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# Compositions with distinct parts

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*Abstract.* The number of compositions $C(n)$ of a positive integer $n$ into distinct parts can be considered as a natural analogue of the number $q(n)$ of distinct partitions of $n$. We obtain an asymptotic estimate for $C(n)$ and in addition show that the sequence $\{C(n, k)\}$ of distinct compositions of $n$ with $k$ distinct parts is unimodal. Our analysis is more complicated than is usual for composition problems. The results imply however that the behaviour of these functions is of comparable complexity to partition problems.

## §1. Introduction

In this note we consider $C(n, k)$, the number of compositions of $n$ with $k$ distinct parts, as well as $C(n) = \sum_k C(n, k)$, the total number of compositions of $n$ with distinct parts. It is clear that $C(n, k) = k!q(n, k)$ where $q(n, k)$ is the number of partitions of $n$ with $k$ distinct parts. Thus $q(n, k)$ is the number of solutions in integers to

$$x_1 + x_2 + \cdots + x_k = n, \qquad 1 \le x_1 < x_2 < \cdots < x_k.$$

The function $q(n, k)$ has been studied in detail by Szekeres in two remarkable papers [Sz1] and [Sz2]. We shall depend very heavily upon the results in [Sz2]. Unrestricted compositions have a well known correspondence with combinations of multi-sets. From this we obtain a further combinatorial interpretation of $C(n, k)$:

Let $S$ be a multi-set with $k$ distinct objects, each with unlimited repetition. Then $C(n, k)$ is the number of $n$-combinations of $S$ in which each object appears at least once and the number of times each of the $k$ objects appears is *different*.

A sequence $a_n(k)$, $k = 1, 2, \ldots K$ is said to be unimodal if there is a $k_0$ such that $a_n(1) \le a_n(2) \le \cdots \le a_n(k_0) \ge a_n(1 + k_0) \ge \cdots \ge a_n(K)$. A surprisingly rich variety

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of methods exist for proving unimodality, see the thorough and fascinating survey by Stanley [S]. Szekeres [Sz2] showed that if $n = m(m+1)/2 + \ell$, $0 \le \ell \le m$, then $q(n, 1) < q(n, 2) < \cdots < q(n, k_0) \ge q(n, 1 + k_0) > \cdots > q(n, m)$. He also showed that if $c = 6^{1/2}\pi$, $b = c^2 \log^2 2$ then $k_0 = 2^{1/2}c(\log 2)n^{1/2} - 1 + 2b(\log 2)^{-1} - \frac{1}{2}b/(1-2b) - 1 + 0(n^{-1/2})$. The paper [Sz2] has led to many further analytic proofs of unimodality, see [S] for references. We prove that $C(n, k)$ is unimodal using [Sz2]. Other sequences such as $m^k k! q(n, k)$, $m^k q(n, k)$ the number of compositions (respectively partitions) with $m$ colours and $k$ parts can be proved to be unimodal by modifying Szekeres's method. We shall identify the required results from [Sz2] for our example so as to facilitate proving that sequences constructed from $q(n, k)$ (or from $p(n, k)$ defined below) are unimodal.

We first prove

THEOREM 1. *Let $n = m(m+1)/2 + \ell$, $0 \le \ell \le m$. Then there is an integer $k_1$, such that*

$$C(n, 1) < C(n, 2) < \cdots < C(n, k_1) \ge C(n, k_1 + 1) > \cdots > C(n, m).$$

*Furthermore, if $y$ is defined as the real solution of eq.* (2.12) *modified by deleting the* 0*-term and replacing $i$ by $y$, and if $i_1$ is the integer part of $y$ then*

$$k_1 = m - i_1 \quad \text{or} \quad m - i_1 - 1 \quad \textit{and} \quad i_1 \sim c^{-2} m/\log^2 m.$$

Here $c = 6^{1/2}/\pi$ and the symbol $\sim$ means that the ratio of the two sides tends to 1 as $n \to \infty$.

Szekeres derived an asymptotic formula for $q(n, k)$ in [Sz2] valid for all $n$ and $k$ tending to infinity so of course there is such a formula for $C(n, k)$. It is not complicated for $k$ near $k_1$, as we now see.

Let $p(n)$ denote the ordinary partition function ($p(n) = P(n, n)$ defined below). The Hardy–Ramunujan formula

$$p(n) \sim \left\{2^{3/2}\pi\left(n - \frac{1}{24}\right)\right\}^{-1/2} \exp\left(2c^{-1}\left(n - \frac{1}{24}\right)^{1/2}\right)\left\{1 - \frac{c}{2n^{1/2}} + O\left(\frac{1}{n}\right)\right\}$$

is derived in [Sz2]. It also follows from the results in [Sz2] or immediately from the results in Erdös–Lehner [EL] that if $k > c\{n - k(k+1)/2\}^{1/2} L + w(n) n^{1/2}$, where $L = \log\{c(n - k(k+1)/2)^{1/2}\}$ and $w(n)$ is an arbitrary function tending to infinity, then

$$q(n, k) \sim p(n - k(k+1)/2),$$

which gives a simple asymptotic formula for $C(n, k)$ when Stirling's formula is applied. (It follows from the estimates in (2.9) that $k_1 - c\{n - k_1(k_1 + 1)/2\}^{1/2}L \sim m \log m/\log m$ so that since $k_1 \sim m$ the Erdös–Lehner condition is satisfied.)

We also prove

THEOREM 2. *Let $n = m\,(m + 1)/2 + \ell$, $0 \le \ell \le m$. Let $C(n) = \sum C(n, k)$. Then, with $c$ and $k_1$ defined in Theorem 1,*

$$C(n) \sim \frac{C(n, k_1)\pi^{1/2}m^{1/2}}{\sqrt{\log m}}.$$

(The proof of Theorem 2 shows that the $C(n, k)/C(n)$ tend to a normal distribution with mean $k_1$ and standard deviation $\sim\sqrt{2}(m/\log m)^{1/2}$.)

The inequalities $mk_1!q(n, k_1) \ge C(n) > m!$ show that

$$\log C(n) \sim (2n)^{1/2} \log n.$$

Thus the restriction that the parts be distinct is strong since if this restriction is dropped we have $2^{n-1}$ compositions.

In addition, using the fact that $C(n) = \sum k!q(n, k)$ and the known generating functions for $q(n, k)$, we obtain the ordinary generating function

$$C(x) = \sum_{n=1}^{\infty} C(n)x^n = \sum_{k=1}^{\infty} \frac{k!x^{\binom{k+1}{2}}}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

It seems simplest to estimate $C(n, k)$ using the $k$-th term of this series and then sum these estimates, rather than dealing directly with $C(x)$.

Although it is not explicitly stated there it follows from the results in [Sz2] that if $k < k_0$ then $q^2(n, k-1) > q(n, k-2)q(n, k)$, i.e. $q(n, k)$ is log-concave for $k < k_0$ while if $k \ge k_0$ then $q(n, k)q(n, k+2) > q^2(n, k+1)$, i.e. $q(n, k)$ is log-convex for these $k$. These inequalities are shown to hold for $C(n, k)$ also.

Some related enumeration problems concerning distinct partition sizes in the contexts of set partitions and permutations are considered in [KORSW].

## §2. Proofs

Let $p(n, k)$ denote the number of partitions of $n$ into $k$ parts and $P(n, k) = \sum_{k \le n} p(n, k)$. Then, as Szekeres points out, $q(n, k) = P(n - k(k+1)/2, k)$. Suppose $\ell$ is defined as in Theorem 1 and $k = m - s$ where $s \le m^{1/5}$. Then

$$n - k(k+1)/2 = \ell + sm - s(s+1)/2,$$

so

$$P(n - k(k+1)/2, k) = P(\ell + sm - s(s+1)/2, m - s) = p(\ell + sm - s(s+1)/2).$$

Hence, if $0 \le s \le m^{1/5}$,

$$\frac{C(n, m-s)}{C(n, m-s-1)} = (m - s)\frac{p(\ell + sm - s(s+1)/2}{p(\ell + (s+1)m - (s+1)(s+2)/2)}.$$

Hence, from the Hardy–Ramanujan formula,

$$\frac{C(n, m)}{C(n, m-1)} = m\frac{p(\ell)}{p(\ell + m + 1)} < \frac{mp(m)}{p(2m+1)} < 1$$

and it is seen that more generally

$$\frac{C(n, m-s)}{C(n, m-s-1)} < 1 \qquad \text{for} \quad 0 \le s \le m^{1/5}.$$

Also if $k$ is bounded then

$$P(n - k(k+1)/2, k) = [x^{n - k(k+1)/2}]\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$$
$$\sim \frac{(n - k(k+1)/2)^k}{k!},$$

as is easily seen from the partial fraction decomposition. Thus Theorem 1 is easily verified for bounded $k$.

We suppose throughout this section that

$$n = m(m+1)/2 + \ell, \quad 0 \le \ell \le m \quad \text{and} \quad c = 6^{1/2}/\pi.$$

Let $k = m - i$ then

$$n - k(k+1)/2 = im - i(i-1)/2 + \ell.$$

We now list some relations from [Sz2] that we need. Define $\alpha$ and $u$ by

$$\alpha k = u \tag{2.1}$$

and [eq. (44), Sz2]

$$\alpha^{-2}\int_0^u \frac{t}{e^t-1}\,dt+\frac{1}{2}\alpha^{-1}\left(\frac{u}{e^u-1}-1\right)=im-(i-1)i/2+\ell+0(1). \tag{2.2}$$

Note that an equivalent definition of $\alpha$ is [eq. 10, Sz2]

$$\sum_{j=1}^{k}\frac{j}{e^{\alpha j-1}}=n-\frac{(k+1)k}{2}=im-i(i-1)/2+\ell$$

which shows, since the left-hand side is monotonic in $\alpha$, that $\alpha$ and $u$ are uniquely determined.

We shall use the method of successive approximations or bootstrapping as Greene and Knuth [GK] say. Since the solutions to (2.1) and (2.2) are unique, the approximations we derive this way are valid for $u$ and $\alpha$. To motivate the following calculations we have these considerations:

We expect the maximum of $C(n, k)$ to be very near the $k_1$, for which

$$\frac{k_1!q(n,k_1)}{(k_1+1)!q(n,1+k_1)}=1 \quad \text{or} \quad \frac{q(n,k_1)}{q(n,1+k_1)}=1+k_1. \tag{2.3}$$

Now [top of page 111 and bottom of page 102, Sz2]

$$\log((q(n,k+1))/1(n,k))=-\log(e^u-1)-\alpha\left(\frac{e^u}{e^u-1}+\frac{1}{2}\frac{A_0^{-1}u^2(e^u+e^{2u})}{(e^u-1)^2}\right.$$
$$\left.-\frac{3}{2}A_0^{-1}\frac{ue^u}{e^u-1}+\frac{1}{2}\frac{A_0^{-2}u^4e^{2u}}{(e^u-1)^3}\right)+O(\alpha^2) \tag{2.4}$$

where

$$A_0=\int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt. \tag{2.5}$$

Clearly we need accurate estimates for $\alpha$, $u$ and $A_0$ to use these relations. We begin by estimating $\alpha$. From [eq. 42, Sz2]

$$\int_0^u\frac{t}{e^t-1}\,dt=c^{-2}-(u+1)e^{-u}+O(ue^{-2u}). \tag{2.6}$$

From (2.2), (2.6) and the formula for the roots of a quadratic we have

$$\frac{1}{\alpha}=\left[-\frac{1}{2}(u(e^u-1)-1)+\left\{\frac{1}{4}(u(e^u-1)-1)^2+4(im-i(i-1)/2+\ell+O(1))\right.\right.$$
$$\left.\left.\times(c^{-2}-(u+1)e^{-u}+O(ue^{-2u}))\right\}^{1/2}\right][2(c^{-2}-(u+1)e^{-u}+O(ue^{-2u}))]^{-1}. \tag{2.7}$$

For our first estimate, we suppose $u\to\infty$ and $i=o(m)$, then from (2.1), (2.7)

$$\alpha^{-1}\sim c(im)^{1/2}\quad\text{hence}\quad \alpha k=u\sim c^{-1}m^{1/2}i^{-1/2}.$$

Now from (2.4) we see that, if $k$ satisfies (2.3),

$$u\sim\log(m-i+1)\quad\text{so}\quad i\sim c^{-2}m/\log^2 m.$$

Hence there is a solution to (2.3), (2.4) and (2.5) that satisfies, with $k_1=m-i_1$,

$$i_1\sim c^{-2}m(\log m)^{-2},$$
$$\alpha\sim c^{-1}(im)^{-1/2}\sim(\log m)/m,\qquad u\sim\log m$$
$$A_0\sim\int_0^\infty t^2e^t(e^t-1)^{-2}\,dt=2c^{-2}, \tag{2.8}$$

which is the unique solution.

We now refine our estimates for $\alpha$ considerably. Note first of all that from eq. (41) of [Sz2] we have that the $O(1)$ term of (2.7) is $1/24+O(\log m/m)$ (using $u\sim\log m$ from (2.8)), which we will use in solving (2.7). We begin by solving (2.2) more precisely.

If we now set $i=c^{-2}m\log^{-2}m+mf(m)$ and substitute this expression into (2.2) we see that all terms involving an $e^u$ or $e^{-u}$ can be neglected and

$$\alpha=\frac{\log m}{m}\left(1+\frac{c^{-2}}{4\log^2 m}-\frac{c^2}{2}\log^2 mf(m)+O(\log^{-2}mf(m))+f^2(m)\right).$$

Hence $\log(m-i+1)=\alpha(m-i+1)$ implies

$$\log m-\frac{i}{m}-\frac{1}{2}\frac{i^2}{m^2}+O\left(\frac{i^3}{m^2}\right)=\left(\log m+\frac{c^{-2}}{4\log m}\right.$$
$$\left.-\frac{c^2}{2}\log^3 mf(m)+O(\log mf(m))\right)\left(1-\frac{i-1}{m}\right)$$

and so

$$f(m) = -\frac{3}{4}\frac{c^{-4}}{\log^4 m} + \frac{2c^{-4}}{\log^5 m} + O(\log^{-6} m).$$

Hence

$$\left.\begin{aligned} i &= c^{-2}m\log^{-2} m - \frac{3}{4}\frac{c^{-4}m}{\log^2 m} + \frac{2c^{-5}m}{\log^5 m} + O(\log^{-6} m) \\ \alpha &= \frac{\log m}{m}\left(1 + \frac{5}{8}\frac{c^{-2}}{\log^2 m} - \frac{c^{-2}}{\log^3 m}\right) + O\left(\frac{1}{m\log^3 m}\right) \\ u &= \alpha k = \alpha(m-i) = \log m - \frac{3}{8}c^{-2}\log^{-1} m - c^{-2}\log^{-2} m + O(\log^{-3} m) \end{aligned}\right\}. \tag{2.9}$$

We now solve (2.7) using (2.9).

Note that

$$4(u+1)e^{-u} = \frac{4}{m}\left(\log m + 1 + \frac{3}{8}c^{-2} + \left(c^{-2} + \frac{9}{128}c^{-4}\right)\log^{-1} m + O(\log^{-2} m)\right).$$

Using similar estimates for $u(e^u - 1)^{-1}$ we find that the expression inside the root of (2.7) when divided by $4(c^{-2} - (u+1)e^{-u} + O(ue^{-2u}))$ equals

$$c\left(im - i(i-1)/2 + \frac{1}{24}\right)^{1/2}\left(1 + \frac{c^2(\log m + 1 - c^{-2} - 3c^{-2}/\log m)}{2m} + O\left(\frac{1}{m\log^2 m}\right)\right)$$

all squared.

We now find that (recall $i \sim c^{-2}m\log^{-2} m$)

$$\alpha = \frac{1}{c\left(im - i(i-1)/2 + \ell + \frac{1}{24}\right)^{1/2}} - \frac{\left(\frac{1}{4} + c/2\right)\log m + c^{-1} + \frac{3}{8}c^{-3} + \left(c^{-3} + \frac{9}{64}c^{-5} + \frac{7}{8}\right)\Big/\log m}{cm\left(im - i(i-1)/2 + \ell + \frac{1}{24}\right)^{1/2}} + O\left(\frac{1}{m^2\log m}\right).$$

(2.10)

Thus

$$
\begin{aligned}
u &= \alpha(m-i)\\
&= \frac{c^{-1}(m-i) - \left(\frac{1}{4}+\frac{c}{2}\right)\log m - c^{-1} - \frac{3}{8}c^{-3} - \left(c^{-3}+\frac{9}{64}c^{-5}+\frac{7}{8}\right)\log^{-1} m}{\left(im - i(i-1)/2 + \ell + \frac{1}{24}\right)^{1/2}}\\
&\quad + O\left(\frac{1}{m\log m}\right).
\end{aligned} \tag{2.11}
$$

Furthermore, using (2.9) in (2.4), we find that .

$$
\log\left(\frac{q(n,k+1)}{q(n,k)}\right) = u - \frac{1}{m} - \frac{1}{c^2 m\log m} - \alpha\left(\frac{c^2}{4}\log^2 m - \frac{3c^2}{4}\log m + 1 - 2c^{-2} + \left(\frac{3}{4} - 2c^{-2}\right)\Big/\log m + O(\log^{-2} m)\right).
$$

Hence when we use (2.10), (2.9) and (2.11) to solve (2.3) we have the equation

$$
\begin{aligned}
&\frac{c^{-1}(m-i) - \left(\frac{c}{2}+\frac{1}{4}\right)\log m - c^{-1} - \frac{3}{8}c^{-3} + \left(c^{-3}+\frac{9}{64}c^{-5}+\frac{7}{8}\right)\log^{-1} m}{\left(im - i(i-1)/2 + \ell + \frac{1}{24}\right)^{1/2}}\\
&\quad + \frac{1}{m} + (c^2 m\log m)^{-1} + m^{-1}\left[\frac{c^2}{4}\log^3 m - \frac{3}{4}c^2\log^2 m + \left(\frac{37}{32} - 2c^{-2}\right)\right.\\
&\quad \left.\times \log m + \left(1 + \frac{3}{4}c^2 - 2c^{-2}\right)\right] + (m\log m)^{-1} = \log(m-i+1).
\end{aligned} \tag{2.12}
$$

Note that the contribution to $y$ of the $O$-term will be $O(\log^{-1} m)$ since $|\Delta \log(m-y+1)| \le (m-y)^{-1}\,\Delta y$. The claims concerning $i_1$ in Theorem 1 follow. One can use Newton's method starting with the estimate given by (2.9) to solve for $y$.

We now turn to verifying the unimodality of $C(n,k)$.

We shall show first that, if $k \ge k_1$,

$$
\frac{C(n,k+1)}{C(n,k)} > \frac{C(n,k+2)}{C(n,k+1)}.
$$

We use the equation at the top of page 111 of [Sz2] to estimate $\Delta\log(C(n,k+1)/C(n,k))$. The analysis to compute $\Delta\log p(n,k)$ starting with eq. (32) and continuing to eq. (46) in [Sz2] shows that "the difference of smaller terms is smaller." (This is also basic to [RS].) Thus to estimate $\Delta\log\{C(n,k+1)/(Cn,k)\}$ is suffices to estimate $\Delta - \log(e^u - 1)$. Hence

$$\Delta\log\{C(n,k+1)/C(n,k)\} \sim \frac{-\Delta u}{1-e^{-u}}.$$

From the equation just after (44) of [Sz2]

$$\Delta\alpha = A_0^{-1}\,\alpha^2 ue^u/(e^u-1) + O(\alpha^3).$$

Since $u = \alpha k$ we have $\Delta u = k\Delta\alpha + \alpha$, so

$$\Delta u \sim A_0^{-1}\alpha^2u^2e^u/(e^u-1).$$

Hence, from (2.8)

$$u(k_1+1) - u(k_1) \sim A_0^{-1}\alpha u^3\,e^u/(e^u-1) \sim \frac{c}{im}(\log^4 m).$$

Now

$$\Delta\log(k_1+1) = \frac{1}{k_1+1} + O\left(\frac{1}{1+k_1}\right)^2$$

and, since $k_1^{-1} \sim 1/m$, we see that

$$\Delta\log\{C(n,k_1+1)/C(n,k_1)\} < 0. \tag{2.13}$$

However since $\Delta^2 u$ is easily seen to be $\sim A_0\alpha^2 3u^2\Delta u \sim 3\alpha^3u^5 > 0$ we see that $\Delta u(k) \geq \Delta u(k_1)$ for $k \geq k_1$. Also $\Delta\log(k+1) = (k+2)^{-1} + O(k^{-2})$ so $\Delta\log(k + \log(k_1+1)$. Hence $\Delta\log\{C(n,k+1)/C(n,k)\} < 0$ for all $k \geq k_1$.

For $k < k_1$ we note that $\Delta u(k-1) < \Delta u(k)$ since $\Delta u > 0$ and that $(k-1)^{-1} > k^{-1}$. Hence

$$\Delta\log\{C(n,k)/C(n,k-1)\} > \Delta\log\{C(n,k_1)/C(n,k_1-1)\}.$$

But it is easily seen that

$$\Delta \log\{C(n, k_1)/C(n, k_1 - 1)\} > 0 \tag{2.14}$$

since here the change is asymptotically the negative of the change in (2.13). Equations (2.13) and (2.14) complete the proof of Theorem 1 (along with eq. (2.12) and the comments following immediately).

We prove Theorem 2. From Stirling's formula

$$\frac{(k_1 + y)!}{k_1!} = \exp\left(y \log(m - i) + y^2/(m - i) + O\left(\frac{y^2}{m - 2}\right)\right).$$

Since from (2.11)

$$\log\left(\frac{q(n, k_1 + 1)}{q(n, k_1)}\right) = -u + O(\alpha \log^2 m)$$

we have

$$\log\left(\frac{q(n, k_1 + 1)}{q(n, k_1)}\right) = -yu + O(y\alpha \log^2 m)$$

$$(yu = -y\alpha(m - i_1 + y)) = -y\alpha(m - i_1) - y^2\alpha + O(y\alpha \log^2 m).$$

Now

$$\alpha(m - i_1) \sim \log(m - i_1), \quad \text{so} \quad \alpha \sim \log(m - i_1)/(m - i_1).$$

Hence, for $y = O(m^{2/3})$,

$$\frac{q(n, k_1 + y)}{q(n, k_1)} = \exp\left(-y \log(m - i_1) - y^2 \frac{\log(m - i_1)}{(m - i_1)} + O\left(\frac{y \log m}{m}\right)\right).$$

Thus

$$\frac{C(n, k_1 + y)}{C(n, k_1)} \sim \exp\left(-y^2 \frac{\log(m - i_1)}{(m - i_1)}\right) \sim \exp\left(-y^2 \frac{\log m}{m}\right).$$

Hence

$$\sum C(n, k) \sim \frac{C(n, k_1)\sqrt{\pi m^{1/2}}}{\sqrt{\log m}}.$$

and Theorem 2 follows.

**Numerical computations**

An exact computation of the sequence $\{C(n, k)\}$ and the numbers $C(n) = \sum_{k \geq 1} C(n, k)$ for reasonably small $n$ is easily accomplished using the recurrence relation $C(n, 1) = 1$,

$$C(n, k) = C(n-k, k) + kC(n-k, k-1).$$

We deduce this by subtracting 1 from each part of the distinct compositions of $n$ into $k$ parts. Then those distinct compositions in which no part is 1 have a one to one correspondence with distinct compositions of $n-k$ into $k$ parts, whereas those distinct compositions with a part 1 correspond to a distinct composition of $n-k$ into $k-1$ parts with an additional zero part which can occur in any of $k$ positions.

We provide below a brief table of the values of $C(n)$ and $\{C(n, k)\}$ for $1 \leq n \leq 20$.

| $n$ | $C(n)$ | $m$ | $\{C(n, k)\}$ |
|---|---|---|---|
| 1 | 1 | 1 | $\{1\}$ |
| 2 | 1 | 1 | $\{1\}$ |
| 3 | 3 | 2 | $\{1, 2\}$ |
| 4 | 3 | 2 | $\{1, 2\}$ |
| 5 | 5 | 2 | $\{1, 4\}$ |
| 6 | 11 | 3 | $\{1, 4, 6\}$ |
| 7 | 13 | 3 | $\{1, 6, 6\}$ |
| 8 | 19 | 3 | $\{1, 6, 12\}$ |
| 9 | 27 | 3 | $\{1, 8, 18\}$ |
| 10 | 57 | 4 | $\{1, 8, 24, 24\}$ |
| 11 | 65 | 4 | $\{1, 10, 30, 24\}$ |
| 12 | 101 | 4 | $\{1, 10, 42, 48\}$ |
| 13 | 133 | 4 | $\{1, 12, 48, 72\}$ |
| 14 | 193 | 4 | $\{1, 12, 60, 120\}$ |
| 15 | 351 | 5 | $\{1, 14, 72, 144, 120\}$ |
| 16 | 435 | 5 | $\{1, 14, 84, 216, 120\}$ |
| 17 | 617 | 5 | $\{1, 16, 96, 264, 240\}$ |
| 18 | 851 | 5 | $\{1, 16, 114, 360, 360\}$ |
| 19 | 1177 | 5 | $\{1, 18, 126, 432, 600\}$ |
| 20 | 1555 | 5 | $\{1, 18, 144, 552, 840\}$ |

In addition we compare the maximum value of $\{C(n, k)\}$ for a few larger values of $n$.

| $n$ | $C(n)$ | $m$ | $k_1$ | $C(n, k_1)$ | $\dfrac{C(n)}{C(n, k_1)}$ | $\left(\dfrac{\pi m}{\log m}\right)^{1/2}$ |
|---|---|---|---|---|---|---|
| 100 | $1.093 \times 10^{12}$ | 13 | 12 | $4.335 \times 10^{11}$ | 2.5232 | 1.2701 |
| 200 | $1.011 \times 10^{20}$ | 19 | 17 | $3.627 \times 10^{19}$ | 2.7884 | 1.4331 |
| 300 | $4.339 \times 10^{26}$ | 24 | 21 | $1.431 \times 10^{26}$ | 3.0316 | 1.5504 |

The asymptotic formula for $k_1$ when $m = 24$ predicts $k_1 = 24 - c^{-2}24/\log^2 24 \doteq 24 - 4.3$ so we expect $k_1$ to be 19 or 20 from the asymptotic formula for $n = 300$. Also the ratio of $C(n)$ to $C(n, k_1)$ is off by 20% for $m = 24$ or $n = 300$. Probably we should not expect better for such small values of $m$.

Finally we have only proved that $C(n, k)$ is a unimodal sequence for $n$ sufficiently large, say $n \geq n_0$. However, $n_0$ is a computable number. It seems very likely that $C(n, k)$ is unimodal for all $n$. We have not found an $n$ such that the maximum is attained for two values of $k$.

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