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Cauchy's equation on Δ^+: further results

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Summary. In a previous paper [*Cauchy's equation on Δ^+*, Aequationes Math. *41* (1991), 192–211], we began the study of Cauchy's equation on Δ^+, the space of probability distribution functions of nonnegative random variables. In this paper we continue this study and extend our previous results to triangle functions of the form $\tau_{T,L}$, where T is a continuous Archimedean t-norm and L a binary operation on R^+, which is iseomorphic to a strict t-conorm. We again use a lattice theoretic approach, and introduce first a theorem on the powers and roots of certain elements of Δ^+ under $\tau_{T,L}$. Under certain additional restrictions we obtain a representation of sup-continuous solutions, similar to the one found in the first paper.

1. Introduction

In a previous paper [8] we began the study of Cauchy's equation on Δ^+, the space of probability distribution functions of nonnegative random variables. In this paper we continue this study and extend our previous results to triangle functions of the form $\tau_{T,L}$, where T is a continuous Archimedean t-norm and L a binary operation on R^+, which is isomorphic to a strict t-conorm. We again use a lattice theoretic approach, and introduce first a theorem on the powers and roots of certain elements of Δ^+ under $\tau_{T,L}$.

This paper is divided into four sections, Section 1 being this introduction. In the next section we introduce some additional notation and preliminary results beyond those of [8]. In Section 3 we prove a theorem about the existence of roots and powers of certain functions in Δ^+. Finally, in Section 4 we prove the main result, a representation of sup-continuous solutions of Cauchy's equation on Δ^+, as well as the corresponding explicit formulas for order automorphism solutions.

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2. Preliminaries

We assume that the reader is familiar with the notation and results from [8] (details can be found in [9]), and introduce here only a few additional items.

We will denote by $\mathscr{T}_s$ the set of strict t-norms and by $\mathscr{T}_A$, the set of continuous Archimedean t-norms (see [9, Definition 5.3.6]). Further, we say that a binary operation L on R^+ is a $*$-composition law if there is a t-norm T and a continuous, strictly increasing bijection ψ from R^+ into I, such that

$$L(u, v) = \psi^{-1}(1 - T(1 - \psi(u), 1 - \psi(v))), \qquad \text{for all } u, v \text{ in } R^+. \tag{2.1}$$

A $*$-composition law L is strict (continuous Archimedean), if the corresponding t-norm T is strict (continuous Archimedean), and we write L in $\mathscr{L}_s$, ($\mathscr{L}_A$).

Since every T in $\mathscr{T}_A$ admits a representation of the form (see [9, Theorem 5.5.2])

$$T(x, y) = g^{(-1)}(g(x) + g(y)), \tag{2.2}$$

(2.1) yields a corresponding representation for every L in $\mathscr{L}_A$:

$$L(u, v) = f^{(-1)}(f(u) + f(v)), \qquad \text{for all } u, v \text{ in } R^+, \tag{2.3}$$

where f is a continuous, strictly increasing function from R^+ into R^+ with $f(0) = 0$ and $f^{(-1)}$ is the pseudo-inverse of f (see [8, Section 2]). In the case where L is in $\mathscr{L}_s$, f is onto R^+ and $f^{(-1)} = f^{-1}$, the ordinary inverse of f.

We are interested in the triangle functions which are induced by a t-norm T and a $*$-composition law L as follows:

$$\tau_{T,L}(F, G)(x) = \sup_{L(u,v) = x} T(F(u), G(v)), \qquad \text{for all } x \text{ in } R^+. \tag{2.4}$$

We are now able to state our main goal: to find solutions of Cauchy's equation for $\tau_{T,L}$, i.e., to find functions $\varphi: \Delta^+ \to \Delta^+$ which satisfy

$$\varphi(\tau_{T,L}(F, G)) = \tau_{T,L}(\varphi(F), \varphi(G)), \qquad \text{for all } F, G \text{ in } \Delta^+. \tag{2.5}$$

3. Properties of $\tau_{T,L}$ and Δ^+

As in our previous paper we will use the lattice theoretic properties of Δ^+ (see [8, Section 3]), to solve (2.5). Again we let $\Delta_\delta^+ = \{\delta_{a,b} \mid a \in [0, \infty], b \in [0, 1]\}$, and note that, since

$$\tau_{T,L}(\delta_{a,b}, \delta_{c,d}) = \delta_{L(a,c),T(b,d)}, \qquad \text{for all } \delta_{a,b}, \delta_{c,d} \text{ in } \Delta_\delta^+, \tag{3.1}$$

$(\Delta_\delta^+, \tau_{T,L})$ is a subsemigroup of $(\Delta^+, \tau_{T,L})$ (compare [8, Lemma 3.4]). Furthermore, every F in Δ^+ can be written as a supremum of elements of Δ_δ^+:

$$F = \sup_{t > b_F} \delta_{t,F(t)}, \tag{3.2}$$

where $b_F = \sup\{t \in R^+ \mid F(t) = 0\}$. This is a refinement of [8, Lemma 3.5], since $\delta_{t,F(t)} \neq \varepsilon_\infty$ for $t > b_F$.

It was shown in [10, 7] that $\tau_{T,L}$ is sup-continuous (see [8, Definitions 2.5, 3.6]), and thus we have

LEMMA 3.1. *A function $\varphi: \Delta^+ \to \Delta^+$ is a sup-continuous solution of Cauchy's equation for $\tau_{T,L}$ on Δ^+ if and only if it is a solution on Δ_δ^+.*

We are now ready to turn to the question of powers and roots under $\tau_{T,L}$, where T is in $\mathscr{T}_A$ and L is in $\mathscr{L}_s$ with generators g and f, respectively. We let

$$\Delta_{T,L}^+ = \{F \in \Delta^+ \mid g \circ F \circ f^{-1} \text{ is convex on } (b_F, \infty)\}.$$

Since $b_{\delta_{a,b}} = a$, and $\delta_{a,b}$ is constant on (a, ∞), we have

$$\Delta_\delta^+ \subset \Delta_{T,L}^+.$$

The set $\Delta_{T,L}^+$ corresponds to the set of T-log-concave elements which R. Moynihan introduced in [5].

The following theorem is an extension of a result by B. Schweizer [9] (see also [5]).

THEOREM 3.2. *Let T be in $\mathscr{T}_A$ with generator g, L in $\mathscr{L}_s$ with generator f, and suppose F is in $\Delta_{T,L}^+ \setminus \{\varepsilon_\infty\}$. Using the abbreviation $\bar{F}$ for $g \circ F \circ f^{-1}$, for $\mu \geq 0$, let F^μ be defined by*

$$F^\mu(x) = g^{(-1)}(\mu \bar{F}(f(x)/\mu)), \qquad \textit{for } 0 < \mu < \infty, \tag{3.3}$$

$$F^0 = \lim_{\mu \to 0} F^\mu = \varepsilon_0, \tag{3.4}$$

$$F^\infty = \lim_{\mu \to \infty} F^\mu = \begin{cases} \varepsilon_0, & \textit{for } F = \varepsilon_0, \\ \varepsilon_\infty, & \textit{for } F \neq \varepsilon_0. \end{cases} \tag{3.5}$$

Then

(*a*) *F^μ is in $\Delta_{T,L}^+$, for all $\mu \geq 0$,*
(*b*) *$F^\mu \leq F^\nu$, whenever $\nu \leq \mu$,*
(*c*) *$\tau_{T,L}(F^\mu, F^\nu) = F^{\mu+\nu}$, for all $\mu, \nu \geq 0$,*
(*d*) *$(F^\mu)^\nu = F^{\mu\nu}$, if either $\mu \leq 1$ or $\mu > 1$ and $\nu \geq 1$.*

Proof. (a) For $\mu = 0$ and $\mu = \infty$, F^μ is ε_0 or ε_∞ and therefore in $\Delta^+_{T,L}$. Now let $0 < \mu < \infty$, then $\mu\bar{F}(x/\mu)$ is convex whenever

$$x > \sup\{t \mid g^{(-1)}(\mu g(F(f^{-1}(t/\mu)))) = 0\} = b_{F^\mu},$$

since it is a composition of convex functions. Hence F^μ is in $\Delta^+_{T,L}$ for all $\mu \geqslant 0$.

(b) Suppose $\nu \leqslant \mu$; since f^{-1} is increasing, F nondecreasing, and g and $g^{(-1)}$ are nonincreasing, we have, for any $x > 0$,

$$g^{(-1)}(\mu\bar{F}(f(x)/\mu)) \leqslant g^{(-1)}(\nu\bar{F}(f(x)/\nu)),$$

whence $F^\mu \leqslant F^\nu$.

(c) This is trivial if μ or ν is either 0 or ∞. Now suppose that $0 < \mu, \nu < \infty$ and let $x > 0$ be given. Then there are two cases to consider:

Case (1). $\tau_{T,L}(F^\mu, F^\nu)(x) = 0$. This is easily seen to hold if and only if $F^{\mu+\nu}(x) = 0$.

Case (2). $\tau_{T,L}(F^\mu, F^\nu)(x) > 0$. Then we have

$$\tau_{T,L}(F^\mu, F^\nu)(x) = \sup_{L(u,v)=x} g^{(-1)}[\mu\bar{F}(f(u)/\mu) + \nu\bar{F}(f(v)/\nu)].$$

Since $g^{(-1)}$ is nonincreasing and continuous, we have

$$\tau_{T,L}(F^\mu, F^\nu)(x) = g^{(-1)}(\inf_{L(u,v)=x} [u\bar{F}(f(u)/\mu) + \nu\bar{F}(f(v)/\nu)]),$$

and, since $\bar{F}$ is convex, we obtain

$$\begin{aligned}[\mu\bar{F}(f(u)/\mu) + \nu\bar{F}(f(v)/\nu)] &\geqslant (\mu+\nu)\bar{F}((f(u)+f(v))/(\mu+\nu)) \\ &= (\mu+\nu)\bar{F}(f(x)/(\mu+\nu))\end{aligned}$$

with equality holding when $f(u) = f(x)\mu/(\mu+\nu)$ and $f(v) = f(x)\nu/(\mu+\nu)$. Consequently,

$$\tau_{T,L}(F^\mu, F^\nu) = F^{\mu+\nu}.$$

(d) The case when $\mu \leqslant 1$ is settled by straightforward calculation. When $\mu > 1$ and $\nu \geqslant 1$ then either $\mu\bar{F}(f(x)/\mu\nu) \leqslant g(0)$, and a simple calculation yields the result, or $\mu\bar{F}(f(x)/\mu\nu) > g(0)$, in which case $(F^\mu)^\nu(x) = 0 = (F^\nu)^\mu$. □

We note that, for any positive integer n, F^n is just the n-fold $\tau_{T,L}$-product of F; and that, for strict T, (d) of the above theorem holds for any $\mu, \nu \geqslant 0$, since in this case F^μ never equals ε_∞.

We also have the following corollary regarding the uniqueness of roots:

COROLLARY 3.3. *Let T, L be as in Theorem 3.2 and let $\mu > 0$ be given. Then for any $\delta_{a,b}$ in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$, there exists a unique $\delta_{c,d}$ in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$ such that*

$$\delta_{a,b} = \delta_{c,d}^{\mu}, \tag{3.6}$$

if and only if $\delta_{a,b}^{1/\mu} \neq \varepsilon_\infty$. In this case $\delta_{c,d} = \delta_{f^{-1}(f(a)/\mu), g^{(-1)}(g(b)/\mu)}$.

Proof. Using (3.3), we get

$$\delta_{a,b} = \delta_{f^{-1}(\mu f(c)), g^{(-1)}(\mu g(d))},$$

and hence $a = f^{-1}(\mu f(c))$, $b = g^{(-1)}(\mu g(d))$. Obviously, $c = f^{-1}(f(a)/\mu)$, and, since, $b \neq 0$ we have $g(b)/\mu = g(d)$. Therefore d is uniquely determined if, and only if, $g(b)/\mu < g(0)$, which is the case if, and only if, $\delta_{a,b}^{1/\mu} \neq \varepsilon_\infty$. □

The next lemma is a direct consequence of (3.1) and of the above (recall that $\varepsilon_a = \delta_{a,1}$).

LEMMA 3.4. *Let $\delta_{a,b} \neq \varepsilon_\infty$, let T be in $\mathcal{T}_A$ with generator g, and let L be in $\mathcal{L}_s$ with generator f. Then, for any c in $(0, 1)$, $\delta_{a,b}$ admits the decomposition,*

$$\delta_{a,b} = \tau_{T,L}(\varepsilon_1^{f(a)/f(1)}, \delta_{0,c}^{g(b)/g(c)}). \tag{3.7}$$

4. Sup-continuous solutions

In this section we discuss the properties of sup-continuous solutions of Cauchy's equation for $\tau_{T,L}$, where T is in $\mathcal{T}_A$ and L is in $\mathcal{L}_s$, and give a representation of such functions. We conclude by giving explicit formulas for order automorphism solutions.

LEMMA 4.1. *Let T be in $\mathcal{T}_A$ and L in $\mathcal{L}_s$. Suppose that φ is a sup-continuous solution of Cauchy's equation for $\tau_{T,L}$ having the property that, if $\delta_{a,b}$ is in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$, then, for all positive integers n, $\varphi(\delta_{a,b}^{1/n})$ is in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$. Then for all $\mu \geqslant 0$,*

$$\varphi(\delta_{a,b}^{\mu}) = [\varphi(\delta_{a,b})]^{\mu}, \tag{4.1}$$

with the understanding that $\varepsilon_\infty^{\mu} = \varepsilon_\infty$.

Proof. Note first that, if $\mu = 0$, then $\delta_{a,b}^{\mu} = \varepsilon_0$ and $\varphi(\varepsilon_0) = \varepsilon_0$. Furthermore, if $\delta_{a,b}^{\mu} = \varepsilon_\infty$, then, since $\varphi(\varepsilon_\infty) = \varepsilon_\infty$ (this is the case for any solution of Cauchy's equation, for a proof see [8]), (4.1) holds for all $\mu > 0$. Now assume $\delta_{a,b}^{\mu} \neq \varepsilon_\infty$.

Let $\mu = n$ be a positive integer. Then this follows by induction from (2.5). Since $\delta_{a,b}^{1/n} \geqslant \delta_{a,b} \neq \varepsilon_\infty$, we have, by (d) of Theorem 3.2,

$$\varphi(\delta_{a,b}) = \varphi((\delta_{a,b}^{1/n})^n) = \varphi(\delta_{a,b}^{1/n})^n.$$

Now we can apply Corollary 3.3 and obtain

$$\varphi(\delta_{a,b}^{1/n}) = (\varphi(\delta_{a,b}))^{1/n},$$

and thus (4.1) holds for all rational $\mu > 0$. To establish the result for all $\mu > 0$, we let $\{r_n\}_{n=1}^{\infty}$ be a sequence of rational numbers with $\inf r_n = \mu$. Since $r_n \geqslant \mu$, we have, by (c) of Theorem 3.2,

$$\delta_{a,b}^{r_n} \leqslant \delta_{a,b}^{\mu};$$

but, if T is not strict, $\delta_{a,b}^{r_n}$ may equal ε_∞ for some r_n. The fact that

$$\delta_{a,b}^{r_n} = \delta_{f^{-1}(r_n f(a)), g^{(-1)}(r_n g(b))}$$

and that g and $g^{(-1)}$ are continuous, implies that there exists a subsequence $\{q_m\}$ of $\{r_n\}$, such that for all m,

$$\delta_{a,b}^{q_m} \neq \varepsilon_\infty.$$

Thus we have

$$\sup_{q_m} \delta_{a,b}^{q_m} = \delta_{a,b}^{\mu}.$$

Using the sup-continuity of φ, we obtain that (4.1) holds for all $\mu > 0$. □

Note that (4.1) implies, if $\varphi(\delta_{a,b}) = \delta_{c,d}$, that we necessarily have $d \leqslant b$.

We are now able to prove the main results of this paper:

THEOREM 4.2. *Let T be in $\mathcal{T}_A$ with generator g and L in $\mathcal{L}_s$ with generator f. Suppose that φ is a sup-continuous solution of Cauchy's equation for $\tau_{T,L}$, having the property that, for some c in $(0, 1)$ and all positive integers n, $\varphi(\delta_{0,c}^{1/n})$ and $\varphi(\varepsilon_1^{1/n})$ are in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$. Then, for all F in Δ^+,*

$$\varphi(F) = \sup_{t > b_F} \tau_{T,L}([\varphi(\varepsilon_1)]^{lf(t)}, [\varphi(\delta_{0,c})]^{kg(F(t))}), \tag{4.2}$$

where $l = 1/f(1)$ and $k = 1/g(c)$.

Proof. By (3.2) and Lemma 3.4 we have that

$$F = \sup_{t > b_F} \delta_{t,F(t)} = \sup_{t > b_F} \tau_{T,L}(\varepsilon_1^{f(t)/f(1)}, \delta_{0,c}^{g(F(t))/g(c)}),$$

whence the sup-continuity of φ, the fact that φ is a solution of Cauchy's equation for $\tau_{T,L}$, and (4.1) yield (4.2). □

The converse of Theorem 4.2 is also true. To establish it we will need the following lattice-theoretic lemma (see, e.g. [3]).

LEMMA 4.3. *Let $\mathcal{M}$ be a complete lattice and let $\{X_{t,\beta} \mid \beta \in B, t \in I_\beta\}$ be a collection of elements of $\mathcal{M}$, where B is an index set and $\{I_\beta\}$ is a family of index sets indexed by B. Then, letting $I = \bigcup_{\beta \in B} I_\beta$, we have*

$$\sup_{t \in I} \sup_{\beta \in B} X_{t,\beta} = \sup_{\beta \in B} \sup_{t \in I_\beta} X_{t,\beta}. \tag{4.3}$$

Proof. An element u is an upper bound of $\{X_{t,\beta} \mid \beta \in B\}$ if and only if u is an upper bound of $\sup_{t \in I_\beta} X_{t,\beta}$. Hence u is an upper bound for every β in B of $\sup_{t \in I_\beta} X_{t,\beta}$ if and only if u is an upper bound of $\sup_{\beta \in B} \sup_{t \in I_\beta} X_{t,\beta}$. Therefore we have

$$\sup_{\beta \in B} \sup_{t \in I_\beta} X_{t,\beta} = \sup\{X_{t,\beta} \mid \beta \in B, t \in I_\beta\}. \tag{4.4}$$

On the other hand, an element u is an upper bound of $\{X_{t,\beta} \mid \beta \in B\}$ if and only if u is an upper bound of $\sup_{\beta \in B} X_{t,\beta}$, where t is in $I = \bigcup_{\beta \in B} I_\beta$. This in turn holds if and only if u is an upper bound of $\sup_{t \in I} \sup_{\beta \in B} X_{t,\beta}$, which implies that

$$\sup_{t \in I} \sup_{\beta \in B} X_{t,\beta} = \sup\{X_{t,\beta} \mid \beta \in B, t \in I_\beta\}. \tag{4.5}$$

Putting (4.4) and (4.5) together yields the result. □

THEOREM 4.4. *Let T be in $\mathcal{T}_A$ with generator g and L in $\mathcal{L}_s$ with generator f. Let c in $(0, 1)$ and $\delta_{a,b}$, $\delta_{d,e}$ in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$ be given (where $e \leqslant c$, if T is non-strict); and define $\varphi: \Delta^+ \to \Delta^+$ by*

$$\varphi(F) = \sup_{t > b_F} \tau_{T,L}(\delta_{a,b}^{lf(t)}, \delta_{d,e}^{kg(F(t))}), \tag{4.6}$$

where $l = 1/f(1)$ and $k = 1/g(c)$. Then φ is a sup-continuous solution of Cauchy's equation for $\tau_{T,L}$. Moreover, $\delta_{a,b} = \varphi(\varepsilon_1)$, $\delta_{d,e} = \varphi(\delta_{0,c})$ and for all positive integers n, $\varphi(\varepsilon_1^{1/n})$ and $\varphi(\delta_{0,c}^{1/n})$ are in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$.

Proof. We first observe that, for $F = \varepsilon_\infty$, we have $b_F = \infty$, whence the supremum in (4.6) is over the empty set. This implies that $\varphi(\varepsilon_\infty) = \varepsilon_\infty$. Now let G

and H be in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$ and suppose $\delta_{a,b} \neq \varepsilon_\infty$. Then we have, with $l = 1/f(1)$ and $k = 1/g(c)$,

$$\tau_{T,L}(G^{lf(t)}, H^{kg(\delta_{a,b}(t))}) = \begin{cases} \tau_{T,L}(G^{lf(t)}, H^{kg(b)}), & \text{for } a < t < \infty, \\ \varepsilon_\infty, & \text{for } t = \infty. \end{cases}$$

Therefore, letting $F = \delta_{a,b}$ in (4.6), we obtain,

$$\varphi(\delta_{a,b}) = \tau_{T,L}(G^{lf(a)}, H^{kg(b)}). \tag{4.7}$$

Using (4.7) we have, in particular, that for all positive integers n,

$$\varphi(\delta_{0,c}^{1/n}) = \varphi(\delta_{0,g^{-1}(g(c)/n)}) = \tau_{T,L}(G^0, H^{1/n}) = \tau_{T,L}(\varepsilon_0, H^{1/n}) = H^{1/n},$$

and, since $g(1) = 0$,

$$\varphi(\varepsilon_1^{1/n}) = \varphi(\delta_{f^{-1}(f(1)/n),1}) = \tau_{T,L}(G^{1/n}, \varepsilon_0) = G^{1/n}.$$

Hence, using Theorem 3.2, we have that $\varphi(\varepsilon_1^{1/n})$ and $\varphi(\delta_{0,c}^{1/n})$ are in $\Delta_\delta^+ \setminus \{\varepsilon_\infty\}$ and for $n = 1$ this yields $G = \varphi(\varepsilon_1)$ and $H = \varphi(\delta_{0,c})$.

To show that φ is sup-continuous, we first note that

$$\begin{aligned} \sup_{t > b_F} \varphi(\delta_{t,F(t)}) &= \sup_{t > b_F} \tau_{T,L}(G^{lf(t)}, H^{kg(F(t))}) \\ &= \varphi(F) = \varphi(\sup_{t > b_F} \delta_{t,F(t)}). \end{aligned}$$

Now let $F = \sup_{\beta \in B} F_\beta$, where F_β is in Δ^+ for all β in some index set B. (Recall that this supremum is pointwise, i.e., $F(x) = \sup_{\beta \in B} F_\beta(x)$.) Using the facts that g and f are continuous, that H is left continuous, inequality (d) of Theorem 3.2 and the fact that g is decreasing, we obtain

$$\sup_{\beta \in B} H^{kg(F_\beta)} = H^{\inf_{\beta \in B} kg(F_\beta)} = H^{kg(\sup_{\beta \in B} F_\beta)}.$$

Hence, using the abbreviation $\sup_\beta$ for $\sup_{\beta \in B}$, we have on the one hand that

$$\begin{aligned} \varphi(F) &= \varphi(\sup_\beta F_\beta) \\ &= \sup_{t > b_F} \sup_\beta \tau_{T,L}(G^{lf(t)}, H^{kg(F_\beta)}); \end{aligned} \tag{4.8}$$

and, writing b_β for b_{F_β}, on the other hand, that

$$\sup_\beta \varphi(F_\beta) = \sup_\beta \sup_{t > b_\beta} \tau_{T,L}(G^{lf(t)}, H^{kg(F_\beta)}). \tag{4.9}$$

In order to apply Lemma 4.3 we note that, since $F(x) = \sup_\beta F_\beta(x)$, we have $b_F = \inf_\beta b_\beta$ and, by writing $t \in (b_\beta, \infty)$ instead of $t > b_\beta$, we obtain that $t \in (b_F, \infty) = \bigcup_\beta (b_\beta, \infty)$. Thus by Lemma 4.3, equations (4.8) and (4.9) are equal and φ is sup-continuous.

It remains to show that φ satisfies Cauchy's equation for $\tau_{T,L}$. Using (4.7), (3.7) and the fact that $\tau_{T,L}$ is commutative and associative, we have that, for all $\delta_{a,c}$ and $\delta_{b,d}$ in Δ_δ^+,

$$\begin{aligned}\tau_{T,L}(\varphi(\delta_{a,c}), \varphi(\delta_{b,d})) &= \tau_{T,L}(\tau_{T,L}(G^{kf(a)}, H^{lg(b)}), \tau_{T,L}(G^{kf(b)}, H^{lg(d)}))\\ &= \tau_{T,L}(\tau_{T,L}(G^{kf(a)}, G^{kf(b)}), \tau_{T,L}(H^{lg(c)}, H^{lg(d)}))\\ &= \tau_{T,L}(G^{kf(L(a,b))}, H^{lg(T(c,d))})\\ &= \varphi(\delta_{L(a,b),T(c,d)})\\ &= \varphi(\tau_{T,L}(\delta_{a,c}, \delta_{b,d})),\end{aligned}$$

whence the conclusion follows from Lemma 3.1. □

We pause here to consider the case when T is actually in $\mathcal{T}_s$. In this case we can replace the set Δ_δ^+ by the larger set $\Delta_{T,L}^+$, wherever it occurs in Corollaries 3.3 and 4.1, Lemma 3.4, and Theorems 4.2 and 4.4.

Finally, we consider functions on Δ^+ which are order-preserving bijections whose inverses also preserve order, the so-called order automorphisms. These were characterized by R. C. Powers in [6]; we state his main result here:

THEOREM 4.5. *A mapping φ is an order automorphism of Δ^+, if and only if, for all F in Δ^+, either*

$$\varphi(F) = \theta \circ F \circ \gamma, \tag{4.10}$$

where θ is a continuous, strictly increasing bijection on I and γ is a continuous, strictly increasing bijection on R^+, or

$$\varphi(F) = \alpha \circ F^\vee \circ \beta, \tag{4.11}$$

where α and β are continuous, strictly decreasing bijections from R^+ to I and $F^\vee$ is the right-continuous quasi-inverse of F which is given by

$$F^\vee(y) = \begin{cases} 0, & \text{for } y = 0,\\ \inf\{x \mid F(x) > y\}, & \text{for } 0 < y < 1,\\ \infty, & \text{for } y = 1.\end{cases} \tag{4.12}$$

From this it is easily seen that order automorphisms of the type (4.10) have to map $\delta_{0,c}$ onto $\delta_{0,a}$ for some a in $(0, 1)$, and ε_1 onto ε_b for some b in $(0, \infty)$. Similarly, order automorphisms of type (4.11) map $\delta_{0,c}$ onto ε_b for some b in $(0, \infty)$, and ε_1 onto $\delta_{0,a}$ for some a in $(0, 1)$. Note also that every order automorphism is sup-continuous. Using these facts and formula (3.3) of Theorem 3.2, we obtain the following.

COROLLARY 4.6. *Let φ be given by* (4.10). *Then φ satisfies Cauchy's equation for $\tau_{T,L}$, where*

(*a*) *T is in $\mathscr{T}_s$, with generator g, and L is in $\mathscr{L}_s$, with generator f, if and only if there exist $k, l > 0$ such that for all x in R^+ and F in Δ^+,*

$$(\varphi(F))(x) = g^{-1}(kg(F(f^{-1}(lf(x))))), \tag{4.13}$$

(*b*) *T is in $\mathscr{T}_A \backslash \mathscr{T}_s$ and L is in $\mathscr{L}_s$ with generator f, if and only if there exists an $l > 0$ such that for all x in R^+ and F in Δ^+,*

$$(\varphi(F))(x) = F(f^{-1}(lf(x))). \tag{4.14}$$

Proof. Suppose φ is of type (4.10) and that φ satisfies Cauchy's equation for $\tau_{T,L}$, then $\varphi(\delta_{0,c}) = \delta_{0,a}$ and $\varphi(\varepsilon_1) = \varepsilon_b$ and, by (4.2) of Theorem 4.1, we have that

$$\begin{aligned}
\varphi(F) &= \sup_{t > b_F} \tau_{T,L}(\varepsilon_b^{f(t)/f(1)}, \delta_{0,a}^{g(F(t))/g(c)}) \\
&= \sup_{t > b_F} \tau_{T,L}(\varepsilon_{f^{-1}(f(b)f(t)/f(1))}, \delta_{0,g^{(-1)}(g(a)g(F(t))/g(c))}) \\
&= \sup_{t > b_F} \delta_{f^{-1}(f(b)f(t)/f(1)),\, g^{(-1)}(g(a)g(F(t))/g(c))} \\
&= \sup_{t > b_F} \theta \circ \delta_{t,F(t)} \circ \gamma = \theta \circ F \circ \gamma,
\end{aligned}$$

where

$$\theta(t) = g^{(-1)}\left(\frac{g(a)}{g(c)} g(t)\right) \qquad \text{for all } t \text{ in } I$$

and

$$\gamma(x) = f^{-1}\left(\frac{f(1)}{f(b)} f(x)\right), \qquad \text{for all } x \text{ in } R^+.$$

It is easily checked that γ is always a continuous, strictly increasing bijection. If T is in $\mathscr{T}_s$, then θ is also a continuous, strictly increasing bijection for any choice of a in $(0, 1)$, and we have $k = g(a)/g(c)$ and $l = f(1)/f(b)$.

In the case where T is non strict, we need, however, that $a \leqslant c$. But, if $a < c$, then we have $g(a)/g(c) < 1$ and θ is not a bijection. Thus we need that $a = c$, therefore $\theta =$ identity on I, and thus $l = f(1)/f(b)$.

Now suppose T is in $\mathscr{T}_s$ and φ is given by (4.13), for some $k, l > 0$. Then it is easily checked that φ is an order automorphism of type (4.10), furthermore

$$\varphi(\delta_{a,b}) = \delta_{f^{-1}(f(a)/l), g^{-1}(kg(b))},$$

thus it maps Δ_δ^+ onto Δ_δ^+. In particular, we have

$$\varphi(\delta_{0,c}) = \delta_{0,g^{-1}(kg(c))} \quad \text{and} \quad \varphi(\varepsilon_1) = \varepsilon_{f^{-1}(f(1)/l)}. \tag{4.15}$$

Thus, if we define $\tilde{\varphi}(F)$ via (4.5), we have a sup-continuous solution of Cauchy's equation for $\tau_{T,L}$. It remains to show that $\tilde{\varphi} = \varphi$. By Lemma 3.1 it suffices to show equality on Δ_δ^+. Using (4.15) and (4.17), we have

$$\begin{aligned}
\tilde{\varphi}(\delta_{a,b}) &= \sup_{t > b_F} \tau_{T,L}\left(\varepsilon_{f^{-1}(f(1)/l)}^{f(t)/f(1)}, \delta_{0,g^{-1}(kg(c))}^{g(\delta_{a,b}(t))/g(c)}\right) \\
&= \tau_{T,L}\left(\varepsilon_{f^{-1}(f(a)/l)}, \delta_{0,g^{-1}(kg(b))}\right) \\
&= \delta_{f^{-1}(f(a)/l), g^{-1}(kg(b))},
\end{aligned}$$

thus $\tilde{\varphi} = \varphi$. In case (b) we have, using (4.14), that $\varphi(\delta_{0,c}) = \delta_{0,c}$ and $\varphi(\varepsilon_1) = \varepsilon_{f^{-1}(f(1)/l)}$, and an argument similar to the above yields the result. □

COROLLARY 4.7. *Let φ given by (4.11) and let T be in $\mathscr{T}_A$ and L in $\mathscr{L}_s$, with generators g and f, respectively. Then φ satisfies Cauchy's equation for $\tau_{T,L}$ if and only if T is in $\mathscr{T}_s$ and there exists a $k > 0$ such that, for all x in R^+ and all F in Δ^+,*

$$(\varphi(F))(x) = g^{-1}(f(F^\vee(g^{-1}(kf(x))))). \tag{4.16}$$

Suppose φ is of type (4.11) and that φ is a solution of Cauchy's equation for $\tau_{T,L}$, then $\varphi(\varepsilon_1) = \delta_{0,a}$ and $\varphi(\delta_{0,c}) = \varepsilon_b$ for some a in $(0, 1)$ and b in $(0, \infty)$ and, by (4.2),

$$\begin{aligned}
\varphi(F) &= \sup_{t > b_F} \tau_{T,L}\left(\delta_{0,a}^{f(t)/f(1)}, \varepsilon_b^{g(F(t))/g(c)}\right) \\
&= \sup_{t > b_F} \tau_{T,L}\left(\delta_{0,g^{(-1)}(f(t)g(a)/g(c))}, \varepsilon_{f^{-1}(g(F(t))f(b)/g(c))}\right) \\
&= \sup_{t > b_F} \delta_{f^{-1}(g(F(t))f(b)/g(c)), g^{(-1)}(f(t)g(a)/g(c))} \\
&= \sup_{t > b_F} \alpha \circ \delta_{t,F(t)^\vee} \circ \beta = \alpha \circ F^\vee \circ \beta,
\end{aligned}$$

where $\alpha(t) = g^{(-1)}(g(a)f(t)/f(a))$ and $\beta(t) = g^{(-1)}(g(c)f(t)/f(1))$. But α, β are one-to-one only if $g^{(-1)} = g^{-1}$, i.e., only if T is in $\mathcal{T}_s$. Since any two generators of L (and any two of T) differ only by a constant multiple (see [8, Theorem 2.2]), we can choose f such that $g(a)/f(1) = 1$. Finally, letting $k = f(b)/g(c)$, we obtain (4.16).

For the converse, assume φ is given by (4.16). This defines an order automorphism only if T is strict, since otherwise $g^{(-1)}f$ is not a strictly decreasing bijection. A simple calculation shows that

$$\varphi(\delta_{a,b}) = \delta_{f^{-1}(g(b)/k),g^{-1}(f(a))},$$

and hence

$$\varphi(\delta_{0,c}) = \varepsilon_{f^{-1}(g(c)/k)} \quad \text{and} \quad \varphi(\varepsilon_1) = \delta_{0,g^{-1}(f(1))}. \tag{4.17}$$

As in the proof of the previous corollary, we use (4.6) of Theorem 4.4 to define a function $\tilde{\varphi}$ which is a solution of Cauchy's equation for $\tau_{T,L}$. This yields

$$\begin{aligned}\tilde{\varphi}(\delta_{a,b}) &= \sup_{t > b_F} \tau_{T,L}(\delta^{f(t)/f(1)}_{0,g^{-1}(f(1))}, \varepsilon^{g(\delta_{a,b}(t))/g(c)}_{f^{-1}(g(c)/k)}) \\ &= \tau_{T,L}(\delta_{0,g^{-1}(f(a))}, \varepsilon_{f^{-1}(g(b)/k)}) \\ &= \delta_{f^{-1}(g(b)/k),g^{-1}(f(a))}.\end{aligned}$$

Therefore $\tilde{\varphi}$ and φ agree on Δ^+_δ and hence on all of Δ^+. □

In conclusion, we note that, for strict T and for $L(u, v) = Sum(u, v) = u + v$ (where the generator is the identity function), the formulas in Theorems 4.2 and 4.4, as well as in Corollaries 4.6 and 4.7, are exactly the ones obtained in [8]. Furthermore, for strict T and for $L(u, v) = f^{-1}(f(u) + f(v))$, the order automorphism Γ of Δ^+, defined by

$$\Gamma(F) = F \circ f^{-1}, \qquad \text{for all } F \text{ in } \Delta^+,$$

let us reduce the problem to that studied in [8]. Since

$$\tau_{T,L}(F, G) = \Gamma^{-1}(\tau_T(\Gamma(F), \Gamma(G))),$$

we have that a function φ is a solution of Cauchy's equation for $\tau_{T,L}$ if and only if $\Gamma\varphi\Gamma^{-1}$ is a solution of Cauchy's equation for τ_T.

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