\alpha(t)$ of bounded variation on $[0, 1]$ with $\alpha(0)=\alpha(0+)=0$ (as in [14, p. 100]). We shall call $\{r_n\}$ a *strict moment sequence* if the function

$$f(x) = \int_0^1 t^x \, d\alpha(t)$$

is strictly decreasing from $f(0)=1$ to $f(\infty)=0$. We define the t-norm generated by $\{r_n\}$ to be the strict t-norm generated by f.

DEFINITION 4. ([3]) A *strict generalized moment sequence* is a collection $\{r(n, y)\}_{y \in I}$ of sequences such that $\{r(n, y)\}$ is a strict moment sequence for each fixed y in $(0, 1)$.

LEMMA 4a. *Suppose* $\{r(n, y)\}_{y \in I}$ *is a strict generalized moment sequence. Let* α_y *be an integrator which corresponds to* $\{r(n, y)\}$ *in the sense of Definition* 3. *Then the sequences* $\{r(n, y)\}_{y \in I}$, *for* $0<y<1$, *all generate the same t-norm T if and only if for each such y there is a number* $u(y)$ *in* $(0, \infty)$ *satisfying* $\alpha_y(t)=\alpha_{1/2}(t^{u(y)})$ *for all t in* $[0, 1]$.

Proof. For $0<y<1$, extend $r(n, y)=\int_0^1 t^n \, d\alpha_y(t)$ to

$$r(x, y) = \int_0^1 t^x \, d\alpha_y(t).$$

Then $r(x, y)$ generates the same T as $r(x, \frac{1}{2})$ if and only if there exists $v(y)$ in $(0, \infty)$ satisfying $r(x, y)=r(v(y)x, \frac{1}{2})$, since, as is easily established by Cauchy's functional equation, if f and g generate the same T, then $g(x)=f(vx)$ for some positive constant v. Hence,

$$\int_0^1 t^x \, d\alpha_y(t) = \int_0^1 t^{v(y)x} \, d\alpha_{1/2}(t)$$

$$= \int_0^1 t^x \, d\alpha_{1/2}(t^{1/v(y)}),$$

so that $\alpha_y(t) = \alpha_{1/2}(t^{1/v(y)})$. (See, for example, [14], p. 63.)

LEMMA 4b. *Suppose T is a strict t-norm generated by each of the strict moment sequences of a strict generalized moment sequence $\{r(n, y)\}_{y \in I}$. Then the mapping $y \to u(y)$ defined in the proof of Lemma* 4a *by $r(x, y) = r(u(y)\,x, \frac{1}{2})$ carries* $(0, 1)$ *onto* $(0, \infty)$.

Proof. Let $u \in (0, \infty)$ and set $y = r(u, \frac{1}{2})$. Then $T(1, y) = r(u, \frac{1}{2})$ and u must be the only solution to the equation $T(1, y) = r(x, \frac{1}{2})$ since the right side is strictly decreasing in x. But $r(x, y)$ must equal $r(sx, \frac{1}{2})$ for some s and all x in $[0, \infty]$, so we conclude that $r(x, y) = r(ux, \frac{1}{2})$ for all x in $[0, \infty)$.

LEMMA 4c. *Suppose α is a nonconstant nondecreasing function on* $[0, 1]$. *Suppose g and h are strictly increasing continuous functions from* $[0, 1]$ *onto* $[0, 1]$ *and that $g(t) = h(t)$ at only* one *point $t = t_0$ in* $(0, 1)$. *Finally, suppose $\alpha[g(t)] = \alpha[h(t)]$ for* every *t in* $(0, 1)$. *Then α has only one point of increase in* $(0, 1)$.

Proof. Writing $k(t) = g[h^{-1}(t)]$, we have $\alpha[k(t)] = \alpha(t)$ for every t in $(0, 1)$. Moreover,

$$\text{case i)} \begin{cases} k(t) < t & \text{for} \quad 0 < t < t_0 \\ k(t) = t & \text{for} \quad t = t_0 \\ k(t) > t & \text{for} \quad t_0 < t < 1 \end{cases} \quad \text{or} \quad \text{case ii)} \begin{cases} k(t) > t & \text{for} \quad 0 < t < t_0 \\ k(t) = t & \text{for} \quad t = t_0 \\ k(t) < t & \text{for} \quad t_0 < t < 1 . \end{cases}$$

Let $k^2(t)$ denote the function $k[k(t)]$ and for $n = 3, 4, \ldots$, let $k^n(t)$ denote the nth iterate $k[k^{n-1}(t)]$ of $k(t)$. Consider the equations

$$\alpha(t) = \alpha[k(t)] = \alpha[k^2(t)] = \cdots = \alpha[k^n(t)]. \tag{11}$$

In case i) we have $\lim_{n\to\infty} k^n(t) = 0$ for $0 \leqslant t < t_0$ and $\lim_{n\to\infty} k^n(t) = 1$ for $t_0 < t \leqslant 1$. Therefore, by (11),

$$\alpha(t_0 -) = \alpha(0 +) \quad \text{and} \quad \alpha(t_0 +) = \alpha(1 -),$$

which is to say that α has only one point of increase in (0, 1), namely t_0. We obtain the same conclusion in case ii), wherein $\lim_{n\to\infty} k^n(t) = k(t_0) = t_0$ for $0 < t \leqslant t_0$ and for $t_0 \leqslant t < 1$.

We now rephrase Theorem 4 as follows:

THEOREM 4. *Suppose a strict t-norm T is admissible over a sequence $\{X_n\}$ of* *random variables whose distribution functions are continuous. Then there exists a strict t-norm* T^* *which is admissible over* $\{-X_n\}$ *if and only if* $T^*=T=$Product. *In other words, if the sets* $[X_n \leqslant x_n]=\{\omega\in\Omega: X_n(\omega)\leqslant x_n\}$ *are jointly distributed by* $T\neq$Product, *then their complements* $[X_n>x_n]$ *are jointly distributed by no t-norm.* (In fact, the proof will show that not even a collection of such sets all having the same probability need be so jointly distributed.)

Proof. Let T be a strict t-norm. Given any generator of T we can easily construct, via Lemma 4a, a strict generalized moment sequence $\{r(n, y)\}_{y\in I}$ each of whose strict moment sequences, for $0<y<1$, generates T. Let α_y be an integrator which corresponds to $\{r(n, y)\}$ as in Definition 3. That is, for all (c, d) in I^2,

$$T(c, d)=f_y[f_y^{-1}(c)+f_y^{-1}(d)],$$

where

$$f_y(x)=r(x, y)=\int_0^1 t^x\, d\alpha_y(t), \quad 0<y<1.$$

We already have $\alpha_y(0+)=\alpha_y(0)$ by Definition 3. Let us note also that $\alpha_y(1-)=\alpha_y(1)$ since $0=f_y(\infty)=\alpha_y(1)-\alpha_y(1-)$. Thus neither 0 nor 1 is a point of increase of α_y.

Suppose $\{X_n\}$ is a sequence of random variables whose distribution functions are continuous and that T is admissible over $\{X_n\}$. Now suppose y in (0, 1) is arbitrary (but we reserve the right to fix its value later). Choose $x_1, x_2,\ldots$ satisfying

$$P[X_n\leqslant x_n]=y, \quad n=1, 2, \ldots.$$

Let Δ^n denote the usual nth order difference operator ([14], p. 101). Then the events $\{[X_n>x_n]\}$ are admissible under the sequence

$$\{\varrho(n, y)\}=\{(-1)^n\Delta^n r(0, y)\}, \tag{12}$$

in the sense that the probability of any n-fold intersection of these events is given by the nth term of (12). An integrator for (12) is

$$\beta_y(t)=1-\alpha_y(1-t).$$

By Lemma 4a, the sequence $\{\varrho(n, y)\}$ generates the same T^* as $\{\varrho(n, \tfrac{1}{2})\}$ if and only if there is a number u in $(0, \infty)$ satisfying

$$\beta_y(t)=\beta_{1/2}(t^u),$$

or equivalently,

$$\alpha_y(1-t)=\alpha_{1/2}(1-t^u) \quad \text{for all} \quad t\in I.$$

Also by Lemma 4a,

$$\alpha_y(t)=\alpha_{1/2}(t^v)$$

for some number v in $(0, \infty)$ so that

$$\alpha_{1/2}(1 - t^u) = \alpha_{1/2}(1 - t)^v \quad \text{for all} \quad t \in I. \tag{13}$$

We shall use (13) to show that $\alpha_{1/2}$ has only one point of increase in (0, 1). Suppose t_0 and t_1 are points with $1-t^u=(1-t)^v$ for $t=t_0$ and $t=t_1$. Then the function

$$g_y(t) = \log_{1-t}(1 - t^u) = \frac{\log(1 - t^u)}{\log(1 - t)}$$

assumes the value v for $t=t_0$ and $t=t_1$. In accord with Lemma 4b, *we now choose* y to satisfy $u(y)=2$. Then

$$g_y(t) = \frac{\log(1 - t) + \log(1 + t)}{\log(1 - t)}.$$

Clearly the one-to-oneness of the function

$$\frac{\log(1 + t)}{\log(1 - t)}$$

is equivalent to that of g_y. Thus the hypothesis of Lemma 4c holds with $\alpha=\alpha_{1/2}$, $g(t)=1-t^u$ and $h(t)=(1-t)^v$. Therefore $\alpha_{1/2}$ has only one point of increase on (0, 1). We noted early in the proof that $\alpha_{1/2}$ has no point of increase at the endpoints 0 and 1. Therefore, as an integrator, $\alpha_{1/2}$ determines a geometric sequence, which makes $T=$Product.

Proof of Theorem 5. *Part* (i). Beginning with the Borel-Cantelli Lemma, if $\sum_{i=1}^{\infty}[1-F_i(x_i)]<\infty$, then

$$\begin{aligned} 0 &= P[X_i > x_i \text{ inf. oft.}] \\ &= 1 - \lim_{m\to\infty}\lim_{n\to\infty} P[X_m \leqslant x_m, \ldots, X_n \leqslant x_n] \\ &= 1 - \lim_{m\to\infty}\lim_{n\to\infty} f\left(\sum_{i=m}^{n} f^{-1}[F_i(x_i)]\right) \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \sum_{i=m}^{n} f^{-1}[F_i(x_i)], \end{aligned}$$

which implies

$$\sum_{i=1}^{\infty} f^{-1}[F_i(x_i)] < \infty.$$

Part (ii). $P[X_i > x_i \text{ inf. oft.}]>0$ is equivalent to

$$\lim_{m\to\infty}\lim_{n\to\infty} \sum_{i=m}^{n} f^{-1}[F_i(x_i)] \neq 0,$$

which is equivalent to

$$\sum_{i=1}^{\infty} f^{-1}[F_i(x_i)] = \infty .$$

Part (iii). Supposing $\sum_{i=1}^{\infty}[1-F_i(x_i)]=\infty$, we have

$$\begin{aligned} \infty &= \sum_{i=1}^{\infty} -\log F_i(x_i) \\ &= \sum_{i=1}^{\infty} \frac{-\log F_i(x_i)}{n} \\ &\leqslant \sum_{i=1}^{\infty} f^{-1}[F_i(x_i)] . \end{aligned}$$

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REFERENCES

[1] Aczél, J., *Lectures on Functional Equations and Their Applications* (Academic Press, New York – London 1966).
[2] Apostol, Tom M., *Mathematical Analysis* (Addison Wesley, Reading, Massachusetts 1957).
[3] Eisenberg, S. M., *Moment Sequences and the Bernstein Polynomials*, Canad. Math. Bull. *12*, 401–411 (1969).
[4] Feller, W., *An Introduction to Probability Theory and Its Applications, Vol. II.* (John Wiley and Sons, New York 1968).
[5] de Finetti, B., *Funzione caratteristica di un fenomeno aleatorio*, Mem. Reale Academia Naz. Lincei, Ser. 6 *4*, 251–299 (1931).
[6] Kimberling, C. H., *On a Class of Associative Functions*, Publ. Math. Debrecen (to appear).
[7] Ling, Cho-Hsin, *Representation of Associative Functions*, Publ. Math. Debrecen *12*, 189–212 (1965).
[8] Menger, K., *Statistical Metrics*, Proc. Nat. Acad. Sci. U.S.A. *28*, 535–537 (1942).
[9] Mostert, P. S. and Shields, A. L., *On the Structure of Semigroups on a Compact Manifold with Boundary*, Ann. of Math. *65*, 117–143 (1957).
[10] Schweizer, B. and Sklar, A., *Associative Functions and Statistical Triangle Inequalities*, Publ. Math. Debrecen *8*, 169–186 (1961).
[11] Schweizer, B. and Sklar, A., *Associative Functions and Abstract Semigroups*. Publ. Math. Debrecen *10*, 69–81 (1963).
[12] Storey, C. R., *The Structure of Threads*, Pacific J. Math. *10*, 1429–1445 (1960).
[13] Tucker, H. G., *A Graduate Course in Probability* (Academic Press, New York 1967).
[14] Widder, D. V., *The Laplace Transform* (Princeton University Press, Princeton 1946).

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