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The combinatorially regular polyhedra of index 2

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Dedicated to Professor Otto Haupt with best wishes on his 100th birthday.

Summary. We investigate polyhedral realizations of regular maps with self-intersections in E^3, whose symmetry group is a subgroup of index 2 in their automorphism group. We show that there are exactly 5 such polyhedra. The polyhedral sets have been more or less known for about 100 years; but the fact that they are realizations of regular maps is new in at least one case, a self-dual icosahedron of genus 11. Our polyhedra are closely related to the 5 regular compounds, which can be interpreted as discontinuous polyhedral realizations of regular maps.

1. Introduction and History

Fifty years ago Coxeter discovered (cf. [3, p. 141]), that two of the 53 uniform polyhedra (cf. [6]) are polyhedral realizations of Kepler-Poinsot-type (for definitions compare Section 2) of the regular maps $\{5,4\}_6$ and $\{5,6\}_4$ (cf. [7, p. 139]). These polyhedra were discovered more than 100 years ago independently by Hess (1878), Pitsch (1881) and Badoureau (1881); (for the history see [4, §6.4], for the figures see [6, figs. 45 and 53], or [4 fig. 6.4a]. Clearly these authors were merely interested in the metrical shape and in the symmetry properties of these polyhedra. The interpretation of the Kepler-Poinsot solids and other self-intersecting polyhedra as Riemann surfaces or as regular maps was suggested about 1930 by Threlfall (cf. [18, pp. 19–21]) and DuVal (cf. [4, p. 116]).

The regular map $\{5,4\}_6$ was discovered first by Gordan [9] (and several times rediscovered). Gordan was interested in it as a Riemann surface and algebraic curve.

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* The author was born on March 5, 1937; so exactly half a century after Otto Haupt.

The regular map $\{5,6\}_4$, which is the Petrie-dual (cf. Section 2 or [7, p. 112] of Gordan's $\{5,4\}_6$, was found by Coxeter [3].

By a recent result of Grünbaum and Shephard [10] the dual of a Kepler-Poinsot-type polyhedron does always exist. So it is easy to show that the dual maps $\{4,5\}_6$ and $\{6,5\}_4$ can be realized as Kepler-Poinsot-type polyhedra, namely the first one as the well-known small stellated triacontahedron (cf. [4, fig. 6.4c]). The realization of $\{6,5\}_4$ is also easy to describe: Its visible part is one of the 59 stellated icosahedra, namely De_2f_2 on plate XI in [5]. The invisible parts are simple added by completing the 20 faces to hexagons. As far as we know the last two polyhedra were first considered as realizations of regular maps in [16].

These 4 Kepler-Poinsot-type polyhedra all have the full icosahedral symmetry group $C_2 \times A_5$, which is a subgroup of index 2 of the automorphism group $C_2 \times S_5$ of the underlying regular map. We call a polyhedral realization of a regular map with this property a (combinatorially) regular polyhedron of index 2.

Because the 5 Platonic solids and the 4 Kepler-Poinsot-solids can be interpreted as regular polyhedra of index 1, the above mentioned 4 polyhedra are closely related to the classical regular polyhedra. So the natural question is: Are there further regular polyhedra of index 2?

The following theorem shows that there is exactly one more regular map which can be realized by a combinatorially regular polyhedron of index 2, namely by a self-dual icosahedron of genus 11. Our main result is the following.

THEOREM: *There are exactly* 5 (*orientable*) *regular maps which can be realized by a combinatorially regular polyhedron of index* 2, *namely the dual maps* $\{4,5\}_6$ *and* $\{5,4\}_6$ *of genus* 4, *the dual maps* $\{6,5\}_4$ *and* $\{5,6\}_4$ *of genus* 9 *and the self-dual map* $\{6,6\}_{6'}$ *of genus* 11. *The realizations are* (*up to dilatations*) *metrically unique.*

REMARK: The notations of the maps are given in Section 2; for the first 4 maps it is the usual notation, whereas for the fifth map the ′ means that this new map was found in a different way (cf. Sections 2 and 4).

In Section 2 we give the necessary definitions and notations. In Section 3 we show that the regular compounds can be interpreted as discontinuous polyhedral realizations of regular maps. In Section 4 we introduce the new polyhedron. In Section 5 (the main part of the paper) we prove the theorem. In Section 6 we describe the flag-diagram.

2. Definitions and Notation

According to [7] a *map* M is a decomposition of a closed real 2-manifold into f_2 simply-connected, non-overlapping regions of M called *faces* by means of f_1 arcs called *edges* of M. The f_0 intersections of the edges are the *vertices* of M. The triplet $f = (f_0, f_1, f_2)$ is called the f-vector of M. A *flag* of M is a set consisting of one vertex, one edge containing this vertex, and one face containing this edge.

M is said to be of type $\{p,q\}$ if all its faces are topological p-gons, q meeting at each vertex.

To every map M there corresponds the dual map M^* having f_0 faces, one surrounding each vertex of M, f_1 edges, one crossing each edge of M, and f_2 vertices, one contained in the interior of each face of M.

In [7] a map is called regular if its (combinatorial) automorphism group $A(M)$ contains two particular automorphisms: one, say ρ, which cyclicly permutes the edges that are successive sides of one face, and another, say σ, which cyclicly permutes the successive edges meeting at one vertex of this face (these are the automorphisms R and S of [7]). Thanks to the connectivity property of maps, for a regular M its group $A(M)$ is transitive on the vertices, on the edges, and on the faces, but need not be flag-transitive. Now, $A(M)$ is flag-transitive if and only if it contains an automorphism which interchanges the vertices of some edge without interchanging the two faces sharing this edge. These are exactly the so-called reflexible regular maps of [7] in contrast to the irreflexible (or chiral) regular maps, where $A(M)$ is not flag-transitive.

According to a more modern definition requiring flag-transitivity for combinatorial regularity we will use the term "regular map" without further qualifications to mean a reflexible regular map in the sense of [7], and also of [11–16,19,20]. From the description it is clear that the order of the automorphism group is $4f_1$. A Petrie-polygon of a map is a "zigzag" along its edges such that every two but no three successive edges of the polygon are edges of a single face. Petrie-polygons do exist for every regular map M, and the various Petrie-polygons are all alike because of the flag-transitivity of the group. This justifies speaking of *the* Petrie-polygon of M.

Every regular map can be derived from its (unique) universal cover $\{p,q\}$ by making suitable identifications [7]. An effective method for the construction of regular maps of type $\{p,q\}$ is the identification of those pairs of vertices of the regular tessellation $\{p,q\}$ which are separated by r steps along a Petrie-polygon (cf. [7, chapter 8.6]). For suitable values of r this identification process gives in fact a regular map denoted by $\{p,q\}_r$. If one replaces the faces of a regular map M by the various specimens of the Petrie-polygon while leaving the vertices and edges of M unchanged, one gets a new map $\{r,q\}_p$ with the same group, but in general on a different surface (cf. [7, p. 112]).

This observation shows that there is one common automorphism group of the six related regular maps

$$\{p,q\}_r,\ \{q,p\}_r,\ \{r,q\}_p,\ \{q,r\}_p,\ \{r,p\}_q,\ \{p,r\}_q.$$

These are called the direct derivates of each other [7,20]. Some of these maps may coincide, if self-duality or self-Petrie-duality occurs (for more details see [7, p. 112] and [20, p. 13 ff]). We mention finally that the underlying 2-manifold for the map $\{p,q\}_r$ is orientable or non-orientable according to whether r is even or not.

In our paper we only consider oriented maps and their realizations. For oriented maps we have the well-known Euler-Poincaré relation

$$f_0 - f_1 + f_2 = 2 - 2g, \tag{1}$$

where g denotes the genus of the manifold.

For a map of type (p,q), (but not necessarily regular) of genus g we also use the short notation $\{p,q;g\}$.

From the regularity follows the simple condition

$$pf_0 = 2f_1 = qf_2 \tag{2}$$

and in the hyperbolic case ($g \geq 2$):

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \tag{3}$$

There are various possibilities of geometric realizations of a given regular map. For certain reasons, but mainly from an intuitive geometrical point of view, we are only interested in polyhedral realizations in the Euclidean 3-space E^3 with or without self-intersections, according to the 5 Platonic solids and the 4 Kepler-Poinsot-solids.

To be exact, by a polyhedron we mean a closed compact 2-manifold in E^3, which can be expressed as a finite union of plane polygonal regions. If these regions are such that no two adjacent ones are coplanar, they are called the faces of P; the vertices and edges of P are the vertices and edges of the faces of P, respectively. This way the set of all vertices, edges and faces of P becomes the set of all vertices, edges and faces of a map on the underlying surface.

We also consider polyhedra with self-intersections, as e.g. the Kepler-Poinsot polyhedra; in particular also with self-intersecting faces (e.g. the pentagram). One reason for this is the existence of the Kepler-Poinsot polyhedra, another one is that regular polyhedra without self-intersections seem to be very rare; in particular there are none of index 2. But the main reason is that for polyhedra without self-intersections the dual may not exist, whereas, by a recent result of Grünbaum and Shephard [10], the dual of a polyhedron with self-intersections does always exist. In the proof of the theorem we do not always distinguish between polyhedra with or without self-intersections. We only distinguish in the case where it is necessary or to avoid confusion.

If the map is regular we call the polyhedron combinatorially regular, or regular with *hidden symmetries*. The definition makes sense because these polyhedra cannot have a flag-transitive symmetry group (except for the 5 Platonic solids, and, with self-intersections, for the 4 Kepler-Poinsot solids). These polyhedra can only have a flag-transitive combinatorial automorphism group. The symmetry group of the polyhedron is a subgroup of the automorphism group of the underlying map. The index of this subgroup is called the *index* of the polyhedral realization. So combinatorially regular polyhedra of index 2 are, in a sense, the closest analogues of the Platonic solids and the Kepler-Poinsot solids.

We emphasize that our polyhedral realizations with or without self-intersections do not include realizations of regular maps, which have two faces or two vertices joined by more than one edge. So, in particular, this rules out the dihedra (and hosohedra) and in particular branched covers of regular polyhedra (e.g. polyhedral realizations of hyperelliptic Riemann surfaces).

The condition that every pair of vertices or every pair of faces are incident with at most one common edge leads to $f_1 \leq \frac{1}{2}f_0(f_0 - 1), f_1 \leq \frac{1}{2}f_2(f_2 - 1)$, and with (1) to

$$f_0 \geq \frac{1}{2}(5 + \sqrt{9 + 16g}), f_2 \geq \frac{1}{2}(5 + \sqrt{9 + 16g}) \tag{4}$$

3. The Regular Compounds

The 5 regular compounds (cf. [4], p. 47) are compounds of equal Platonic polyhedra and are clearly no polyhedra. But in our context they occur in a natural way and can be interpreted as discontinuous combinatorially regular polyhedra. This is described in detail for Kepler's stella octangula [20], which can be interpreted as a discontinuous polyhedral realization of the toroidal maps $\{3,6\}_4$ and $\{6,3\}_4$. We denote these realizations by $C_{3,6}$ and $C_{6,3}$. For the other regular compounds we give a short survey to show their relationship to our polyhedra of index 2.

The regular compound consisting of 5 cubes has the 20 vertices of the (regular) dodecahedron and in each of these 20 vertices lie 2 of the 3-valent vertices of a cube. So each of these 20 vertices can be interpreted as a 6-valent vertex, where the hexagonal vertex-figure splits into 2 triangles. Hence this compound can be interpreted as a discontinuous polyhedron with $p = 4$, $q = 6$ and $f = (20,60,30)$, i.e. the 30 squares of the 5 cubes. For brevity we denote it by $C_{4,6}$.

In the same way the regular compound consisting of 5 octahedra can be interpreted, where the role of vertices and faces changes. So here we have a discontinuous polyhedron with $p = 6$, $q = 4$ and $f = (30,60,20)$ which we denote by $C_{6,4}$.

Finally the regular compound consisting of 10 tetrahedra has (with this interpretation) 20 hexagons, which split into two triangles each, and 20 vertices which split into two 3-valent vertices each. We denote it by $C_{6,6}$.

To complete the list we add two further compounds:

(*a*) The compound consisting of the (regular) dodecahedron and the (regular) great stellated dodecahedron $\{\frac{5}{2},3\}$ (cf. [4, p. 96]) with common vertices can be interpreted as a discontinuous polyhedron with $p = 5$, $q = 6$ and $f = (20,60,24)$; the 24 faces are the 12 pentagons and the 12 pentagrams. We denote it by $C_{5,6}$.

(*b*) The compound consisting of the (regular) icosahedron and the (regular) great icosahedron $\{3,\frac{5}{2}\}$ (cf. [4, p. 96]) with common face-planes can be interpreted as a discontinuous polyhedron with $p = 6$, $q = 5$ and $f = (24,60,20)$. We denote it by $C_{6,5}$.

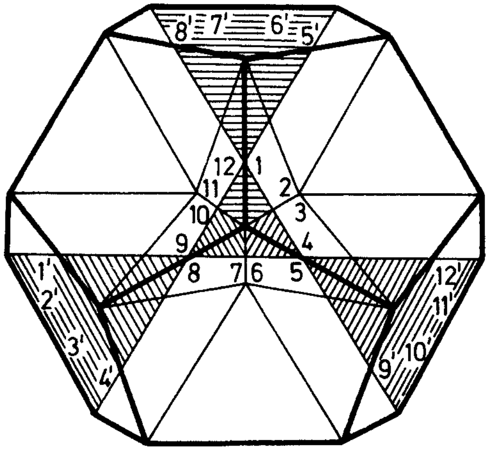
These 5 discontinuous polyhedra are shown in the flag-diagram in Figure 2. They are symbolized by the hexagonal stars, built up of two triangles. The most important property of $C_{4,6}$,$C_{6,4}$,$C_{5,6}$ and $C_{6,6}$ in our context is that they all have 60 edges and the icosahedral symmetry group. This shows their close relation to the 5 combinatorially regular polyhedra of index 2 in the theorem, which all have these properties, too.

The fact that $C_{6,6}$ splits in its faces and vertex figures leads to the well-known property that $C_{6,6}$ itself can be decomposed into two regular compounds of 5 tetrahedra each, which are chiral (left and right).

4. A Self-Dual Icosahedron of Genus 11

In the following we describe the polyhedron shown in Figure 1 and prove that it is the polyhedral realization of a regular map.

The polyhedron is of type $\{6,6\}$, oriented and of genus 11, so it has 20 6-valent vertices and 20 self-intersecting hexagonal faces. One face is hatched in Figure 1. The 20 vertices are the vertices of the regular dodecahedron. The visible part of the polyhedron is one of the 59 stellated icosahedra (cf. [5, fig. Ef_1g_1 on Plate IX]); it traces back at least to Brückner (cf. [2, fig. 26, Tafel VIII]), but is perhaps much older. The invisible part of the polyhedron is the icosahedron. The symmetry group of the polyhedron is the full icosahedral group; it acts transitively on the 20 vertices and the 20 faces. The 60 edges split into two orbits (long edges and short edges) under the



$\{6,6\}_6'$ fig. 1 f=20(1,3,1)

symmetry group. The main point is that the 240 flags split into two orbits only under the symmetry group, which is easy to see. So, if the polyhedron is combinatorially regular, it is of index 2. To prove the regularity we need an "outer automorphism" as mentioned by Coxeter ([3, p. 142]) for the other regular maps of index 2.

In our case the outer automorphism is very geometric and easy to find by using the self-duality which interchanges the role of the 20 vertices and the 20 faces. This self-duality interchanges also the two flag orbits (of the 30 long edges and the 30 short edges) while preserving all incidence properties, which is easy to verify. For this we labelled in Figure 1 the 12 flags incident with the front vertex by 1, . . ., 12 and the 12 corresponding flags incident with the corresponding (hatched) face by 1′, . . ., 12′. So the polyhedron is combinatorially regular of index 2.

We add some remarks:

(1) The surprising fact that the regularity properties of the polyhedron were never observed before is due to the fact that the faces are not metrically regular.

(2) The underlying regular map for the polyhedron is relatively new: It was discovered in 1976 by Wilson in his thesis ([21, p. 151; regular map no. (60,57)]). Wilson only investigated regular maps; he did not consider polyhedral realizations. We denote the map by $\{6,6\}_6'$ because of $p = q = 6$ and the length 6 of its Petrie-polygon. The ′ indicates that Wilson found it in a different way as the $\{p,q\}_r$ are usually constructed (cf. [7, p. 111] or Section 2).

(3) Another regularity proof for the polyhedron by standard methods of comparing the incidence properties of the polyhedron and the underlying regular map has been proposed by E. Schulte (personal communication).

(4) The polyhedron has no "false" edges; i.e. the only points of self-intersection are the 12 points where the visible and the invisible part of the polyhedron intersect.

(5) If one considers the "holes" (cf. [20, p. 5 and p. 151]) of $\{6,6\}'_6$ then the 2-holes H_2 form exactly the regular compound $C_{5,6}$ and the 3-holes H_3 form the degenerate compound of 15 dihedra, described in Lemma 7.

5. Proof of the Theorem

In this section we prove that no other regular map besides the 5 mentioned in the theorem is realizable as a polyhedron with or without self-intersections of index 2. Moreover we show that these realizations are unique up to dilatations. We need 6 lemmas. The final part of the theorem is after Lemma 6.

LEMMA 1: *There are no combinatorially regular polyhedra of index* 2 *with the cyclic or dihedral symmetry group.*

Proof. If such a polyhedron exists, its vertices lie on k orbits with respect to the rotation. We consider three cases: $k = 1$, $k = 2$, $k \geq 3$.

(1) $k = 1$. The "polyhedral" realization is "flat", i.e. lies in a plane. Hence no polyhedral realization exists.

(2) $k = 2$. There is at least one face which contains vertices of both orbits. This face cannot contain more than two vertices of each orbit. So it is either a triangle or a quadrangle. If it is a triangle it belongs to flags of at least three different orbits (with respect to the full symmetry group). If it is a quadrangle, then it must be a rectangle or a degenerate rectangle (with 2 parallel edges replaced by the two diagonals).

In both cases one needs further faces from another orbit to build up a polyhedron. Hence there are more than two flag orbits and the polyhedron has an index ≥ 3.

(3) $k \geq 3$. There is at least one edge which joins a vertex of an "outer" orbit and a vertex of one "inner" orbit. This edge belongs to two flags which are from different orbits (with respect to the symmetry group). Further there is at least one edge which does not belong to the same orbit as the first one. Hence there are at least three flag-orbits and the polyhedron has an index ≥ 3.

LEMMA 2. *There are no combinatorially regular polyhedra of index* 2 *and either the rotation group or the full symmetry group of the tetrahedron or the octahedron.*

Proof. If such a polyhedron exists, its automorphism group has the order 24, 48 or 96, hence $f_1 = 6$ or 12 or 24. Without restriction we can assume $p \leq q$, so $f_0 \leq f_2$.

From (4) follows for $g = 1$: $f_0 \geq 5$. But the only regular map with $g = 1$ and $f_1 = 6$ or 12 or 24 is $\{3,6\}_4$ and has $f_0 = 4$ (cf. [7, p. 104, 108]). So let $g \geq 2$.

From (4) follows for $g \geq 2$: $f_0 \geq 6$, and for $g \geq 3$: $f_0 \geq 7$.

From (3) and $p \leq q$ follows $q \geq 5$ and so from (2): $f_0 \leq \left[\frac{2}{5} f_1\right]$.

From this follows for $f_1 \leq 12$: $f_0 \leq 4$. This contradicts $f_0 \geq 6$, and so the only remaining case is $f_1 = 24$. From (2) we have

$$qf_0 = pf_2 = 48.$$

From this and $q \geq 5, f_0 \geq 6$ it follows that the only possible values for q and f_0 are:

(1) $q = 8$ and $f_0 = 6$ for $g = 2$
(2) $q = 6$ and $f_0 = 8$ for $g \geq 3$.

If in case (2) we have $g \geq 6$, then we get from (1): $f_0 - f_1 + f_2 = -16 + f_2 \leq 2 - 12 = -10$ or $f_2 \leq 6$, which contradicts $f_2 \geq f_0 = 8$.

So it is easy to check from (1) and (2) and the complete lists of regular maps for $g = 2,3,4,5$ in [7 p. 140], [17 p. 475] and [8 pp. 53, 54] that the only possible regular maps are the following, which are uniquely determined by their f-vector:

$g = 2$ $f = (6,24,16)$

$g = 3$ $f = (8,24,12)$

$g = 5$ $f = (8,24,8)$.

All these maps have the property that two adjacent vertices are joined by more than one edge, which can either be seen from [7,8,17] and the literature mentioned there, or from Wilson's list [20 p. 130], where all these maps and their properties are listed.

LEMMA 3. *There are no combinatorially regular polyhedra of index* 2 *which have the rotation group of the icosahedron.*

Proof. If such a polyhedron exists, then $f_1 = 30$. From the lists of Coxeter-Moser ([17, pp. 104, 108 and 140]), Sherk ([17, p. 475]) and Garbe ([8, pp. 53, 54]) follows that for $g \leq 6$ the only regular map with $f_1 = 30$ is the one of type $\{5,5\}$ and of genus 4, which is just the map for the two Kepler-Poinsot polyhedra of genus 4. So we can assume that a regular map with $g \geq 7$ and $f_1 = 30$ exists. Then

$qf_0 = pf_2 = 2f_1 = 60.$

Without restriction $p \leq q$, so $f_0 \leq f_2$.

Further we have from the icosahedral rotation group $12 \leq f_0 \leq f_2$ and hence $p \leq q \leq 5$. From this and (3) it follows that the only possible pairs (p,q) are (4,5) and (5,5). The corresponding pairs (f_0,f_2) are (15,12) and (12,12). So we have from Euler's relation (1):

$$-6 \leq f_0 - f_1 + f_2 = 2 - 2g$$

or $g \leq 4$, which contradicts $g \geq 7$.

So we only have to consider the map with $f_1 = 30$ and $g = 4$ mentioned above. This map has the two realizations $\{5,\frac{5}{2}\}$ and $\{\frac{5}{2},5\}$ which are two of the Kepler-Poinsot-solids with the full icosahedral symmetry group. We have to show that no realization with the icosahedral rotation group is possible. For this we note that the 12 vertices of a realization must be the vertices of a regular icosahedron.

We consider one rotation axis through two opposite vertices V and $\bar{V}$. Then the other 10 vertices lie on two orbits, say X and Y, of 5 vertices each. Because each vertex is 5-valent, the edges of V meet either all vertices of X or of Y. No other distributions of the 30 edges is possible. But these two possibilities yield $\{5,\frac{5}{2}\}$ and $\{\frac{5}{2},5\}$ and another polyhedral realization of this map with icosahedral rotation group does not exist.

LEMMA 4. *Combinatorially regular polyhedra of index* 2, *and* $f_1 = 60$ *are at most possible for the following triplets* $\{p,q;g\}$:

$\{4,5;4\}$, $\{4,6;6\}$, $\{5,5;7\}$, $\{5,6;9\}$, $\{6,6;11\}$,

$\{5,4;4\}$, $\{6,4;6\}$, $\{6,5;9\}$.

Proof. If such a polyhedron exists, then it has the full icosahedral symmetry group. So $f_1 = 60$ and

$qf_0 = pf_2 = 2f_1 = 120.$

Again let $p \leq q$, so $f_0 \leq f_2$.

(1) From the lists of the regular maps for $g \leq 6$ (cf. [7, pp. 104, 108, 140], [8, pp. 53, 54]; [17, p. 475]) follows that the only regular maps with $g \leq 6$ and $f_1 = 60$ are:

(*a*) $\{4,5;4\} = \{4,5\}_6$, $\{4,6;6\} = \{4,6/3\}$ and their duals $\{5,4;4\} = \{5,4\}_6$, $\{6,4;6\} = \{6,4/3\}$.
(These maps are realizable, but not all of index 2, as we will see later.)

(*b*) $\{3,10;5\}$ and $\{10,3;5\}$. These maps are hyperelliptic (cf. [8] or [20]), i.e. two adjacent vertices are joined by two edges. So these maps are not realizable in our sense.

(2) In the following let $g \geq 7$. Then (4) implies $8 \leq f_0 \leq f_2$. From $qf_0 = 120$ it follows that f_0 is a divisor of 120.

So we obtain the following values for f_0 and the corresponding $q \geq 3$:

$$f_0 = 8,\ 10,\ 12,\ 15,\ 20,\ 24,\ 30,\ 40$$

$$q = 15,\ 12,\ 10,\ 8,\ 6,\ 5,\ 4,\ 3.$$

Because of the icosahedral symmetry the cases $f_0 = 8$, 10 and 15 are not possible.

We show that also $f_0 = 12$ is not possible. If $f_0 = 12$, then the vertices are exactly the 12 vertices of the regular icosahedron. Because no 6 vertices of the icosahedron lie in a plane, we have $p = 3,4$ or 5. There are exactly 12 planes which contain 5 vertices of the icosahedron. This yields $pf_2 = 60$, which contradicts $pf_2 = 120$. This rules out $p = 5$.

Now let $p = 4$. A quadrangle cannot lie in one of the 12 planes which contain 5 vertices of the icosahedron, because then for symmetry reasons 5 quadrangles lie in each of these planes, which violates the properties of a polyhedron and of a regular map. So these quadrangles are not possible. There are exactly 15 planes which contain exactly 4 vertices of the icosahedron. This yields $pf_2 = 60$, which contradicts $pf_2 = 120$ and rules out $p = 4$.

If $p = 3$, then $f_2 = 40$, and from (1) follows

$$f_0 - f_1 + f_2 = -8 = 2 - 2g,$$

and so $g = 5$, which contradicts $g \geq 7$.

So the only remaining values are now: $f_0 = 20,24,30$ and 40, and we have $p \leq q \leq 6$. From (3) and $q \leq 6$ follows $p \geq 4$.

(3) So we have from (1) and (2) that for $f_1 = 60$ the only possible pairs (p,q) with $p \leq q$ and their duals with $p \geq q$ are:

(4,5), (5,5), (4,6), (5,6), (6,6)

(5,4) (6,4), (6,5).

Because the genus g is uniquely determined by (p,q) and already mentioned in (a) for $g \leq 6$, we have all triplets $\{p,q;g\}$ mentioned in the lemma.

LEMMA 5. *There is no combinatorially regular polyhedron* $\{5,5;7\}$ *of index* 2.

Proof. If such a polyhedron exists, it follows from the Euler-relation that $f = (24,60,24)$. Its symmetry group is the full icosahedral group, which is a subgroup of index 2 in its automorphism group $C_2 \times S_5$. So its vertices and its faces lie on exactly two orbits of 12 each, say the orbits x,X and y,Y. Further there exist edges which join vertices of different orbits, and hence faces which contain vertices of different orbits. Because the faces are pentagons (or pentagrams) they cannot have the same number of vertices from both orbits. Hence the faces from the face-orbit X must have more vertices from, say vertex-orbit x, than the faces from face-orbit Y.

From this it follows that there are at least four orbits of flags and so the polyhedron has at least the index 4.

LEMMA 6. *There are exactly* 2 *combinatorially regular polyhedra of index* 2, *which are realizations of regular maps of the type* $\{4,6;6\}$, $\{5,6;9\}$ *or* $\{6,6;11\}$ *and have* $f_0 = 20$.

Proof. Let P be such a polyhedron of index 2. From Lemma 4 we know that $f_1 = 60$ and that its symmetry group is the full icosahedral group. So the 20 vertices lie on one orbit, i.e. they are the vertices of a regular dodecahedron.

We consider a rotation-axis through one vertex v and its opposite vertex $\bar{v}$. Then the remaining 18 vertices of P split under this rotation into two orbits of 3 vertices each and two orbits of 6 vertices each.

We denote the orbits by O_i, $i = 1,2,3,4$, where i is the length of the shortest path from v to a vertex of O_i along the edges of the dodecahedron.

So O_1 and O_4 contain 3 vertices each, and O_2,O_3 6 vertices each.

Now let v_{ij} denote the vertices of O_i, i.e.

$$O_1 = \{v_{11},v_{12},v_{13}\},\ O_2 = \{v_{21},\ldots,v_{26}\}$$

$$O_3 = \{v_{31}\ldots,v_{36}\},\ O_4 = \{v_{41},v_{42},v_{43}\}.$$

From $q = 6$ and the icosahedral symmetry follows that for the 6 edges incident with v there are only the following three possibilities:

(I) $\overline{vv_{ij}}$ $i = 1,4,\ j = 1,2,3$
(II) $\overline{vv_{2j}}$ $j = 1,\ldots,6$
(III) $\overline{vv_{3j}}$ $j = 1,\ldots,6$

(I), (II) and (III) define uniquely the three only possible 1-skeletons (=edge-graphs) for P, and we have to check all possible (orientable) 2-manifolds with $p = 4,5$ or 6, which can be constructed from these. This needs much space to write down in detail, but is easy to check for the few possible cases, if one takes a model of a regular dodecahedron:

(I) $p = 4$. One obtains 30 rectangles which form 15 dihedra, i.e. a highly degenerate compound, but no polyhedron.
$p = 5$. One obtains the regular compound $C_{5,6}$ described in Section 3.
$p = 6$. One obtains the unique polyhedral realization of index 2 of $\{6,6\}'_6$.
(II) $p = 4$. One obtains the regular compound $C_{4,6}$ described in Section 3.
$p = 5$. One obtains the unique polyhedral realization of index 2 of $\{5,6\}_4$.
$p = 6$. No 6 edges form a plane hexagon.
(III) $p = 4,5,6$. No polyhedra are possible. It can easily be seen that the only possible polygons are 40 regular triangles which yield the well-known regular compound built up of 10 regular tetrahedra, i.e. $C_{6,6}$ of Section 3.

Proof of the theorem. After Lemma 6 we are now able to complete the proof of the theorem: From Lemmas 1,2,3 it follows that such realizations exist (if at all) for $f_1 = 60$, i.e. for the full icosahedral symmetry group. These cases are investigated in Lemmas 4,5 and 6. From Lemmas 4 and 5 follows that such realizations exist (if at all) only for the following triplets $p,q;g$:

$\{4,5;4\}$, $\{4,6;6\}$, $\{5,6;9\}$, $\{6,6;11\}$
$\{5,4;4\}$, $\{6,4;6\}$, $\{6,5;9\}$.

For $g = 4$ there are exactly the two regular maps (cf. [8]) ($\{4,5\}_6$ and $\{5,4\}_6$, and their polyhedral realizations with icosahedral symmetry group (i.e. of index 2) are unique, which was proved e.g. in [4, p. 102].

For the 3 triplets $\{4,6;6\}$, $\{5,6;9\}$ and $\{6,6;11\}$, it follows from Lemma 7 that exactly the regular maps $\{5,6\}_4$ and $\{6,6\}'_6$ are realizable by a polyhedron of index 2 and that these realizations are metrically unique (up to dilatations).

From Grünbaum's and Shephard's duality theorem this follows also for the dual map $\{6,5\}_4$ and the theorem is proved.

6. The Flag-Diagram

Regular maps and their various polyhedral realizations can be shown in a flag-diagram (Fig. 2) i.e. in a p,q-diagram for $p \geq 3, q \geq 3$ (or $p \geq 2, q \geq 2$, if one wants to include dihedra and hosohedra). The labels in the (p,q)-fields denote the genus g.

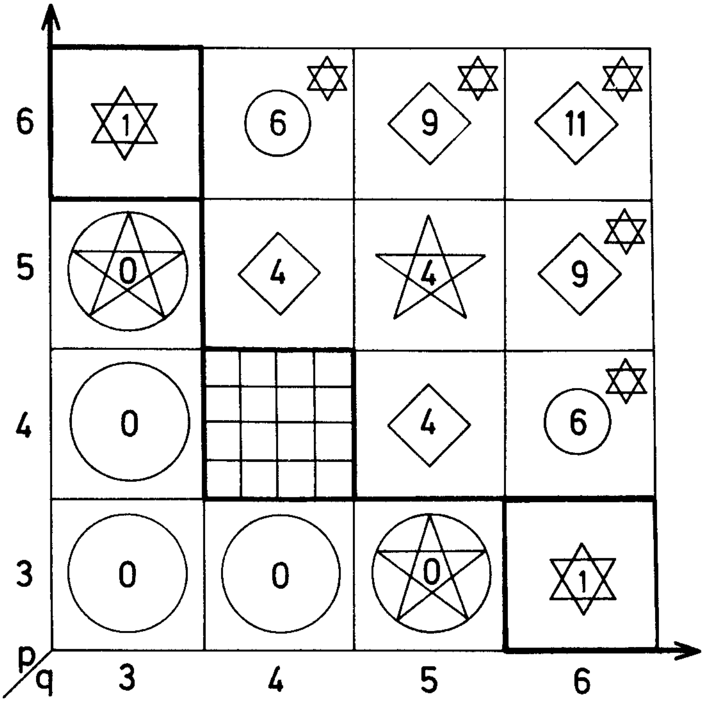


fig. 2

The number of flags at a vertex is $2q$, and in a face is $2p$.
The bold lines separate the three regions

(1) Elliptic; $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, $g = 0$

(2) Parabolic; $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $g = 1$

(3) Hyperbolic; $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, $g \geq 2$.

The 5 large circles denote the 5 Platonic polyhedra; the 3 large pentagrams denote the 4 Kepler-Pointsot polyhedra ($\{5,\frac{5}{2}\}$ and $\{\frac{5}{2},5\}$ are realizations of the same map); bold squares denote the 3 regular tilings of the plane. The 2 small circles denote Coxeter's regular maps $\{4,6/3\}$ and $\{6,4/3\}$ of genus 6, which can be realized in E^3 without self-intersections [12], but not as polyhedra of index 2, as shown in Lemma 6.

The hexagonal stars denote the regular compounds

$$C_{3,6}, C_{6,3}, C_{4,6}, C_{6,4}, C_{5,6}, C_{6,5}, C_{6,6}$$

described in Section 3. Finally the 5 diamonds denote the 5 combinatorially regular polyhedra of index 2 of our theorem.

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