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General theory of the translation equation

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Summary. This paper gives a survey of the results of the general theory of translation equation which appeared after 1973.

Introduction

In [41] I listed several mathematical domains (abstract geometric and algebraic objects, abstract automata, groups of transformations, iterations, linear representations of groups, dynamical systems) in which the translation equation appears. I showed there also some directions of the development of the general theory of this equation. The bibliography in [41] contains papers in these directions published until 1973. So, in this survey we consider papers concerning the general theory of the translation equation which appeared after 1973 (both in the directions mentioned in [41] and in new domains of this theory).

The equation

$$F(F(\alpha, x), y) = F(\alpha, x \cdot y), \tag{1}$$

where the unknown function F has its values in an arbitrary set Γ and is defined on a subset of the Cartesian product $\Gamma \times G$, where G is a set with a binary operation " $\cdot$ ", defined for some pairs $(x, y) \in G \times G$, is called the translation equation (or the transformation equation).

If the function F is defined on the whole set $\Gamma \times G$ and the operation " $\cdot$ " is defined for all pairs $(x, y) \in G \times G$, then the notion that F satisfies the translation equation needs no comments: both sides of the equation are defined and equal for

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any $\alpha \in \Gamma$ and $x, y \in G$. The situation changes when we deal with the general case, as it has been formulated above. One can define in various ways what it means that the translation equation is satisfied, and in various fields of mathematics we do use various definitions.

To this area belong the papers [27], [62], [82], [83], from which I quoted the first two in [41] with incomplete data. They give a complete system of definitions stating when the translation equation is satisfied, comparison of these definitions and their consequences, e.g. for the problem of extending solutions.

I. Structure of solutions

The papers [60], [76], [65], [63], [64], [15], [80], [14], [5], [24], contain constructions of the general solution of the translation equation on some algebraic structures. No additional assumptions are supposed. I present here the result of one of these papers. The paper [80] gives some simplification of the construction of the general solution of (1) in the case where G is a group (see [41]). It reads as follows.

Let $\{G_k\}_{k \in K}$ denote an arbitrary family of subgroups of the group G. We do not assume that the map $k \to G_k$ is one-to-one, thus the map $k \to G/G_k$, where $G/G_k = \{G_k x : x \in G\}$, is also not necessarily one-to-one. We shall introduce the so called indexed quotient structure $(G/G_k, k) := \{(G_k x, k) : x \in G\}$ for $k \in K$. In this way we shall obtain a one-to-one map $k \to (G/G_k, k)$.

The multiplication of indexed cosets by elements of the group G is defined in the natural way:

$$(G_k x, k)y := (G_k xy, k) \qquad \text{for } x, y \in G,\ k \in K.$$

We choose

(1) an arbitrary family $\{G_k\}_{k \in K}$ of subgroups of G such that

$$\operatorname{card} \bigcup_{k \in K} (G/G_k, k) \le \operatorname{card} \Gamma;$$

(2) an arbitrary one-to-one map $\phi\colon \bigcup_{k \in K} (G/G_k, k) \to \Gamma$ and
(3) an arbitrary function $g\colon \Gamma \to \Gamma$ such that $g(g) = g$ and

$$\phi\Big(\bigcup_{k \in K} (G/G_k, k)\Big) = g(\Gamma).$$

We define the function F by the formula

$$F(\alpha, x) := \phi[\phi^{-1}(g(\alpha))x] \qquad \text{for } \alpha \in \Gamma, x \in G. \tag{2}$$

If we know all solutions of (1) on a structure G, it was shown in [60] how to get all solution of this equation on the structure obtained by adjoining a zero 0 (i.e. we set $0 \cdot x = x \cdot 0$ for $x \in G \cup \{0\}$, where $0 \notin G$).

In Chapter VIII of [76] the general structure of the solution of (1) on $G \cup \{0\}$ was given in the case where G is an Ehresmann groupoid.

In [63] the general solution of (1) was given in the case where $\text{card}(G \cdot G) = 1$ or $ab = b$ for all $a, b \in G$ or $ab = a$ for all $a, b \in G$. The paper [5] generalizes these results to the case of equation

$$F(\ldots F(F(\alpha, x_1), x_2), \ldots), x_n) = F(\alpha, x_1 \cdot x_2 \ldots x_n).$$

Under the assumption that $x_1 \cdot x_2 \ldots x_n = x_n$ for all $x_1, \ldots, x_n \in G$ we have the following result.

A mapping $F: \Gamma \times G \to \Gamma$ is a solution of the equation

$$F(\ldots F(F(\alpha, x_1), x_2)\ldots), x_n) = F(\alpha, x_n)$$

iff there exist a partition $\{\Gamma_i\}_{i \in I}$ of Γ, a family J of functions $f: \Gamma \to \Gamma$ and a function $h: G \to J$ such that the following conditions are satisfied:

(1) if $f \in J$ then $f^{n-1}(\Gamma_i) \subset \Gamma_i$ and $\text{card} f(\Gamma_i) = 1$ for every $i \in I$;
(2) $f_1(f_2) = f_1(f_3)$ for all $f_1, f_2, f_3 \in J$:
(3) $F(\alpha, x) = (h(x))(\alpha)$ for all $(\alpha, x) \in \Gamma \times G$.

In [14] the construction of the general solution of (1) was given in the case where G is a commutative semigroup, $G \cdot G = G$ and the order $\leq$ defined by $x \leq y \leftrightarrow x + y = y$ is linear, complete and G possesses a minimal element.

The long paper [24] concerns, among other things, the construction of the general solution of (1) on some not necessarily associative structures called QD-groupoids.

A groupoid $(G, \cdot)$ is called a groupoid with quasi-division (QD-groupoid), if the condition

$$\forall a, b \in G\ \exists c, d \in G \qquad (\overline{a, c, b} = a \text{ or } \overline{a, d, b} = b)$$

is satisfied, where, by definition,

$\overline{a, c, b}$ equals either $a(cb)$ or $(ac)b$ $\qquad (a, b, c \in G)$.

Obviously, $\overline{a, c, b}$ is uniquely defined iff $(G, \cdot)$ is a semigroup.

It is easy to verify that every groupoid with division (D-groupoid) is a QD-groupoid. Consequently, quasigroups, loops and groups are QD-groupoids.

For QD-groupoids the solution of the translation equation can be characterized by means of partitions P of the set G satisfying the condition

$$\forall a \in G\ \forall A \in P\ \exists B \in P \qquad (A \cdot a \subset B), \tag{3}$$

and the following condition

$$(Aa)b \subset B \rightarrow A(ab) \subset B \qquad \text{for } a, b \in G;\ A, B \in P.$$

Such partitions are called the translative partitions of $(G, \cdot)$.

In [25] it is proved that, in the case where $(G, \cdot)$ is a quasigroup, every solution of (1) can be characterized by means of the translative partitions of $(G, \cdot)$. Moreover, it has been shown that each translative partition of $(G, \cdot)$ is of the form $\{Ha: a \in G)$, where H is some special subquasigroup of $(G, \cdot)$, called translative subquasigroup of $(G, \cdot)$. In the paper [24] we generalize this result. In the class of QD-groupoids we characterize the solution of equation (1) by means of the translative partitions, and we characterize also the translative partitions by means of some special QD-groupoids in $(G, \cdot)$, called translative QD-groupoids in $(G, \cdot)$. In this case the main problem is not the form of solutions of (1) but the characterization of the translative partitions. This is related to the fact that we do not assume associativity and that there need not exist local identities and inverses. It compels to distinguish some elements, the role of which is similar to that of local identities and inverses. The translative QD-groupoids play for the QD-groupoids an analogous role to that of the subgroups for groups.

Several more papers were published in the considered period which give only some classes of the solutions of equation (1), for instance [6], [7], [42], [15], [39], [33], [50], [54], [2].

The papers [6], [7], [39] generalize results of [4]. For example, in [39] the following theorem was proved.

If

(1) *Γ is a Hausdorff topological space and a function $F: \Gamma \times R \rightarrow \Gamma$ satisfies the translation equation*

$$F(F(\alpha, x), y) = F(\alpha, x + y),$$

(2) *for some* $\alpha_0 \in \Gamma$ *the function* $g(y) = F(\alpha_0, y)$ *is continuous and not constant,*
(3) F *is transitive, i.e.*

$$\forall \alpha, \beta \in \Gamma\ \exists x \in R \qquad F(\alpha, x) = \beta,$$

then the function $g\colon R \to \Gamma$ *is a bijection such that*

$$F(\alpha, x) = g[g^{-1}(\alpha) + x]$$

on the set $\Gamma \times R$ *or there exists a constant* $c > 0$ *such that the function* $g_{|[0,c)}\colon [0, c) \to \Gamma$ *is a bijection with which*

$$F(\alpha, x) = g_{|[0,c)}\left[g^{-1}_{|[0,c)}(\alpha) \underset{\bmod c}{+} c\right]$$

on the set $\Gamma \times R$ *(here* $\underset{\bmod c}{+}$ *denotes the addition modulo* c*).*

In [42] all the solutions $F\colon \Gamma \times G^+ \to \Gamma$ of equation (1) were given, where Γ is an arbitrary set and G^+ is a semigroup of positive elements of an Archimedean group G, satisfying the condition

$$\forall \alpha, \beta \in F(\Gamma, G^+) \qquad [F(\alpha, G^+) \cap F(\beta, G^+) \neq \varnothing$$
$$\Rightarrow F(\alpha, G^+) \subset F(\beta, G^+) \text{ or } F(\beta, G^+) \subset F(\alpha, G^+)]. \tag{4}$$

These solutions are given by the following *construction* (C):

(1) Let $f\colon \Gamma \to \Gamma$ be a mapping satisfying the condition

$$\forall \alpha \in \Gamma\colon f(f(\alpha)) = f(\alpha).$$

(2) We decompose $f(\Gamma)$ into a disjoint union of non-empty sets Γ_k $(k \in K)$ such that for each $k \in K$ there exists an invariant decomposition $\{W_{ik}\}_{i \in I_k}$ (i.e. satisfying (3) for $P = \{W_{ik}\}_{i \in I_k}$) of the interval $\Delta_k \subset G$ such that $G^+ \subset \Delta_k$ and $\operatorname{card} I_k = \operatorname{card} \Gamma_k$.
(3) Let $h_k\colon \{W_{ik}\}_{i \in I_k} \to \Gamma_k$ be a bijection and let us define $h_k^*\colon \Delta_k \to \Gamma_k$ in the following way

$$h^*(x) = h_k(W_{ik}) \qquad \text{for } x \in W_{ik}.$$

(4) We put

$$F(\alpha, x) = h_k^*(h_k^{*-1}(f(\alpha))x) \qquad \text{for } f(\alpha) \in \Gamma_k.$$

The decompositions mentioned above were given in [43], [32], [35].

Some generalizations of the construction (C) were given in [15] and [33].

The papers [50], [54], [2] generalize, among other, the particular form of the solution of (1), known from [3], [1], to the following:

$$F(\alpha, x) = f^{-1}(f(\alpha)l(x)), \tag{a}$$

where f is a bijection of Γ onto group G_1 isomorphic to the group G, l is a homomorphism of G into G_1,

$$F(\alpha, x_1, \ldots, x_m) = f^{-1}(f(\alpha) + c_1x_1 + \cdots + c_mx_m), \tag{b}$$

where f is a homomorphism of Γ into R and $(c_1, \ldots, c_m) \in R^m$,

$$F(\alpha, x) = k^{-1}(k(\alpha) + Cx), \tag{c}$$

where $F: R^n \times R^m \to R^n$, $k: R^n \to R^n$ is a bijection and C is a constant $n \times m$ matrix of rank $\min(n, m)$,

$$F(\alpha, x) = \Phi^{-1}(\Phi(\alpha) + c(x)), \tag{d}$$

where $F: \Gamma \times G \to \Gamma$, Φ is an injection of Γ into a group H, c is a homomorphism of the group G into H,

$$F(\alpha, x) = f^{-1}[(u + vB(\alpha))J_\varrho + f(\alpha)], \tag{e}$$

where $F: R^n \times R^m \to R^n$, f is a bijection of R^n onto itself, $\varrho = \min(n, m)$, $x = (x_1, \ldots, x_\varrho, \ldots, x_n)$, $u = (x_1, \ldots, x_\varrho)$, $v = (x_{\varrho+1}, \ldots, x_n)$, $B(\alpha)$ for α from R^n is a $(m - \varrho) \times \varrho$ matrix satisfying

$$B(F(\alpha, x)) = \beta(\alpha),$$

$J_\varrho = (1_\varrho, \mathbf{0})$, 1_ϱ is a unit $\varrho \times \varrho$ matrix and $\mathbf{0}$ denotes $\varrho \times (n - \varrho)$ zero matrix.

Moreover, in these papers necessary and sufficient conditions are given for a solution of (1) to be of one of the forms given above.

The form (e) of the solution of (1) and the problem, considered in [2], when the composition of two solutions of form (e) is of this form as well, generated the papers [9], [10], [11] in which these problems are considered locally. In [21], the more general translation equation of "Pexider's type"

$$F_1(F_2(\alpha, x), y) = F_3(\alpha, xy),$$

is considered, where the unknown functions F_1, F_2, F_3 are defined on subsets of a set $\Gamma \times E$, where Γ is an arbitrary set, E is in particular an Ehresmann groupoid [78] and F_1, F_2, F_3 take their values in Γ.

E. J. Jasińska and M. Kucharzewski [28], [29] began to define accurately the notion of the Klein geometry. These considerations and their continuation in [72] led to an interesting possibility of expressing a solution of the translation equation in terms of a particular solution. Generally this result can be formulated as follows [78].

Given a solution $f: \Gamma_1 \times G \to \Gamma_1$ of (1), where G is a group and e its unit, satisfying the identity condition

$$\forall \alpha \in \Gamma: f(\alpha, e) = \alpha,$$

and the effectivity condition

$$\forall x \in G \qquad [\forall \alpha \in \Gamma: f(\alpha, x) = \alpha \to x = e],$$

we can express with aid of f every solution $F: \Gamma_2 \times G \to \Gamma_2$ of equation (1) satisfying the identity condition if

$$\text{card}\, \Gamma_2 < \sup(\text{card}\, \Gamma_1, \text{card}\, 2^{\Gamma_1}, \text{card}\, 2^{2^{\Gamma_1}}, \ldots).$$

Considerable applications of the general theory of equation (1) to the Klein geometry are possible (see e.g. [28], [29] and references in [41] and [76], [52]).

II. Regular solutions (continuous, differentiable, analytic, monotonic)

Continuous solutions (with respect to one variable or to both variables) of (1) appear natural in different applications of this equation, in particular to iteration theory. Therefore this domain contains many interesting papers concerning continuous solutions (e.g. by J. Aczél, J. A. Baker, C. Blanton, D. Gronau, E. Jabotinsky, H. Michel, M. Sablik, A. Sklar, G. Targoński, J. Weitkämper, M. C. Zdun) in which variables run over subsets of the set of real numbers with natural topology.

Since results of iteration theory is discussed in another survey, by G. Targonski, in this issue, I omit them in the rest of my survey and in the bibliography. Similarly, the theory of dynamical systems is a theory of regular solutions of (1) with some additional properties. It is impossible to discuss here the results of this theory, even those achieved in the last twenty years.

Therefore we present only the papers [46], [18], [17], [19], [55] briefly.

I quote from [17] the following theorem concerning differentiable solutions of equation (1), where Γ is a real interval and $(G, \cdot) = (R, +)$.

Let F be a solution of (1) *on $\Gamma \times R$ (where Γ is a real interval) which is differentiable with respect to each coordinate. Then F is of the structure*

$$F(\alpha, x) = \begin{cases} h_m(h_m^{-1}(\alpha) + x) & \text{when } \alpha \in J_m, \\ \alpha & \text{otherwise}, \end{cases}$$

where $\{J_m\}_{m \in M}$ are the components of an open set O, $O = \bigcup_{m \in M} J_m$, $h_m: R \to J_m$ is a differentiable homeomorphism from R onto J_m such that $h'_m(u) \neq 0$ for all $u \in R$ and for each $m \in M$.

Conversely, any function F of the above structure is a solution of (1), *differentiable with respect to the second variable. It is differentiable with respect to the first variable iff each of the functions $\alpha \to F(\alpha, x)$ is differentiable at any point $\alpha \in \delta O$, the border of O.*

Further, the author finds conditions on O or on the functions h_m which make F differentiable with respect to the first coordinate at the points $\alpha \in \delta O$.

The paper [19] deals with locally differentiable solutions of (1) in a Banach space, defined in the following way.

Let Γ be a real or complex Banach space. A Γ-valued function F is said to be a local solution of the equation (1) if there exist open neighbourhoods U, U' in Γ of $0 \in \Gamma$ and real (open or half open) intervals I, I' or open connected subsets I, I' of complex numbers, each containing the number 0 and where U' is a subset of U, I' a subset of I, such that F is defined on $U \times I$ and

$$F(\alpha, x) \in U \qquad \text{for } (\alpha, x) \in U' \times I'$$

and

$$F(F(\alpha, x), y) = F(\alpha, x + y)$$

holds for all $x, y \in I$ with $x + y \in I$ and $\alpha \in U$, whenever $F(F(\alpha, x), y)$ is defined.

The following theorem was proved there:

Let F be a solution of (1) (*in the sense of the above definition*) *with* $F(0,0) = 0$ *and let* $F(\alpha, 0)$ *be continuously differentiable near zero. Then there exist closed subspaces* Y *and* K *of* Γ *and a local diffeomorphism* T *with fixed point zero, such that* $\Gamma = Y \oplus K$.

The conjugate

$$F^*(\alpha, x) = T(F(T^{-1}(\alpha), x)),$$

which is again a solution of (1), defined on some neighbourhood $W \times I$ of $(0, 0)$ in $\Gamma \times I$, satisfies

(i) $F^*(\beta + \gamma, 0) = \beta$ for $\beta + \gamma \in Y \oplus K$;

(ii) $\operatorname{Im} F^*$ is contained in Y;

(iii) $F^*(\beta + \gamma, x) = F^*(\beta, x)$ for $\beta \in Y \cap W, \gamma \in K \cap W$, $x \in I$.

If $F(\alpha, x)$ is continuously differentiable with respect to the variable α for all x, then so is $F^*(\alpha, x)$. And if, with $x \in I$, also $-x$ is contained in I, then $F^*(\alpha, x)$ restricted to Y is a local diffeomorphism on Y with its inverse $F^*(\alpha, -x)$.

The paper [18] concerns monotonic and continuous solutions of (1), where Γ is a linearly ordered set and G a linearly ordered group. We say that a mapping $F: \Gamma \times G \to \Gamma$ is monotonic if

(1) for each $x \in G$ the mapping $F(\cdot, x)$ is monotonic (in the same direction) and

(2) for each $\alpha \in \Gamma$ the mapping $F(\alpha, \cdot)$ is monotonic (in the same direction).

As we know [41] that the general transitive (i.e. $\forall \alpha, \beta \in \Gamma\ \exists x \in G; F(\alpha, x) = \beta$) solution of (1) is given by the formula

$$F(\alpha, x) = g(g^{-1}(\alpha)x), \tag{5}$$

where g is a bijection of G/G^*—the set of right cosets of G modulo some of its subgroup G^*, onto Γ.

One can prove easily that, if G^* is a convex subgroup of G, then G/G^* is linearly ordered by the following relation

$$A \leq B \leftrightarrow \exists x \in A\ \exists y \in B: x \leq y \qquad \text{for } A, B \in G/G^*.$$

The general transitive and monotic solution of (1) is given by (5), where it is additionally required that G^* be convex and g be monotonic.

Assume now that Γ and G are endowed with topologies induced by the orders in Γ and G, respectively. Then every transitive and monotonic solution of (1) is continuous.

A characterization of continuous solutions of (1) in the general case, where Γ is a topological space and G a topological structure, is very difficult even in the case where G is a topological group and F is a non-transitive solution. These difficulties are considered in [46]. In that paper a condition is given under which continuity of a (not necessarily transitive) solution of (1) on a topological group is equivalent to the continuity of the parameters used in the construction of the solution.

At the University of Graz (D. Gronau, G. H. Mehring, L. Reich, J. Schwaiger) the theory of formal power series has been developed in which a condition called also the translation equation appears, however it is not an equation of the type (1) (see [55]). For this reason I am not going to present the interesting results of this group.

In [55] analytic solutions of (1) are considered, which are defined as follows:

Let $\{F(\alpha, x)\}_{x \in K}$, where $K = R$ or $K = C$, be a family of functions $F(\cdot, x)\colon C(x) \to K$ for all $x \in K$, where $C(x) = \{\alpha \in K\colon |\alpha| < \varrho(x)\}$ for $\varrho(x)$ being real positive ($\varrho(x) = \infty$ is not excluded for some x), analytic with respect to α and such that

$$\forall x \in K\colon F(0, x) = 0,$$

i.e., such that

$$F(\alpha, x) = a_1(x)\alpha + a_2(x)\alpha^2 + \cdots$$

for each $x \in K$ and each $\alpha \in C(x)$, where $a_\nu(x)\colon K \to K$.

The family $\{F(\alpha, x)\}_{x \in K}$ is an analytic solution of (1) provided that there exists an open interval I, symmetric with respect to zero or an open ball I with the centre at zero ($I = K$ is also possible) such that (1) is valid for all $\alpha \in I$ and $x, y \in K$.

In [55], among other things, it is proved that $F(\alpha, x) = a_1(x)\alpha$, where $a_1(x)$ is an exponential function, is the unique analytic solution of (1) in the case where $I = K$ or if

$$K = C \quad \text{and} \quad |a_1(x)| \neq 1 \quad \text{or} \quad a_1(x) = 1 \quad \text{or} \quad F(I, C) \subset I.$$

It has also been shown there that in the case $K = C$ the function

$$F(\alpha, x) = \frac{1}{2} e^{irex}\left[2\alpha\, e^{irex} + (1 - e^{irex})\left(\binom{\frac{1}{2}}{1} 4\alpha + \binom{\frac{1}{2}}{2}(4\alpha)^2 + \cdots\right)\right]$$

$$= \frac{1}{2} e^{irex}[e^{irex} - 1 + 2\alpha\, e^{irex} + (1 - e^{irex})(1 + 4\alpha)^{1/2}]$$

is an analytic solution of (1), which is not of the form $a_1(x)\alpha$.

At the University of Innsbruck (W. Förg-Rob, K. Kuhnert, R. Liedl, H. Reitberger) a theory has been developed, by the method of the so-called Pilgerschritt transformation, which enables to determine one-parameter subgroups of a topological group and which has some applications to characterizations of regular solutions of the translation equation. The discussion of the results of this theory and of papers related to it (see, for example, the bibliography in [31]) is beyond the framework of this topic.

III. The problem of extension

For a given set Γ^* such that $\Gamma \subset \Gamma^*$ and a given structure G^*, such that G is a substructure of G^*, the problem arises of extending a solution $F: \Gamma \times G \to \Gamma$ of equation (1) to a solution $F^*: \Gamma^* \times G^* \to \Gamma^*$ of this equation. Different modifications of this problem can also be considered (see [44]). To this area belong the papers [27], [59], [74], [75], [76], [58], [64], [83], [13], [44], [47], [16], [38], [84], [34]. Below I give some results of some of these papers.

In [59] the following theorem was proved about the extendability in the case where $\Gamma^* = \Gamma$, G and G^* are groups and we do not assume the transitivity of F.

A function of the form (2) *is extendable from the set* $\Gamma \times G$ *to the set* $\Gamma \times G^*$ *iff*

$$\forall G_k\ \exists G_k^* \subset G^* \qquad (G_k^* \cap G = G_k)$$

and there exists a decomposition $\{K_l\}_{l \in L}$ *of the set* K *such that*

$$\forall l \in L\ \exists q \in K_l\ \forall k \in K_l\ \forall a_k \in G^* \quad \Big[G_k = a_k^{-1} G_q a_k \text{ and the family}$$

$$\{A_p\}_{p \in K_l}, \text{ where } A_p = \bigcup_{b \in G} G_q^* a_k b \text{ is a decomposition of } G^* \Big].$$

Let A be an arbitrary set and G an arbitrary group. Consider the translation equation

$$F[F(\alpha; (a, b, x)), (b, c, y)] = F(\alpha; a, c, xy)) \tag{6}$$

for all $a, b, c \in A$, $\alpha \in \Gamma$ and all $x, y \in G$. A binary inner operation in $A \times A \times G$ is also defined as follows:

$$(a, b, x) \cdot (d, c, y) = (a, c, xy) \leftrightarrow b = d$$

(from the theory of geometric objects).

The general solution of this equation can be obtained in the following way [74], [76]:

(1) for every $a \in A$ we choose arbitrarily sets Γ_a and Γ_a^* such that $\Gamma_a^* \subset \Gamma_a \subset \Gamma$ and card $\Gamma_a =$ card Γ_a^*,
(2) for each $a \in A$ we construct a function f_a mapping Γ_a onto Γ_a^* such that $f_a(\alpha) = \alpha$ for $\alpha \in \Gamma_a^*$,
(3) for a fixed element $a_0 \in A$ and for every $a \in A$ we construct a bijection h_a of the set Γ_a^* onto the set $\Gamma_{a_0}^*$,
(4) we choose an arbitrary function H satisfying the translation equation on $\Gamma_{a_0}^* \times G$,
(5) we put

$$F[\alpha; (a, b, x)] = h_a^{-1} H[h_b f_b(\alpha), x] \qquad \text{for } \alpha \in \Gamma_b. \tag{7}$$

The following theorem is valid [74]:

Let F satisfy equation (6) and let A^* be a superset of A (i.e. $A \subset A^*$) and G^* a supergroup of G. The function F can be extended to a solution of equation (6) on the set $\Gamma \times A^* \times A^* \times G^*$ iff there exists a representation of F in form (7) such that the function H can be extended to a function $H^*: \Gamma_{a_0}^* \times G^* \to \Gamma_{a_0}^*$ satisfying equation (1) on the set $\Gamma_{a_0}^* \times G^*$.

Extensions of solutions of equation (1) on an Ehresmann groupoid are discussed also in chapter VIII of [76].

A different question is the problem of extendability of regular (e.g. continuous, open, differentiable) solutions of equation (1). I quote from [75] the following results.

If G is an open subgroup of a topological group G^* and Γ is a topological space then every extension on the set $\Gamma \times G^*$ of a transitive regular solution F of (1) (i.e. F is continuous and for each $\alpha \in \Gamma$ the mapping $x \to F(\alpha, x)$ is open), is transitive and regular, too.

If Γ is a topological T_1-space, G an algebraic subgroup of a topological group G^* and F a transitive, regular solution of (1) on $\Gamma \times G$, which is regularly extendable on the set $\Gamma \times G^*$, then, for every so called stability subgroup $G_\alpha := \{x \in G: F(\alpha, x) = \alpha\}$ of the solution F, there exists a topological subgroup G_α^* of G^* such that

$$G^* = G_\alpha^* \cdot G \quad \text{and} \quad G_\alpha = G_\alpha^* \cap G.$$

The following theorem was proved in [47].

If P is semigroup of the positive elements of a linearly ordered Archimedean group G and $F: \Gamma \times P \to \Gamma$ is a solution of (1) then F is extendable to a solution

$F^*: \Gamma \times G \to \Gamma$ *of* (1) *iff*

(A) *for an arbitrary* α *from* Γ *the cardinality of the set* $E_\alpha(b) := \{x \in P: F(\alpha, x) = \beta\}$ *does not depend on* β *from* $F(\alpha, P)$,

(B) *the relation* R *defined on the set* $F(\Gamma, e)$ *in the following way*

$$\alpha R \beta \leftrightarrow \exists x \in P[F(\alpha, x) = \beta \text{ or } F(\beta, x) = \alpha]$$

is translative,

(C) $\forall x \in P: \quad F(\Gamma, e) \subset F(\Gamma, x)$.

From [58] we quote the following theorem:

The conjunction of the conditions (A) and (B) mentioned above is necessary and sufficient for the existence of a set $\Gamma^* \supset \Gamma$ and a solution $F^*: \Gamma^* \times G \to \Gamma^*$ of (1) which is an extension of F.

The following was proved in [38].

If P *is a subsemigroup of the group* G *such that* $G = P \cup P^{-1}$ *and* $F: \Gamma \times P \to \Gamma$ *is a solution of* (1) *then there exists a solution* F^*: $\Gamma \times G \to \Gamma$ *of* (1) *which is an extension of* F *iff for all* x *from* P *the function* $F(\cdot, x)_{|F(\Gamma, e)}$ *is a bijection onto* $F(\Gamma, e)$.

We quote also the following theorem from [34]:

If P *is a subsemigroup of the group* G *such that* $G = P \cup P^{-1}$ *and* $F: \Gamma \times P \to \Gamma$ *is a solution of* (1) *then there exists a set* $\Gamma^* \supset \Gamma$ *and a solution* $F^*: \Gamma^* \times G \to \Gamma^*$ *of* (1) *which is an extension of* F *iff for all* x *from* P *the function* $F(\cdot, x)_{|F(\Gamma, e)}$ *is an injection into* $F(\Gamma, e)$.

IV. Additional properties of solutions

Various applications of solutions of equation (1) require that they satisfy some additional properties on them. It turns out that possession of these properties by solutions, in the case where G is a group, can be characterized by the parameters $g(\alpha)$ and G_k (the so called stability subgroups of the solution) determining solutions of the form (2). I am going to give, following [56], definitions of some of these properties and after each of them an equivalent condition formulated in terms of the parameters $g(\alpha)$ and G_k.

(1) The identity condition: $\forall \alpha \in \Gamma$: $F(\alpha, e) = \alpha$,
—it is equivalent to $g(\alpha) = \alpha$.

(2) The transitivity: $\forall \alpha, \beta \in \Gamma \quad \exists x \in G$: $F(\alpha, x) = \beta$,
—it is equivalent to $g(\alpha) = \alpha$ and card $K = 1$.

(3) The quasi-transitivity: $\forall \alpha, \beta \in \Gamma \quad \exists x \in G$: $F(\alpha, x) = \beta$ or $F(\beta, x) = \alpha$,
—it is equivalent to $g(\alpha) = \alpha$ and card $K = 1$.

(4) The simple transitivity: the transitivity and the injectivity of $F(\alpha, \cdot\,)$ for each $x \in G$,
—it is equivalent to $g(\alpha) = \alpha$ and card $K = 1$ and $G_k = \{e\}$.

(5) The injectivity of $F(\,\cdot\,, x)$ for each $x \in G$,
—it is equivalent to $g(\alpha) = \alpha$.

(6) The effectivity: $\forall x \in G\ [\forall \alpha \in \Gamma$: $F(\alpha, x) = \alpha \to x = e]$,
—it is equivalent to $N := \bigcap_{k \in K} \bigcap_{a \in G} a^{-1}Ga = \{e\}$.

(7) G acts freely on Γ by F, i.e. $\forall x \in G\ [\exists \alpha \in \Gamma$: $F(\alpha, x) = \alpha \to x = e]$,
—it is equivalent to $G_k = \{e\}$ for each $k \in K$.

(8) F is disjoint at a point α_0, i.e.

$$\forall x, y \in G \qquad [F(\alpha_0, x) = F(\alpha_0, y) \to \forall \alpha \in \Gamma\colon F(\alpha, x) = F(\alpha, y)].$$

—it is equivalent to $G_{k_0} = N$ for k_0 such that $g(\alpha_0) \in \Gamma_{k_0}$.

(9) The commutativity of F, i.e.

$$\forall x, y \in G\ \forall \alpha \in \Gamma\colon F(F(\alpha, x), y) = F(F(\alpha, y), x),$$

—it is equivalent to the condition that the quotient group G/N is abelian.

(10) F is maximal, i.e.

$$\forall p\colon \Gamma \to \Gamma \qquad [\forall \alpha \in \Gamma\ \forall x \in G\colon p(F(\alpha, x)) = F(p(\alpha), x) \to$$
$$\exists x_0 \in G\ \forall \alpha \in \Gamma\colon p(\alpha) = F(\alpha, x_0)],$$

—it is equivalent to the following:

$$g(\alpha) = \alpha \quad \text{and} \quad G_k \neq G \text{ for each } k \in K$$

$$\text{and} \quad \bigcap_{k \in K} G_k a(k) \neq \varnothing \text{ for each function } a\colon K \to G.$$

(11) F is called parallelizable if there exists such a function $\tau\colon \Gamma \to G$ that $\tau(F(\alpha, x)) = \tau(\alpha)x^{-1}$,
—it is equivalent to the following: G is abelian and $G_k = \{e\}$ for each $k \in K$.

The above equivalences were proved in the papers [60], [8], [51], [53], [56]. In the case where G forms a structure more general than a group, the conditions in these equivalences are more complicated. The commutativity of a solution, in the case where G is e.g. an Ehresmann groupoid, is discussed in [30]. The papers [73] and [77] refer to commutativity as well. For a structure with a zero element, in place of G, some of these conditions are discussed in papers [60] and [51] and for the case where G is a semigroup of positive elements of an Archimedean group in [51], [36], [37]. In [37] I give, as an example, a condition equivalent to the effectivity of a solution of (1) in the case where G^+ is a semigroup of positive elements of an Archimedean group G and the solution satisfies the condition (4).

Referring to the above mentioned construction (C) we give an arbitrary invariant decomposition of the interval Δ_k:

(i) there exists an element u from the complement of G such that points of the interval $J_k := \{x \in \Delta_k : x \lll u\}$, where $\lll = \leq$ or $\lll = <$, form components of this decomposition ($J_k = \varnothing$ is not excluded),
(ii) the remaining components of the decomposition are restrictions of cosets G/G_k to the set $\Delta_k \backslash J_k$, where G_k is a subgroup of G.

Then a condition equivalent to the effectivity of a solution of (1) is the following:

$$N = \bigcap_{k \in K} G_k = \{e\} \quad \text{or} \quad G^+ \subset \bigcup_{k \in K} \{xy^{-1} : x, y \in J_k\}.$$

V. Translation equation on the products of structures

In the papers [40], [30], [61], [12], [48] a solution of the translation equation on the direct product or on the semi-direct product of some structures is expressed by solutions of this equation on the components of the product. The results of the paper [40] have been generalized in [12] and [48].

VI. Homomorphisms and solutions of the translation equation

A function $F: \Gamma \times G \to \Gamma$ is a solution of equation (1) iff the mapping $x \to F(\cdot, x)$ of the structure $(G, \cdot)$ into the family of functions mapping Γ into itself (with the composition as a binary operation), is a homomorphism. So, every solution of (1) is a homomorphism. The converse is also true. Each homomorphism h of the structure G into an associative structure S dictates a solution F of equation (1), with S in place of the fibre Γ. This homomorphism is defined by the following condition: $F(\alpha, x) = \alpha h(x)$. This means that a correlation exists between the solutions of (1) and homomorphisms, which has its consequences in the theory of the translation equation. There are some papers establishing also these consequences. To this area belong the papers [20], [22], [23], [26], [13], [45], [49]. Good examples are the results of the paper [21], which are consequences of the theorems concerning homomorphisms, given in [20] and [13].

VII. Set-valued iteration semigroups

Let A, X, Y and Z be nonempty sets with $A \subset X$ and let F be a set-valued function from X into Y, i.e. the values of F are subsets of Y. The image of A under F is the set

$$F(A) = \bigcup \{F(x): x \in A\}.$$

Moreover, if G is a set-valued function from Y into Z then one can define the composition $G(F)$ of F and G:

$$(G(F))(x) := G(F(x)).$$

A family $\{F^t, t > 0\}$ of set-valued functions F^t from X into X is said to be an iteration semigroup if the equation

$$F^s(F^t) = F^{s+t} \tag{8}$$

holds for every $s, t > 0$ and sets $F^s(x) \neq \emptyset$ for every $s > 0, x \in X$.

Let (X, ϱ) be a separable metric space, then the set $c(X)$ of all nonempty compact subsets of X is a separable metric space with respect to the Hausdorff metric.

An iteration semigroup of set-valued functions $F^t: X \to c(X)$ is said to be measurable (continuous) if the set-valued functions

$$t \to F^t(x) \qquad (x \in X) \tag{9}$$

are measurable (continuous) with respect to the Hausdorff metric.

If $F^t: X \to c(X)$ is an iteration semigroup of set-valued functions and X is a locally compact space and $F^t(x) \subset g(t, x)$ for $t > 0$, $x \in X$, where g is an upper semi-continuous compact-valued function, or X is compact or the $F^t(x)$ are lipschitzian in x, then the measurability of F^t implies its continuity (see [67] and see also [79] for single-valued iteration semigroups).

We say that the family F^t fulfils functional equation (8) almost everywhere (a.e.) if the set of all pairs (s, t) $(s, t > 0)$ for which equation (8) does not hold, is a set of Lebesgue measure zero.

Suppose that X is a nonempty and closed subset of a separable Banach space and assume that $F^t: X \to c(X)$ is a family of lipschitzian set-valued functions such that every function (9) is Lebesgue measurable for $x \in X$. If equation (8) is fulfilled for a.e. $s, t > 0$ in X, then there exists a continuous iteration semigroup $\{G^t, t > 0\}$ of lipschitzian set-valued functions on X such that $G^t = F^t$ a.e. on $(0, \infty)$ (see [69]).

Let X be a locally compact and separable metric space. If $F^t: X \to c(X)$ is an iteration semigroup of contractions such that the set-valued functions (9) are upper semi-continuous then there exists a minimal semi-continuous iteration semigroup $\{G^t, t > 0\}$ of contractions $G^t: X \to c(X)$ such that $G^t(x) \subset F^t(x)$ for $t > 0$ and $x \in X$. This iteration semigroup is continuous (see [68]).

Let X be a nonempty subset of a linear space and let $\phi: X \to R$. A set-valued function F from X into X is said to be ϕ-increasing if, for every $x, y \in X$ with $\phi(x) \le \phi(y)$ and every $w \in F(y)$, there exists a $u \in F(x)$ such that $\phi(u) \le \phi(w)$.

Let X be a non-empty convex subset of a normed linear space and let ϕ be a real strictly convex and lower semicontinuous function defined on X. If $\{F^t, t > 0\}$ is an iteration semigroup of ϕ-increasing set-valued functions from X into X with convex, compact values, then there exists an iteration semigroup $\{f^t, t > 0\}$ of single-valued functions from X into X such that $f^t(x) \in F^t(x)$ and $\phi(f^t(x)) = \inf\{\phi(y): y \in F^t(x)\}$ for every $x \in X$ and $t > 0$. If ϕ and the set-valued functions $x \to F^t(x)$ $(t \to F^t(x))$ are continuous then the functions $x \to f^t(x)$ $(t \to f^t(x))$ are continuous (see [67]).

We note that set-valued solutions of the generalized translation equation, analogous to (8), occur in the theory of abstract, nondeterministic automata (see [70] or [71]) as functions of passage.

VIII. Some open problems

(1) Comparison of various definitions of the local solution of the translation equation (e.g. in [11], [19], [55]) and establishing conditions for extendability of these local solutions to global ones (see [57]).

(2) Constructions of solutions of translation equation on various algebraic structures by means of independent parameters.

(3) Problem of stability of the translation equation.

Let (Γ, ϱ) be a metric space and let G be a group. Does there exist for each $\varepsilon > 0$ a $\delta > 0$ such that for each $H: \Gamma \times G \to \Gamma$ satisfying the condition

$$\forall \alpha \in \Gamma, \forall x, y \in G: \varrho(H(H(\alpha, x), y), H(\alpha, x \cdot y)) < \delta$$

there exists a solution F of the translation equation such that

$$\forall \alpha \in \Gamma \, \forall x \in G: \varrho(H(\alpha, x), F(\alpha, x)) < \varepsilon?$$

For the equation $F(F(\alpha)) = F(\alpha)$ the answer is positive.

Some open problems are formulated in [56].

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