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On a problem of Chern–Akivis–Shelekhov on hexagonal three-webs

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Summary. In the following we discuss the infinitesimal theory of analytic multidimensional hexagonal three-webs. It is known that the structure of any such three-web in a neighborhood of an arbitrary point is uniquely determined by four tensor values at the given point. We prove that three of them are sufficient and we analyze the corresponding integrability conditions.

There are four well known classes of multidimensional differentiable three-webs generally attributed correspondingly to Thomsen, Reidemeister, Moufang and Bol. Hexagonal three-webs form the fifth one. Three-webs of the first four classes and their differentiable local (coordinate) loops were largely investigated [1–8]. It is known about differentiable hexagonal three-webs [1], [9] that a given three-web is locally hexagonal if and only if

$$\mathbf{b}^i_{(jkl)} = 0.$$

Here $\mathbf{b}^i_{jkl}$ is a curvature tensor field on the manifold of the web and the round brackets denote the symmetrization with respect to the indices j, k, l. The structure of any analytic hexagonal three-web in a neighborhood of an arbitrary point is uniquely determined by the values of torsion $\mathbf{a}^i_{jk}$, curvature $\mathbf{b}^i_{jkl}$ and covarient derivatives (with respect to Chern connection) $\mathbf{c}^i_{jklm} = \mathbf{D}^1_m \mathbf{b}^i_{jkl}$ and $\mathbf{d}^i_{jklm} = \mathbf{D}^2_m \mathbf{b}^i_{jkl}$ at a given point (Shelekhov's theorem, [4], [10]). The problem of finding integrability conditions for the pointwise tensors $\mathbf{a}^i_{jk}$, $\mathbf{b}^i_{jkl}$, $\mathbf{c}^i_{jklm}$, $\mathbf{d}^i_{jklm}$ (defining relations of algebras tangent to hexagonal three-webs) remains open.

In what follows we deduce algebraic expressions for tensor fields $\mathbf{c}^i_{jklm}$ and $\mathbf{d}^i_{jklm}$ of a hexagonal three-web in terms of $\mathbf{a}^i_{jk}$, $\mathbf{b}^i_{jkl}$, $\mathbf{d}^i_{[j|k|l]m}$, where the square brackets

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denote anti-symmetrization with respect to inserted indices. In this way we formulate a more economic version of Shelekhov's theorem and we make some progress in investigation of the corresponding integrability conditions.

An explicit description of the whole collection of integrability conditions and their interpretation in terms of the theory of Lie algebras (in the spirit of [5], [6]) remains an open problem. Its solution could conclude the series of infinitesimal theories corresponding to the five classical varieties of three-webs and could allow to describe the variety of analytic hexagonal three-webs. As far as we know the progress in this matter is reduced to separate examples. In this connection let us mention the Griffits' conjecture [11] on representation of hexagonal three-webs of dimension k in $\mathbf{R}^{2k}$ by cubic hypersurfaces in projective space $\mathbf{PR}^{2k+1}$, see [2] for counter-examples.

Let us remark that analytic hexagonal three-webs allow an algebraic treatment [4] via analytic local loops $\langle \mathbf{Q}, \cdot, \backslash, /, e \rangle$ satisfying the following identity:

$$(\mathbf{x}/\mathbf{a})(\mathbf{b}\backslash((\mathbf{x}/\mathbf{a})(\mathbf{b}\backslash\mathbf{x}))) = (((\mathbf{x}/\mathbf{a})(\mathbf{b}\backslash\mathbf{x}))/\mathbf{a})(\mathbf{b}\backslash\mathbf{x}).$$

Thus the above considerations have an interpretation in terms of the general theory of differentiable composition laws with identities. For the algebraic interpretation of tensors $\mathbf{a}^i_{jk}$, $\mathbf{b}^i_{jkl}$, $\mathbf{d}^i_{[j|k|l]m}$ see [12].

The proofs of propositions 1–6 are omitted in the present note. They are calculational and were executed via a system of analytic calculus.

Let $\mathbf{M}$ be an analytic manifold of dimension $2N$ equipped with analytic foliations f_1, f_2, f_3 of dimension N such that every point of $\mathbf{M}$ belongs to exactly one leaf of each foliation and the tangent spaces of the leaves through every point of $\mathbf{M}$ are in general position. Then $(\mathbf{M}, f_1, f_2, f_3)$ is called a (multidimensional) three-web [1–8].

Let us introduce on $\mathbf{M}$ co-frames $(\Omega^i_1, \Omega^j_2)_{i,j=1,\dots,N}$ such that the foliations f_1, f_2, f_3, correspond to the following system of Pfaffian equations

$$\Omega^i_1 = 0, \quad \Omega^i_2 = 0, \quad \Omega^i_1 + \Omega^i_2 = 0, \qquad i = 1, \dots, N.$$

Then the following structure equations of the three-web $\mathbf{M}$ hold [1–4]:

$$\left.\begin{aligned} d\Omega^i_1 &= \Omega^j_1 \wedge \Omega^i_j + \mathbf{a}^i_{jk}\Omega^j_1 \wedge \Omega^k_1 \\ \partial\Omega^i_2 &= \Omega^j_2 \wedge \Omega^i_j - \mathbf{a}^i_{jk}\Omega^j_2 \wedge \Omega^k_2 \end{aligned}\right], \tag{1}$$

$$d\Omega^i_j = \Omega^k_j \wedge \Omega^i_k + \mathbf{b}^i_{jkl}\Omega^k_1 \wedge \Omega^l_2. \tag{2}$$

Here the tensor fields $\mathbf{a}^i_{jk}, \mathbf{b}^i_{jkl}$ $(i, j, k, l = 1, \ldots, N)$ are called the torsion and the curvature of the three-web, respectively. Obviously the torsion satisfies the condition

$$\mathbf{a}^i_{(jk)} = 0. \tag{3}$$

Throughout this note we assume that the following hexagonal condition

$$\mathbf{b}^i_{(jkl)} = 0 \tag{4}$$

is satisfied. (Hereafter the round brackets denote symmetrization with respect to inserted indices and the square brackets denote anti-symmetrization.) For geometric characterization of hexagonal three-webs see [1–4].

By differentiation of (1) we get [2]

$$\mathbf{b}^i_{[jkl]} = \tfrac{2}{3}(\mathbf{a}^p_{jk}\mathbf{a}^i_{pl} + \mathbf{a}^p_{lj}\mathbf{a}^i_{pk} + \mathbf{a}^p_{kl}\mathbf{a}^i_{pj}), \tag{5}$$

$$\mathbf{D}^1_l \mathbf{a}^i_{jk} = \mathbf{b}^i_{[j|l|k]}, \qquad \mathbf{D}^2_l \mathbf{a}^i_{jk} = \mathbf{b}^i_{[jk]l}. \tag{6}$$

Here $(\mathbf{D}^1, \mathbf{D}^2)$ is the canonical connection in the space of the three-web $\mathbf{M}$ (called Chern connection [1]) defined by the rule:

$$\begin{aligned}&\mathbf{D}^1_q \mathbf{\Phi}^i_{j_1 \ldots j_m}\Omega^q_1 + \mathbf{D}^2_q \mathbf{\Phi}^i_{j_1 \ldots j_m}\Omega^q_2 \\ &\quad = \mathbf{d}\mathbf{\Phi}^i_{j_1 \ldots j_m} + \mathbf{\Phi}^p_{j_1 \ldots j_m}\Omega^i_p - \mathbf{\Phi}^i_{p \ldots j_m}\Omega^p_{j_1} - \cdots - \mathbf{\Phi}^i_{j_1 \ldots p}\Omega^p_{j_m},\end{aligned}$$

for tensor fields $\mathbf{\Phi}^i_{j_1 \ldots j_m}$, $i, j_1, \ldots, j_m = 1, \ldots, N$, on the manifold $\mathbf{M}$. Differential operators $\mathbf{D}^1$ and $\mathbf{D}^2$ satisfy the following commutation relations

$$\begin{aligned}&[\mathbf{D}^1_p, \mathbf{D}^1_q]\mathbf{\Phi}^i_{j_1 \ldots j_m} = -2\mathbf{D}^1_t \mathbf{\Phi}^i_{j_1 \ldots j_m}\mathbf{a}^t_{pq}, \\ &[\mathbf{D}^2_p, \mathbf{D}^2_q]\mathbf{\Phi}^i_{j_1 \ldots j_m} = 2\mathbf{D}^2_t \mathbf{\Phi}^i_{j_1 \ldots j_m}\mathbf{a}^t_{pq}, \\ &[\mathbf{D}^1_p, \mathbf{D}^2_q]\mathbf{\Phi}^i_{j_1 \ldots j_m} = \mathbf{\Phi}^t_{j_1 \ldots j_m}\mathbf{b}^i_{tpq} - \mathbf{\Phi}^i_{t \ldots j_m}\mathbf{b}^t_{j_1 pq} - \cdots - \mathbf{\Phi}^i_{j_1 \ldots t}\mathbf{b}^t_{j_m pq}.\end{aligned} \tag{7}$$

Note that the pair of identities (4)–(5) is equivalent to the following:

$$\sigma_{jkl}(\mathbf{b}^i_{jkl} - 2\mathbf{a}^p_{jk}\mathbf{a}^i_{pl}) = 0, \tag{8}$$

where σ_{jkl} denotes the cyclic sum with respect to indices j, k, l.

We introduce on $\mathbf{M}$ tensor fields

$$\mathbf{c}^i_{jklm} = \mathbf{D}^1_m \mathbf{b}^i_{jkl}, \qquad \mathbf{d}^i_{jklm} = \mathbf{D}^2_m \mathbf{b}^i_{jkl}. \tag{9}$$

Then we get the integrability conditions for the system (1)–(3), (6)–(8) in the following form [9]

$$\mathbf{c}^i_{j[k|l|m]} = \mathbf{b}^i_{jpl}\,\mathbf{a}^p_{km}, \tag{10}$$

$$\mathbf{d}^i_{jk[lm]} = -\mathbf{b}^i_{jkp}\,\mathbf{a}^p_{lm}, \tag{11}$$

$$\mathbf{c}^i_{[jk]lm} - \mathbf{d}^i_{[j|m|k]l} = \mathbf{a}^p_{jk}\,\mathbf{b}^i_{pml} - \mathbf{a}^i_{pk}\,\mathbf{b}^p_{jml} - \mathbf{a}^i_{jp}\,\mathbf{b}^p_{kml}, \tag{12}$$

$$\sigma_{jkl}(\mathbf{c}^i_{jklm} - 2\mathbf{b}^t_{[j|m|k]}\,\mathbf{a}^i_{tl} - 2\mathbf{a}^t_{jk}\,\mathbf{b}^i_{[t|m|l]}) = 0, \tag{13}$$

$$\sigma_{jkl}(\mathbf{d}^i_{jklm} - 2\mathbf{b}^t_{[jk]m}\,\mathbf{a}^i_{tl} - 2\mathbf{a}^t_{jk}\,\mathbf{b}^i_{[tl]m}) = 0. \tag{14}$$

The identities (10)–(14) allow a partial solution. More precisely, the following proposition holds.

PROPOSITION 1. *The following are true:*

$$\begin{aligned} 3\mathbf{c}^i_{jklm} = {} & 3\mathbf{c}^i_{[jkl]m} - 6\mathbf{c}^i_{[jkm]l} + 3\mathbf{c}^i_{[jlm]k} + 3\mathbf{c}^i_{[klm]j} - 4\mathbf{c}^i_{k[l|j|m]} + 4\mathbf{c}^i_{j[k|l|m]} \\ & + 4\mathbf{c}^i_{m[j|k|l]} + 2\mathbf{c}^i_{j[k|m|l]} + 2\mathbf{c}^i_{l[k|j|m]} + 2\mathbf{c}^i_{k[j|m|l]} + 2\mathbf{c}^i_{m[j|l|k]} + 2\mathbf{c}^i_{l[m|k|j]} \\ & - 4\mathbf{c}^i_{[lm]kj} + 4\mathbf{c}^i_{[jm]lk} + 2\mathbf{c}^i_{[kl]jm} + 2\mathbf{c}^i_{[jk]ml}, \end{aligned} \tag{15}$$

where

$$\mathbf{c}^i_{[jkl]m} = \tfrac{2}{3}\sigma_{jkl}(\mathbf{b}^p_{[j|m|k]}\,\mathbf{a}^i_{pl} + \mathbf{a}^p_{jk}\,\mathbf{b}^i_{[p|m|l]}),$$

$$\mathbf{c}^i_{j[k|l|m]} = \mathbf{b}^i_{jpl}\,\mathbf{a}^p_{km},$$

$$\mathbf{c}^i_{[jk]lm} = \mathbf{d}^i_{[j|m|k]l} + \mathbf{a}^p_{jk}\,\mathbf{b}^i_{pml} - \mathbf{a}^i_{pk}\,\mathbf{b}^p_{jml} - \mathbf{a}^i_{jp}\,\mathbf{b}^p_{kml},$$

$$\begin{aligned} 3\mathbf{d}^i_{jklm} = {} & 3\mathbf{d}^i_{[jkl]m} - 6\mathbf{d}^i_{[jml]k} + 3\mathbf{d}^i_{[jmk]l} + 3\mathbf{d}^i_{[lmk]j} - 4\mathbf{d}^i_{lj[km]} + 4\mathbf{d}^i_{jk[lm]} \\ & + 4\mathbf{d}^i_{ml[jk]} + 2\mathbf{d}^i_{jm[lk]} + 2\mathbf{d}^i_{kj[lm]} + 2\mathbf{d}^i_{lm[jk]} + 2\mathbf{d}^i_{mk[jl]} + 2\mathbf{d}^i_{kl[mj]} \\ & - 4\mathbf{d}^i_{[k|l|m]j} + 4\mathbf{d}^i_{[j|k|m]l} + 2\mathbf{d}^i_{[l|j|k]m} + 2\mathbf{d}^i_{[j|m|l]k}, \end{aligned} \tag{16}$$

and where

$$\mathbf{d}^i_{[jkl]m} = \tfrac{2}{3}\sigma_{jkl}(\mathbf{b}^p_{[jk]m}\,\mathbf{a}^i_{pl} + \mathbf{a}^p_{jk}\,\mathbf{b}^i_{[pl]m}),$$

$$\mathbf{d}^i_{jk[lm]} = -\mathbf{b}^i_{jkp}\,\mathbf{a}^p_{lm}.$$

The proof is done by direct computation. □

The identities (15)–(16) will be considered as the algebraic expressions of the tensor fields $\mathbf{c}^i_{jklm}$ and $\mathbf{d}^i_{jklm}$ in terms of basic vector fields $\mathbf{a}^i_{jk}$, $\mathbf{b}^i_{jkl}$ and $\mathbf{d}^i_{[j|k|l]m}$.

PROPOSITION 2. *The identities* (10)–(14) *are reduced, in virtue of* (3), (8), (15) *and* (16), *to the following identities*:

$$\sigma_{jlm}(\mathbf{d}^i_{[j|k|l]m} + \mathbf{b}^i_{jkp}\,\mathbf{a}^p_{lm}) = 0, \tag{17}$$

$$\sigma_{jkl}(\mathbf{d}^i_{[j|k|l]m} - 2\mathbf{b}^p_{[jk]m}\,\mathbf{a}^i_{pl} - 2\mathbf{a}^p_{jk}\,\mathbf{b}^i_{[pl]m} = 0. \tag{18}$$

□

As a result of these considerations we state the following theorem.

THEOREM A. *The integrability conditions of the system* (3), (6), (8), (9), (15)–(16) *are reduced to* (17)–(18). □

We pass to an analysis of the system of equations (3), (6), (8), (9), (15)–(18). For that we introduce on the manifold $\mathbf{M}$ tensor fields

$$\begin{aligned} &\mathbf{D}^1_n\,\mathbf{c}^i_{jklm} = \mathbf{X}^i_{jklmn}, \qquad \mathbf{D}^1_n\,\mathbf{d}^i_{jklm} = \mathbf{U}^i_{jklmn},\\ &\mathbf{D}^2_n\,\mathbf{c}^i_{jklm} = \mathbf{Y}^i_{jklmn}, \qquad \mathbf{D}^2_n\,\mathbf{d}^i_{jklm} = \mathbf{Z}^i_{jklmn}. \end{aligned} \tag{19}$$

Tensor fields $\mathbf{X}, \mathbf{U}, \mathbf{Y}, \mathbf{Z}$ satisfy the following relations (commutation relations (7) applied to $\mathbf{b}^i_{jkl}$):

$$\mathbf{X}^i_{jkl[mn]} = \mathbf{c}^i_{jklp}\,\mathbf{a}^p_{mn}, \tag{20}$$

$$\mathbf{Z}^i_{jkl[mn]} = -\mathbf{d}^i_{jklp}\,\mathbf{a}^p_{mn}, \tag{21}$$

$$\mathbf{U}^i_{jklmn} = \mathbf{Y}^i_{jklmn} + \mathbf{b}^p_{jkl}\,\mathbf{b}^i_{pmn} - \mathbf{b}^i_{pkl}\,\mathbf{b}^p_{jmn} - \mathbf{b}^i_{jpl}\,\mathbf{b}^p_{kmn} - \mathbf{b}^i_{jkp}\,\mathbf{b}^p_{lmn}. \tag{22}$$

The following identities are derived from (10)–(14) by applying $\mathbf{D}^1$- and $\mathbf{D}^2$-derivatives:

$$\begin{aligned} 3\mathbf{X}^i_{jklmn} = \mathbf{D}^1_n(&3\mathbf{c}^i_{[jkl]m} - 6\mathbf{c}^i_{[jkm]l} + 3\mathbf{c}^i_{[jlm]k} + 3\mathbf{c}^i_{[klm]j} - 4\mathbf{c}^i_{k[l|j|m]} + 4\mathbf{c}^i_{j[k|l|m]}\\ &+ 4\mathbf{c}^i_{m[j|k|l]} + 2\mathbf{c}^i_{j[k|m|l]} + 2\mathbf{c}^i_{l[k|j|m]} + 2\mathbf{c}^i_{k[j|m|l]} + 2\mathbf{c}^i_{m[j|l|k]} + 2\mathbf{c}^i_{l[m|k|j]}\\ &- 4\mathbf{c}^i_{[lm]kj} + 4\mathbf{c}^i_{[jm]lk} + 2\mathbf{c}^i_{[kl]jm} + 2\mathbf{c}^i_{[jk]ml}), \end{aligned} \tag{23}$$

where

$$\mathbf{D}^1_n \mathbf{c}^i_{[jkl]m} = \tfrac{2}{3}\sigma_{jkl}(\mathbf{c}^p_{[j|m|k]n}\,\mathbf{a}^i_{pl} + \mathbf{a}^p_{jk}\,\mathbf{c}^i_{[p|m|l]n} + \mathbf{b}^p_{[j|m|k]}\,\mathbf{b}^i_{[p|n|l]} + \mathbf{b}^p_{[j|n|k]}\,\mathbf{b}^i_{[p|m|l]}),$$

$$\mathbf{D}^1_n \mathbf{c}^i_{j[k|l|m]} = \mathbf{c}^i_{jpln}\,\mathbf{a}^p_{km} + \mathbf{b}^i_{jpl}\,\mathbf{b}^p_{[k|n|m]},$$

$$\mathbf{D}^1_n \mathbf{c}^i_{[jk]lm} = \mathbf{U}^i_{[j|m|k]ln} + \mathbf{b}^p_{[j|n|k]}\,\mathbf{b}^i_{pml} - \mathbf{b}^i_{[p|n|k]}\,\mathbf{b}^p_{jml} - \mathbf{b}^i_{[j|n|p]}\,\mathbf{b}^p_{kml}$$
$$+ \mathbf{a}^p_{jk}\,\mathbf{c}^i_{pmln} - \mathbf{a}^i_{pk}\,\mathbf{c}^p_{jmln} - \mathbf{a}^i_{jp}\,\mathbf{c}^p_{kmln},$$

$$\mathbf{U}^i_{jk[lm]n} = -\mathbf{c}^i_{jkpn}\,\mathbf{a}^p_{lm} - \mathbf{b}^i_{jkp}\,\mathbf{b}^p_{[l|n|m]}, \tag{24}$$

$$\sigma_{jkl}(\mathbf{U}^i_{jklmn} - 2\mathbf{c}^p_{[jk]mn}\,\mathbf{a}^i_{pl} - 2\mathbf{a}^p_{jk}\,\mathbf{c}^i_{[pl]mn} - 2\mathbf{b}^p_{[jk]m}\,\mathbf{b}^i_{[p|n|l]} - 2\mathbf{b}^p_{[j|n|k]}\,\mathbf{b}^i_{[pl]m} = 0, \tag{25}$$

$$3\mathbf{Y}^i_{jklmn} = \mathbf{D}^2_n(3\mathbf{c}^i_{[jkl]m} - 6\mathbf{c}^i_{[jkm]l} + 3\mathbf{c}^i_{[jlm]k} + 3\mathbf{c}^i_{[klm]j} - 4\mathbf{c}^i_{k[l|j|m]} + 4\mathbf{c}^i_{j[k|l|m]}$$
$$+ 4\mathbf{c}^i_{m[j|k|l]} + 2\mathbf{c}^i_{j[k|m|l]} + 2\mathbf{c}^i_{l[k|j|m]} + 2\mathbf{c}^i_{k[j|m|l]} + 2\mathbf{c}^i_{m[j|l|k]} + 2\mathbf{c}^i_{l[m|k|j]}$$
$$- 4\mathbf{c}^i_{[lm]kj} + 4\mathbf{c}^i_{[jm]lk} + 2\mathbf{c}^i_{[kl]jm} + 2\mathbf{c}^i_{[jk]ml}), \tag{26}$$

where

$$\mathbf{D}^2_n \mathbf{c}^i_{[jkl]m} = \tfrac{2}{3}\sigma_{jkl}(\mathbf{d}^p_{[j|m|k]n}\,\mathbf{a}^i_{pl} + \mathbf{a}^p_{jk}\,\mathbf{d}^i_{[p|m|l]n} + \mathbf{b}^p_{[j|m|k]}\,\mathbf{b}^i_{[pl]n} + \mathbf{b}^p_{[jk]n}\,\mathbf{b}^i_{[p|m|l]}),$$

$$\mathbf{D}^2_n \mathbf{c}^i_{j[k|l|m]} = \mathbf{d}^i_{jpln}\,\mathbf{a}^p_{km} + \mathbf{b}^i_{jpl}\,\mathbf{b}^p_{[km]n},$$

$$\mathbf{D}^2_n \mathbf{c}^i_{[jk]lm} = \mathbf{Z}^i_{[j|m|k]ln} + \mathbf{b}^p_{[jk]n}\,\mathbf{b}^i_{pml} - \mathbf{b}^i_{[pk]n}\,\mathbf{b}^p_{jml} - \mathbf{b}^i_{[jp]n}\,\mathbf{b}^p_{kml} + \mathbf{a}^p_{jk}\,\mathbf{d}^i_{pmln}$$
$$- \mathbf{a}^i_{pk}\,\mathbf{d}^p_{jmln} - \mathbf{a}^i_{jp}\,\mathbf{d}^p_{kmln},$$

$$3\mathbf{Z}^i_{jklmn} = \mathbf{D}^2_n(3\mathbf{d}^i_{[jkl]m} - 6\mathbf{d}^i_{[jml]k} + 3\mathbf{d}^i_{[jmk]l} + 3\mathbf{d}^i_{[lmk]j} - 4\mathbf{d}^i_{lj[km]} + 4\mathbf{d}^i_{jk[lm]}$$
$$+ 4\mathbf{d}^i_{ml[jk]} + 2\mathbf{d}^i_{jm[lk]} + 2\mathbf{d}^i_{kj[lm]} + 2\mathbf{d}^i_{lm[jk]} + 2\mathbf{d}^i_{mk[jl]} + 2\mathbf{d}^i_{kl[mj]})$$
$$- 4\mathbf{Z}^i_{[k|l|m]jn} + 4\mathbf{Z}^i_{[j|k|m]ln} + 2\mathbf{Z}^i_{[l|j|k]mn} + 2\mathbf{Z}^i_{[j|m|l]kn} \tag{27}$$

and where

$$\mathbf{D}^2_n \mathbf{d}^i_{[jkl]m} = \tfrac{2}{3}\sigma_{jkl}(\mathbf{d}^p_{[jk]mn}\,\mathbf{a}^i_{pl} + \mathbf{a}^p_{jk}\,\mathbf{d}^i_{[pl]mn} + \mathbf{b}^p_{[jk]m}\,\mathbf{b}^i_{[pl]n} + \mathbf{b}^p_{[jk]n}\,\mathbf{b}^i_{[pl]m}),$$

$$\mathbf{D}^2_n \mathbf{d}^i_{jk[lm]} = -\mathbf{d}^i_{jkpn}\,\mathbf{a}^p_{lm} - \mathbf{b}^i_{jkp}\,\mathbf{b}^p_{[lm]n},$$

$$\sigma_{jkl}(\mathbf{Z}^i_{[j|k|l]mn} - 2\mathbf{d}^p_{[jk]mn}\,\mathbf{a}^i_{pl} - 2\mathbf{a}^p_{jk}\,\mathbf{d}^i_{[pl]mn} - \mathbf{b}^p_{[jk]m}\,\mathbf{b}^i_{[pl]n} - \mathbf{b}^p_{[jk]n}\,\mathbf{b}^i_{[pl]m}) = 0, \tag{28}$$

$$\sigma_{jlm}(\mathbf{Z}^i_{[j|k|l]mn} + \mathbf{d}^i_{jkpn}\,\mathbf{a}^p_{lm} + \mathbf{b}^i_{jkp}\,\mathbf{b}^p_{[lm]n}) = 0. \tag{29}$$

In virtue of (22), (23), (26) and (27), the tensor fields **X**, **Y**, **Z** and **U** will be considered as algebraic combinations of tensor fields $\mathbf{a}^i_{jk}$, $\mathbf{b}^i_{jkl}$, $\mathbf{d}^i_{[j|k|l]m}$, $\mathbf{Z}^i_{[j|k|l]mn}$. By a partial solution of the system of equations (20), (21), (24), (25), (28) and (29) we get the following expression for $\mathbf{Z}^i_{[j|k|l]mn}$:

$$
\begin{aligned}
6\mathbf{Z}^i_{[j|k|l]mn} = {} & 6\mathbf{Z}^i_{[mkn][jl]} + 4\mathbf{Z}^i_{[n|m|k][jl]} - 6\mathbf{Z}^i_{mkn[jl]} - 6\mathbf{Z}^i_{[mjn]kl} + 3\mathbf{Z}^i_{[mjk]nl} + 3\mathbf{Z}^i_{[njk]ml} \\
& + 6\mathbf{Z}^i_{[mln]kj} - 3\mathbf{Z}^i_{[mlk]nj} - 3\mathbf{Z}^i_{[nlk]mj} - 4\mathbf{Z}^i_{nm[kj]l} + 4\mathbf{Z}^i_{mk[nj]l} + 4\mathbf{Z}^i_{jn[mk]l} \\
& + 4\mathbf{Z}^i_{nm[kl]j} - 4\mathbf{Z}^i_{mk[nl]j} - 4\mathbf{Z}^i_{ln[mk]j} + 2\mathbf{Z}^i_{mj[nk]l} + 2\mathbf{Z}^i_{km[nj]l} - 2\mathbf{Z}^i_{nj[mk]l} \\
& + 2\mathbf{Z}^i_{jk[mn]l} - 2\mathbf{Z}^i_{ml[nk]j} - 2\mathbf{Z}^i_{km[nl]j} - 2\mathbf{Z}^i_{nl[mk]j} - 2\mathbf{Z}^i_{lk[mn]j} + 2\mathbf{Z}^i_{kn[jm]l} \\
& - 2\mathbf{Z}^i_{kn[lm]j} - 8\mathbf{Z}^i_{[k|n|j][ml]} + 16\mathbf{Z}^i_{[m|k|j][nl]} + 4\mathbf{Z}^i_{[m|j|n][kl]} + 8\mathbf{Z}^i_{[k|n|l][mj]} \\
& - 16\mathbf{Z}^i_{[m|k|l][nj]} - 4\mathbf{Z}^i_{[m|l|n][kj]} + 8\mathbf{Z}^i_{mk[jl]n} + 8\mathbf{Z}^i_{jk[lm]n} - 8\mathbf{Z}^i_{lk[jm]n} \\
& + 2\mathbf{Z}^i_{nj[lm]k} + 2\mathbf{Z}^i_{lj[mn]k} + 2\mathbf{Z}^i_{mj[nl]k} - 2\mathbf{Z}^i_{nl[jm]k} - 2\mathbf{Z}^i_{jl[mn]k} - 2\mathbf{Z}^i_{ml[nj]k} \\
& - 4\mathbf{Z}^i_{[j|n|l][mk]} + 6\mathbf{Z}^i_{[mjl]nk} + 6\mathbf{Z}^i_{[jnl]mk} + 4\mathbf{Z}^i_{[j|k|l][mn]} + 2\mathbf{Z}^i_{nml[jk]} \\
& - 2\mathbf{Z}^i_{nmj[lk]} + 2\mathbf{Z}^i_{nm[jl]k} + 6\mathbf{Z}^i_{[nkj]lm} - 2\mathbf{Z}^i_{kjn[lm]} + 2\mathbf{Z}^i_{kln[jm]} - 2\mathbf{Z}^i_{jm[nl]k} \\
& - 6\mathbf{Z}^i_{[nkl]jm} + 2\mathbf{Z}^i_{njk[lm]} - 2\mathbf{Z}^i_{nlk[jm]} + 2\mathbf{Z}^i_{lm[nj]k} + 8\mathbf{Z}^i_{[k|n|m][jl]} + 2\mathbf{Z}^i_{jmn[lk]} \\
& - 2\mathbf{Z}^i_{lmn[jk]} + 2\mathbf{Z}^i_{jn[kl]m} - 2\mathbf{Z}^i_{ln[kj]m} + 2\mathbf{Z}^i_{jk[nl]m} - 2\mathbf{Z}^i_{lk[nj]m} + 2\mathbf{Z}^i_{nk[lj]m} \\
& + 2\mathbf{Z}^i_{kn[lj]m} - 4(\mathbf{Y}^i_{[jm]nkl} - \mathbf{Z}^i_{[j|k|m]nl}) + 4(\mathbf{Y}^i_{[km]jnl} - \mathbf{Z}^i_{[k|n|m]jl}) \\
& + 4(\mathbf{Y}^i_{[lm]nkj} - \mathbf{Z}^i_{[l|k|m]nj}) - 4(\mathbf{Y}^i_{[km]lnj} - \mathbf{Z}^i_{[k|n|m]lj}) + 2(\mathbf{Y}^i_{[nj]kml} - \mathbf{Z}^i_{[n|m|j]kl}) \\
& - 2(\mathbf{Y}^i_{[nl]kmj} - \mathbf{Z}^i_{[n|m|l]kj}) - 2(\mathbf{Y}^i_{[kn]mjl} - \mathbf{Z}^i_{[k|j|n]ml}) \\
& - 2(\mathbf{Y}^i_{[kn]mlj} - \mathbf{Z}^i_{[k|l|n]mj}) + 3(\mathbf{Y}^i_{knjml} - \mathbf{U}^i_{knjlm}) + 3(\mathbf{Y}^i_{knlmj} - \mathbf{U}^i_{knljm}) \\
& - 6\mathbf{U}^i_{kn[jl]m} + 3\mathbf{Y}^i_{[knj]ml} - 6\mathbf{Y}^i_{[knm]jl} + 3\mathbf{Y}^i_{[kjm]nl} + 3\mathbf{Y}^i_{[njm]kl} - 3\mathbf{Y}^i_{[knl]mj} \\
& + 6\mathbf{Y}^i_{[knm]lj} - 3\mathbf{Y}^i_{[klm]nj} - 3\mathbf{Y}^i_{[nlm]kj} - 4\mathbf{Y}^i_{n[j|k|m]l} + 4\mathbf{Y}^i_{k[n|j|m]l} \\
& + 4\mathbf{Y}^i_{m[k|n|j]l} + 4\mathbf{Y}^i_{n[l|k|m]j} - 4\mathbf{Y}^i_{k[n|l|m]j} - 4\mathbf{Y}^i_{m[k|n|l]j} + 2\mathbf{Y}^i_{k[n|m|j]l} \\
& + 2\mathbf{Y}^i_{j[n|k|m]l} + 2\mathbf{Y}^i_{n[k|m|j]l} + 2\mathbf{Y}^i_{m[k|j|n]l} - 2\mathbf{Y}^i_{k[n|m|l]j} - 2\mathbf{Y}^i_{l[n|k|m]j} \\
& - 2\mathbf{Y}^i_{n[k|m|l]j} - 2\mathbf{Y}^i_{m[k|l|n]j} + 2\mathbf{Y}^i_{j[m|n|k]l} - 2\mathbf{Y}^i_{l[m|n|k]j},
\end{aligned}
\tag{30}
$$

where the tensor fields $\mathbf{Z}^i_{[jkl]mn}$, $\mathbf{Z}^i_{jk[lm]n}$, $\mathbf{Z}^i_{jkl[mn]}$, $(\mathbf{Y}^i_{[jk]lmn} - \mathbf{Z}^i_{[j|m|k]ln})$, $(\mathbf{Y}^i_{jklmn} - \mathbf{U}^i_{jklnm})$, $\mathbf{U}^i_{jk[nm]l}$, $\mathbf{Y}^i_{[jkl]mn}$, $\mathbf{Y}^i_{j[k|l|m]n}$ are expressed in terms of $\mathbf{a}^i_{jk}$, $\mathbf{b}^i_{jkl}$ and $\mathbf{d}^i_{[j|k|l]m}$ (see above). As a result of these considerations we state the following theorem.

THEOREM B. *Equations* (3), (6), (8), (9), (15), (16), (22), (23), (26), (27), *and* (30) *form a Pfaffian system defining structure tensor fields* $\mathbf{a}^i_{jk}$ *and* $\mathbf{b}^i_{jkl}$ *of the*

***hexagonal** three-web. Any solution of the system in a neighborhood of an arbitrary point* 0 *is uniquely determined by the initial values, i.e., by the values of* $\mathbf{a}^i_{jk}(0)$, $\mathbf{b}^i_{jkl}(0)$ *and* $\mathbf{d}^i_{[j|k|l]m}(0)$ *at that point.* □

This theorem was proved by A. M. Shelekhov [4], [10]. In Shelekhov's formulation the local structure of a hexagonal three-web is uniquely determined by initial values $\mathbf{a}^i_{jk}(0)$, $\mathbf{b}^i_{jkl}(0)$, $\mathbf{c}^i_{jklm}(0)$ and $\mathbf{d}^i_{jklm}(0)$.

Expressions (15)–(16), discussed above, allow us to formulate a more economic version of the theorem and to make some progress in the investigation of the integrability conduction of this system of structure equations (see below).

The following propositions hold:

PROPOSITION 3. *Equations* (21), (24), (25) *are algebraic corollaries of the system of equations* (3), (8), (22), (23), (26), (27), (30) *and* (17), (18), (20), (28), (29). □

PROPOSITION 4. *Equations* (28) *and* (29) *are equivalent in virtue of* (3), (8), (17), (18), (30) *to the following identity*:

$$\begin{aligned}
\sigma_{jkl}\{[&-6\mathbf{d}^p_{[k|m|l]n}\,\mathbf{a}^i_{pj} - 6(\mathbf{d}^i_{[m|l|p]n} + \mathbf{d}^i_{[m|n|l]p})\mathbf{b}^p_{jk} - 6(\mathbf{d}^i_{[k|n|l]p} + \mathbf{d}^i_{[k|p|l]n})\mathbf{a}^p_{jm}\\
&+ 3\mathbf{b}^p_{lmn}(-\mathbf{b}^i_{[pk]j} + \mathbf{b}^i_{[pj]k}) + 3\mathbf{b}^p_{mnl}\,\mathbf{b}^i_{p[jk]} + 3\mathbf{b}^p_{[kl]m}\,\mathbf{b}^i_{jnp}\\
&- 3\mathbf{b}^p_{m[kl]}\,\mathbf{b}^i_{npj} - 3\mathbf{b}^p_{[l|m|k]}\,\mathbf{b}^i_{jpn} - 3\mathbf{b}^p_{m[kl]}\,\mathbf{b}^i_{jpn} - 3\mathbf{b}^p_{m[kl]}\,\mathbf{b}^i_{pnj} - 3\mathbf{b}^p_{[jkl]}\,\mathbf{b}^i_{pmn}\\
&+ \mathbf{a}^p_{jk}\,\mathbf{a}^t_{pm}(-12\mathbf{b}^i_{lnt} - 6\mathbf{b}^i_{nlt} + 6\mathbf{b}^i_{[tln]}) + 4\mathbf{a}^p_{jk}\,\mathbf{a}^i_{pt}\,\mathbf{b}^t_{[lm]n} + 3\mathbf{a}^p_{jk}\,\mathbf{a}^i_{tl}\,\mathbf{b}^t_{m[np]}\\
&+ \mathbf{a}^p_{jk}\,\mathbf{a}^i_{tm}(-2\mathbf{b}^t_{pnl} - 2\mathbf{b}^t_{lnp} - 4\mathbf{b}^t_{pln} + 3\mathbf{b}^t_{lpn}) - 6\mathbf{a}^i_{pj}\,\mathbf{a}^t_{mk}\,\mathbf{b}^p_{lnt} + 6\mathbf{a}^i_{pj}\,\mathbf{a}^t_{ml}\,\mathbf{b}^p_{knt}\\
&+ 6\mathbf{a}^p_{jm}\,\mathbf{a}^t_{kn}(\mathbf{b}^i_{lpt} - \mathbf{b}^i_{ltp})] + [m \leftrightarrow n]\} = 0,
\end{aligned} \tag{31}$$

where $[m \leftrightarrow n]$ *denotes the expression obtained by the permutation of the indices* m *and* n *in the expression into the square brackets.* □

PROPOSITION 5. *Under conditions of* (3), (8), (17), (18), (22), (23), (26), (27), (28), (29), (30) *equation* (20) *is equivalent to equation* (32) *by*

$$\begin{aligned}
\{[&\mathbf{a}^i_{pj}(2\mathbf{d}^p_{[m|k|n]l} - \mathbf{d}^p_{[m|l|n]k}) - \mathbf{a}^i_{pl}\,\mathbf{d}^p_{[m|j|n]k} + 2\mathbf{a}^i_{pm}\,\mathbf{d}^p_{[l|n|j]k} + 3\mathbf{a}^p_{jl}\,\mathbf{d}^i_{[m|p|n]k}\\
&+ 2\mathbf{a}^p_{jm}(\mathbf{d}^i_{[k|n|p]l} + \mathbf{d}^i_{[k|n|l]p}) + 2\mathbf{a}^p_{lm}\,\mathbf{d}^i_{[p|n|j]k} + \mathbf{a}^i_{pj}\,\mathbf{a}^t_{mn}(\mathbf{b}^p_{ltk} + \mathbf{b}^p_{tkl}) - 2\mathbf{a}^p_{mn}\,\mathbf{d}^i_{[j|k|l]p}\\
&+ \mathbf{a}^i_{pj}\,\mathbf{a}^t_{ml}(2\mathbf{b}^p_{[knt]} + \mathbf{b}^p_{tnk} + \mathbf{b}^p_{tkn} - 2\mathbf{b}^p_{knt} - \tfrac{8}{3}\mathbf{b}^p_{[nt]k} - \tfrac{8}{3}\mathbf{b}^p_{[nk]t}) + \mathbf{a}^i_{pj}\,\mathbf{a}^p_{tk}(\mathbf{b}^t_{mnl} + \mathbf{b}^t_{mln})\\
&+ \mathbf{a}^i_{pj}\,\mathbf{a}^t_{km}(4\mathbf{b}^p_{[nlt]} + \tfrac{20}{3}\mathbf{b}^p_{[tn]l} + \tfrac{4}{3}\mathbf{b}^p_{[lt]n}) + (\mathbf{a}^p_{lj}\,\mathbf{a}^i_{pt} - \mathbf{a}^i_{pj}\,\mathbf{a}^p_{tl})(\mathbf{b}^t_{mnk} + \mathbf{b}^t_{mkn})\\
&+ \tfrac{1}{3}\mathbf{a}^i_{pt}\,\mathbf{a}^p_{lm}(\mathbf{b}^t_{jkn} - \mathbf{b}^t_{jnk}) - \mathbf{a}^i_{pm}\,\mathbf{a}^p_{lt}(3\mathbf{b}^t_{jnk} + \mathbf{b}^t_{jkn}) + \mathbf{a}^p_{jm}\,\mathbf{a}^i_{tl}(-10\mathbf{b}^t_{[nkp]} - 2\mathbf{b}^t_{nkp} + \mathbf{b}^t_{pnk}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{b}^t_{pkn} + \tfrac{8}{3}\mathbf{b}^t_{[nk]p} + \tfrac{8}{3}\mathbf{b}^t_{[np]k}) + \mathbf{a}^t_{tm}\,\mathbf{a}^p_{jn}(-12\mathbf{b}^t_{[plk]} + \tfrac{4}{3}\mathbf{b}^t_{[pl]k} + \tfrac{4}{3}\mathbf{b}^t_{[pk]l}) \\
&+ \mathbf{a}^i_{tm}\,\mathbf{a}^p_{jl}(4\mathbf{b}^t_{[np]k} - \mathbf{b}^t_{pkn} - \mathbf{b}^t_{pnk}) + \mathbf{a}^i_{pl}\,\mathbf{a}^t_{mn}\,\mathbf{b}^p_{jkt} + \tfrac{3}{2}\mathbf{a}^p_{mn}\,\mathbf{a}^i_{pt}\,\mathbf{b}^t_{ljk} \\
&+ \mathbf{a}^p_{tj}\,\mathbf{a}^i_{pm}(12\mathbf{b}^t_{[kln]} - \mathbf{b}^t_{lnk}) + \mathbf{a}^p_{jm}\,\mathbf{a}^i_{pt}(\tfrac{2}{3}\mathbf{b}^t_{[kn]l} - \tfrac{2}{3}\mathbf{b}^t_{[nl]k} + \mathbf{b}^t_{knl}) \\
&+ \mathbf{a}^p_{mt}\,\mathbf{a}^i_{pj}(+12\mathbf{b}^t_{[kln]} - 2\mathbf{b}^t_{nkl} - \mathbf{b}^t_{lkn}) + \mathbf{a}^p_{jl}\,\mathbf{a}^t_{mn}(\mathbf{b}^i_{tpk} + \mathbf{b}^i_{ktp} - \mathbf{b}^i_{kpt} - \mathbf{b}^i_{pkt}) \\
&+ \mathbf{a}^p_{jl}\,\mathbf{a}^t_{pk}(\mathbf{b}^i_{mtn} + \mathbf{b}^i_{mnt}) + \mathbf{a}^p_{jm}\,\mathbf{a}^t_{kl}(+15\mathbf{b}^i_{[tpn]} + 3\mathbf{b}^i_{tnp} + 3\mathbf{b}^i_{ntp} - \mathbf{b}^i_{pnt}) \\
&+ \mathbf{a}^p_{jm}\,\mathbf{a}^t_{pk}(+13\mathbf{b}^i_{[ntl]} - \mathbf{b}^i_{tnl} + \mathbf{b}^i_{nlt} + \mathbf{b}^i_{lnt}) + \mathbf{a}^p_{jl}\,\mathbf{a}^t_{pm}(\mathbf{b}^i_{[tkn]} - 3\mathbf{b}^i_{ntk} - \mathbf{b}^i_{ktn}) \\
&+ \mathbf{a}^p_{lm}\,\mathbf{a}^t_{pj}(-\mathbf{b}^i_{[tkn]} - \mathbf{b}^i_{knt} - \mathbf{b}^i_{nkt}) + \mathbf{a}^p_{mj}\,\mathbf{a}^t_{pl}(12\mathbf{b}^i_{[tkn]} - \mathbf{b}^i_{tnk}) \\
&+ \mathbf{a}^p_{jm}\,\mathbf{a}^t_{kn}(11\mathbf{b}^i_{[tlp]} - \tfrac{1}{2}\mathbf{b}^i_{lpt} + \tfrac{1}{2}\mathbf{b}^i_{ltp} + \mathbf{b}^i_{ptl} - \mathbf{b}^i_{tpl}) \\
&+ \mathbf{a}^p_{jm}\,\mathbf{a}^t_{ln}(14\mathbf{b}^i_{[pkt]} - 2\mathbf{b}^i_{ktp} + \mathbf{b}^i_{tpk} + \mathbf{b}^i_{tkp}) \\
&+ \mathbf{a}^p_{mn}\,\mathbf{a}^t_{pj}(-3\mathbf{b}^i_{[tkl]} - \tfrac{1}{2}\mathbf{b}^i_{ltk} + \mathbf{b}^i_{tlk} - \tfrac{1}{2}\mathbf{b}^i_{lkt} + 2\mathbf{b}^i_{ktl}) - \mathbf{a}^p_{mn}\,\mathbf{a}^t_{pl}(2\mathbf{b}^i_{jtk} + \tfrac{3}{2}\mathbf{b}^i_{tjk}) \\
&+ \tfrac{4}{3}\mathbf{a}^p_{lm}\,\mathbf{a}^i_{tn}(\mathbf{b}^t_{pjk} - \mathbf{b}^t_{jpk}) + \tfrac{1}{2}(\mathbf{b}^i_{lmp}\,\mathbf{b}^p_{jnk} - \mathbf{b}^i_{mpl}\,\mathbf{b}^p_{kjn} - \mathbf{b}^i_{mlp}\,\mathbf{b}^p_{nkj} + \mathbf{b}^i_{lpm}\,\mathbf{b}^p_{jnk}) \\
&- \tfrac{1}{2}(\mathbf{b}^i_{npm}\,\mathbf{b}^p_{jkl} + \mathbf{b}^i_{mnp}\,\mathbf{b}^p_{ljk} + \mathbf{b}^i_{pmn}\,\mathbf{b}^p_{klj}) + \tfrac{3}{2}\mathbf{b}^i_{[mlp]}\,\mathbf{b}^p_{jnk} - \tfrac{3}{4}\mathbf{b}^i_{[mnp]}\,\mathbf{b}^p_{ljk} \\
&- \tfrac{3}{2}\mathbf{b}^p_{mnl}\,\mathbf{b}^i_{jkp} + \tfrac{1}{2}\mathbf{b}^p_{nlm}\,\mathbf{b}^i_{pjk} + \mathbf{b}^p_{lmn}\,\mathbf{b}^i_{kpj} + \tfrac{3}{4}\mathbf{b}^p_{[lmn]}\,\mathbf{b}^i_{jpk} + \tfrac{9}{4}\mathbf{b}^p_{[lmn]}\,\mathbf{b}^i_{jkp} \\
&- \mathbf{b}^i_{jmp}\,\mathbf{b}^p_{kln} - \mathbf{b}^i_{jpm}\,\mathbf{b}^p_{lnk} + \mathbf{b}^p_{nkl}\,\mathbf{b}^i_{mjp} + \mathbf{b}^p_{nlk}\,\mathbf{b}^i_{pjm} + \tfrac{1}{2}\mathbf{b}^p_{lnk}(\mathbf{b}^i_{pjm} - \mathbf{b}^i_{mpj}) \\
&+ \mathbf{b}^i_{[mpj]}(\tfrac{3}{2}\mathbf{b}^p_{knl} - 3\mathbf{b}^p_{lnk}) - \tfrac{1}{2}\mathbf{b}^p_{nkl}\,\mathbf{b}^i_{pjm} + \tfrac{1}{2}\mathbf{b}^p_{lkn}\,\mathbf{b}^i_{jmp} - \tfrac{1}{2}\mathbf{b}^p_{nlk}\,\mathbf{b}^i_{mjp} \\
&+ \mathbf{b}^p_{knl}(-\tfrac{1}{2}\mathbf{b}^i_{jmp} + \tfrac{1}{2}\mathbf{b}^i_{mpj} + \tfrac{1}{2}\mathbf{b}^i_{jpm}) + 3\mathbf{b}^p_{[kln]}\,\mathbf{b}^i_{jpm} - \tfrac{15}{2}\mathbf{b}^i_{[mpj]}\,\mathbf{b}^p_{[kln]} \\
&+ \mathbf{b}^p_{mnk}(+\tfrac{1}{4}\mathbf{b}^i_{lpj} + \tfrac{1}{4}\mathbf{b}^i_{ljp} - \tfrac{1}{2}\mathbf{b}^i_{plj}) + \mathbf{b}^p_{nkm}(\tfrac{1}{4}\mathbf{b}^i_{pjl} - \tfrac{1}{4}\mathbf{b}^i_{jlp} + \tfrac{5}{4}\mathbf{b}^i_{jpl} - \tfrac{1}{4}\mathbf{b}^i_{plj}) \\
&- \mathbf{b}^p_{kmn}(\tfrac{1}{4}\mathbf{b}^i_{lpj} + \tfrac{3}{4}\mathbf{b}^i_{ljp}) - 3\mathbf{b}^p_{[kmn]}\,\mathbf{b}^i_{jpl} + \tfrac{3}{2}\mathbf{b}^p_{nkm}\,\mathbf{b}^i_{[jlp]} \\
&+ \tfrac{9}{2}\mathbf{b}^p_{[kmn]}\,\mathbf{b}^i_{[jlp]}] - [m \leftrightarrow n]\} + \{j \leftrightarrow k\} = 0, \qquad (32)
\end{aligned}
$$

where $[m \leftrightarrow n]$ *denotes the expression obtained by the permutation of the indices* m *and* n *in the expression into the square brackets* $\{j \leftrightarrow k\}$*—analogically.* □

PROPOSITION 6. *Equation* (31) *is an algebraic corollary of equations* (3), (8), (17), (18), (32).

(The operator of cyclic summation with respect to indices j, k, l must be applied to (32) in order to prove the proposition. The resulting identity is equivalent to (31)). □

Finally, we have the following theorem.

THEOREM C. *The integrability conditions of the system* (3), (6), (8), (9), (15), (16), (22), (23), (26), (27), (30) *are reduced to* (17), (18), (32). □

Concluding remarks

In order to get the differentiable extension of system (3), (6), (8), (9), (15), (16), (17), (18), (22), (23), (26), (27), (30), (32) one has to consider the commutation relations (7) applied to $\mathbf{d}^i_{[j|k|l]m}$ and $\mathbf{D}^1$-, $\mathbf{D}^2$-derivatives of equation (32). Three of these five equations follow from the following:

$$\mathbf{D}^1_p(\mathbf{X}^i_{jkl[mn]} - \mathbf{c}^i_{jklt}\,\mathbf{a}^t_{mn}) = 0,$$

$$\mathbf{D}^1_p\,\mathbf{Z}^i_{[j|k|l]mn} - \mathbf{D}^2_n\,\mathbf{U}^i_{[j|k|l]mp} = \mathbf{d}^t_{[j|k|l]m}\,\mathbf{b}^i_{tpn} - \mathbf{d}^i_{[t|k|l]m}\,\mathbf{b}^t_{jpn} - \mathbf{d}^i_{[j|t|l]m}\,\mathbf{b}^t_{kpn} - \mathbf{d}^i_{[j|k|t]m}\,\mathbf{b}^t_{lpn} - \mathbf{d}^i_{[j|k|l]t}\,\mathbf{b}^t_{mpn}. \tag{33}$$

Thus further analysis of the system is reduced to the computation of equations (33) and an investigation of their independence from (3), (8), (17), (18), (32).

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