Extensions of Certain Homomorphisms of Subsemigroups to Homomorphisms of Groups

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Dedicated to L. Rédei on his 70th Birthday

1. In the Fall of 1969, at the Functional Equations Seminar of the Division of Mathematical Research of the University of Waterloo, several proofs of a theorem of J. Aczél and P. Erdös [2, 1] were presented. The theorem states that every solution of Cauchy's functional equation on the positive quarter plane,

$$f(x+y)=f(x)+f(y) \quad \text{for} \quad x>0, y>0, \tag{1}$$

can be extended to a solution g of the same equation on the whole plane,

$$g(x+y)=g(x)+g(y) \quad \text{for all} \quad (x,y)\in R^2,$$

so that

$$g(x)=f(x) \quad \text{for all} \quad x>0.$$

Then we have noticed, that our proofs can be applied to extensions of more general homomorphisms of subsemigroups ($R^+=\{x \mid x>0, x\in R\}$ is of course a subsemigroup of the addition group of reals) and tried to generalize further the results thus obtained. The results of our endeavours are contained in this paper. The problem is by no means settled. It would be desirable, for instance, to know under exactly what conditions can a homomorphism of a subsemigroup be extended to a group generated by it (not necessarily abelian). The section 1, 2, 3, 4 and 5 are works of J. Aczél, J. A. Baker, F. Radó, D. Ž. Djoković and PL. Kannappan, respectively.

We will write

$$f(xy)=f(x)f(y), \qquad x,y\in S \tag{2}$$

instead of (1). We first prove the following theorem.

THEOREM 1. *Let S be a subsemigroup of the group G, such that for every element $x\in G$, different from the unit element of G, either $x\in S$ or $x^{-1}\in S$ (or both). Then every homomorphism of S into a group H can be extended in a unique way to a homomorphism of G in the following sense:*

$$g(x)=f(x) \quad \textit{for all} \quad x\in S \tag{3}$$

and

$$g(xy)=g(x)g(y) \quad \textit{for all} \quad x,y\in G. \tag{4}$$

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Proof. Define

$$g(e)=E \quad \text{(the unit element of } H) \tag{5}$$

and

$$g(x)=f(x) \quad \text{if} \quad x\in S, \qquad g(x)=f(x^{-1})^{-1} \quad \text{if} \quad x^{-1}\in S. \tag{6}$$

By (6), evidently, (3) is satisfied. Here we have supposed that $e\notin S$ and if $x\in S$, then $x^{-1}\notin S$. Else (5) and the second part of (6) necessarily holds for f instead of g and g can be defined for e and for such x as $g(x)=f(x)$. Everything else goes in the same way. – We prove (4).

Evidently, (5) and (6) imply

$$g(x^{-1})=g(x)^{-1} \quad \text{for all} \quad x\in G. \tag{7}$$

LEMMA 1. *If*

$$g(xy)=g(x)\,g(y) \tag{8}$$

holds for a pair of elements x, y of G, then it holds also for y^{-1} and x^{-1} and vice versa.

Proof of the Lemma 1.

$$g(y^{-1}x^{-1}) = g[(xy)^{-1}] \overset{(7)}{=} g(xy)^{-1} \overset{(8)}{=} [g(x)\,g(y)]^{-1} = g(y)^{-1}\,g(x)^{-1} \overset{(7)}{=} g(y^{-1})\,g(x^{-1}).$$

Conclusion of the Proof of Theorem 1. We distinguish seven cases.

1) $x, y\in S$. Then (4) holds by (2).

2) $x=e$ or $y=e$. Then (4) holds by (5).

3) $x^{-1}\in S$, $y^{-1}\in S$. Then (4) holds by (2) and by the Lemma 1.

4) $x\in S$, $y^{-1}\in S$, $xy\in S$.

$$g(x) \overset{(6)}{=} f(x) = f(xy\cdot y^{-1}) \overset{(2)}{=} f(xy)\,f(y^{-1}) \overset{(6)}{=} g(xy)\,g(y)^{-1}.$$

Thus (4) holds again:

$$g(x)\,g(y)=g(xy).$$

5) $x^{-1}\in S$, $y\in S$, $xy\in S$. Proof similar to 4).

6) $x\in S$, $y^{-1}\in S$, $(xy)^{-1}\in S$. Then $y^{-1}\in S$, $(x^{-1})^{-1}\in S$, $y^{-1}x^{-1}\in S$, so that for y^{-1} and x^{-1} we have the case 4). Now, (4) follows from the Lemma 1.

7) $x^{-1}\in S$, $y\in S$, $(xy)^{-1}\in S$. Proof similar to 6).

This exhausts all possible cases and (4) is thus proved.

Finally, the uniqueness of the extension is evident: every homomorphism of G into H satisfies (5) and (7), further (3) and (7) imply (6).

This concludes the proof of Theorem 1. A similar (but not identical) theorem was proved in [3].

2. THEOREM 2. *Let the subsemigroup S generate the abelian group G and let f be a homomosphism of S into the abelian group H. Then there exists a unique extension g of f to a homomorphism of G into H.*

Proof. The semigroup S generates the abelian group G iff

$$G = S \cdot S^{-1} = \{xy^{-1} \mid x \in S, y \in S\}. \tag{9}$$

Now define

$$g(xy^{-1}) = f(x) f(y)^{-1}, (x, y \in S). \tag{10}$$

By this formula (10), g is well defined on G, because

$$xy^{-1} = uv^{-1}, (x, y, u, v \in S)$$

implies $xv = uy$, thus [(2)]

$$f(x) f(v) = f(u) f(y) \quad \text{or} \quad f(x) f(y)^{-1} = f(u) f(v)^{-1}.$$

Also, g is a homomorphism of G into H, since for every $z \in G$, $w \in G$, there exist $x, y, u, v \in S$, such that $z = xy^{-1}$, $w = uv^{-1}$ and

$$g(zw) = g(xy^{-1}uv^{-1}) = g[(xu)(vy)^{-1}]$$
$$\overset{(10)}{=} f(xu) f(vy)^{-1} \overset{(2)}{=} f(x) f(u) f(y)^{-1} f(v^{-1}) = g(z)\, g(w).$$

It is also an extension of f, since for all $x \in S$

$$g(x) = g(xx \cdot x^{-1}) = f(xx) f(x)^{-1} = f(x) f(x) f(x)^{-1} = f(x). \tag{11}$$

Finally, the extension g is unique, that is, all homomorphisms of G, equal to f on S, are of the form (10), because for an arbitrary such homomorphism h

$$h(xy^{-1}) = h(x) h(y)^{-1} = f(x) f(y)^{-1} = g(xy^{-1}) \quad \text{for all} \quad x, y \in S.$$

This concludes the proof of Theorem 2. A similar (but not identical) theorem was proved in [3]. The present proof has been used in a less general context in functional analysis, for example [4].

3. If G is not abelian, then (9) is only a very special way of S generating G. Also, most of the steps in the proof of Theorem 2 are not valid anymore. However, one succeeds to avoid these difficulties at least in the case (9) in the following way.

THEOREM 3. *Let G, H be groups, S a subsemigroup of G such that*

$$G = S \cdot S^{-1} = \{xy^{-1} \mid x \in S, y \in S\} \tag{9}$$

and let f be a homomorphism of S into H. Then f can be extended to a homomorphism g of G into H in a unique way.

Proof. The assumption (9) will be applied also in the following form:

$$\text{for all } x \in G \text{ there exist } y \in S \text{ such that } xy \in S. \tag{12}$$

Notice that

$$x\in S,\ x^{-1}y\in S \Rightarrow f(x^{-1}y)=f(x)^{-1}f(y). \tag{13}$$

Indeed, putting $z=x^{-1}y$, we have $y=xz$ and, by (2),

$$f(y)=f(x)f(z)$$

which gives (13).

Now we need the following.

LEMMA 2. *Whenever*

$$x_1,y_1,z_1,x_2,y_2,z_2\in S,\ x_1y_1^{-1}z_1=x_2y_2^{-1}z_2, \tag{14}$$

then

$$f(x_1)f(y_1)^{-1}f(z_1)=f(x_2)f(y_2)^{-1}f(z_2). \tag{15}$$

Proof of Lemma 2. Choose $s_1\in S$ so that $y_1^{-1}z_1s_1\in S$. This is possible by (12). Similarly, it is now possible to choose an $s_2\in S$ so that $y_2^{-1}z_2s_1s_2\in S$. Denoting $s=s_1s_2$, we have

$$s\in S,\ y_1^{-1}z_1s\in S,\ y_2^{-1}z_2s\in S.$$

From (14)

$$x_1y_1^{-1}z_1s=x_2y_2^{-1}z_2s$$

follows and, by (2),

$$f(x_1)f(y_1^{-1}z_1s)=f(x_2)f(y_2^{-1}z_2s),$$

or, by (13),

$$f(x_1)f(y_1)^{-1}f(z_1s)=f(x_2)f(y_2)^{-1}f(z_2s).$$

By applying (2) again and cancelling $f(s)$, we get (15) which proves the Lemma 2.

Conclusion of the Proof of Theorem 3. Choose in Lemma 2 $x_1=x$, $y_1=yy$, $z_1=y$, $x_2=u$, $y_2=vv$, $z_2=v$ in order to get, with aid of (2),

$$xy^{-1}=uv^{-1}\Rightarrow f(x)f(y)^{-1}=f(u)f(v)^{-1} \tag{16}$$

as in the proof of Theorem 2. So the definition

$$g(xy^{-1})=f(x)f(y)^{-1}\ (x,y\in S) \tag{10}$$

again defines g unambiguously on G. Again, (11) shows that g is an extension of f.

Also, g is a homomorphism of G into H. Let again $z\in G$, $w\in G$ be arbitrary. Then, by (9), there exist x, y, u, v, s, t so, that

$$z=xy^{-1},\ w=uv^{-1},\quad\text{and}\quad xy^{-1}uv^{-1}=zw=st^{-1} \tag{17}$$

with $x, y, u, v, s, t\in S$. The last equation in (17) gives $xy^{-1}u=st^{-1}v$, and this, by Lemma 2,

$$f(x)f(y)^{-1}f(u)=f(s)f(t)^{-1}f(v),$$

or

$$f(x)\,f(y)^{-1}\,f(u)\,f(v)^{-1}=f(s)\,f(t)^{-1}.$$

By (10) and (17), this goes over into

$$g(z)\,g(w)=g(zw),$$

as claimed. Finally, the uniqueness of g is proved exactly as in Theorem 2, and this concludes the proof of Theorem 3.[1])

4. In order to get extensions of homomorphisms of subsemigroups to groups under more general conditions, we would have to examine the cases $G=S\cdot S^{-1}\cdot S$, $G=S\cdot S^{-1}\cdot S\cdot S^{-1}$ and so on. However, even in the first two cases, we can prove theorems similar to Theorem 3 only under further assumptions. (Of course the Øre conditions [5, cf. 6] would be obviously sufficient.)

Let S be a subsemigroup of a group G and $f:S\to H$ a semigroup homomorphism into a group H. We assume that S generates G. Theorems 3–5 give some sufficient conditions in order that f has an extension to a group homomorphism $g:G\to H$. The following example shows that they are not necessary. Let G be the free group on two free generators a and b and let S be the subsemigroup (with identity or not) generated by a and b. It is clear that in this case every f can be extended to a group homomorphism. Of course, G is generated as a group by S but the conditions of Theorem 3 are not satisfied, and neither are those in the following Theorems 4 and 5.

One can choose G, S, H, f so that *f cannot be extended to a group homomorphism $G\to H$*. For instance, let G be as above and let S be the subsemigroup (with identity or not) generated by a, b and $ab^{-1}a$. Then S is freely generated by these three elements. Let $H=G$ and let $f:S\to G$ be the semigroup homomorphism which is uniquely defined by

$$f(a)=a\,,\quad f(b)=b\,,\quad f(ab^{-1}a)=1\,.$$

It is evident that this f cannot be extended to an endomorphism of G. So, contrary to the abelian case, in general the homomorphisms of subsemigroups can not be extended to the groups generated by them.

On the other hand, let S be a subsemigroup of a group G and let S generate G. If S is invariant, i.e. $x^{-1}Sx\subset S$ for all $x\in G$ then one can easily show that $G=SS^{-1}$ and, hence, Theorem 3 can be applied. Other sufficient conditions are obtained in the following theorems.

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[1]) It was pointed out by Pl. Kannappan and, independently, by one of the referees that Theorem 3 can also be proved in a way similar to the proofs of Theorems 4 and 5. – Of course, Theorems 1 and 2 are special cases of Theorem 3.

THEOREM 4. *Let G, H be groups, S a subsemigroup of G such that*

$$G = S \cdot S^{-1} \cdot S = \{xy^{-1}z \mid x, y, z \in S\}, \tag{18}$$

and let f be a homomorphism of S into H. The map f can be extended to a homomorphism g of G into H iff

$$x, y, u, v \in S, \; xy^{-1} = uv^{-1} \Rightarrow f(x) f(y)^{-1} = f(u) f(v)^{-1}. \tag{19}$$

When the extension exists, then it is unique.

As we see, (16), which was a consequence of the suppositions in Theorems 2 and 3, had to be postulated here additionally.

Proof of Theorem 4. We first prove, as in Lemma 2,

$$x, y, z, u, v, w \in S, \; xy^{-1}z = uv^{-1}w \Rightarrow f(x) f(y)^{-1} f(z) = f(u) f(v)^{-1} f(w). \tag{20}$$

In fact, by (18) there exist r, s, t in S such that

$$y^{-1}zw^{-1} = rs^{-1}t \tag{21}$$

and the hypothesis of (20) implies

$$uv^{-1} = xy^{-1}zw^{-1} = xrs^{-1}t. \tag{22}$$

From (21) and (22)

$$xrs^{-1} = uv^{-1}t^{-1} \quad \text{and} \quad yrs^{-1} = zw^{-1}t^{-1}$$

follow. Again (2) and (19) imply

$$f(x) f(r) f(s)^{-1} = f(u) f(tv)^{-1} = f(u) f(v)^{-1} f(t)^{-1}$$

and

$$f(y) f(r) f(s)^{-1} = f(z) f(w)^{-1} f(t)^{-1}.$$

So

$$f(x)^{-1} f(u) f(v)^{-1} = f(r) f(s)^{-1} f(t) = f(y)^{-1} f(z) f(w)^{-1},$$

which indeed gives (20)

$$f(x) f(y)^{-1} f(z) = f(u) f(v)^{-1} f(w).$$

Because of (20), the following mapping g is well defined on G.

$$g(xy^{-1}z) = f(x) f(y)^{-1} f(z) \; (x, y, z \in S). \tag{23}$$

This is an extension of f. Indeed, take $z = y$ in order to get

$$g(x) = f(x) \quad \text{for all} \quad x \in S.$$

We proceed to show that g is a group homomorphism. For arbitrary p, $q \in G$ there exist, by (18), $x, y, z, u, v, w, r, s, t, a, b, c$ in S such that

$$p = xy^{-1}z, \quad q = uv^{-1}w, \quad pq = xy^{-1}zuv^{-1}w = rs^{-1}t, \tag{24}$$

and

$$y^{-1}zuv^{-1} = ab^{-1}c.$$

Thus

$$xab^{-1}cw = rs^{-1}t, \qquad zuv^{-1} = yab^{-1}c$$

and, by (20) and (2),

$$f(x)f(a)f(b)^{-1}f(c)f(w) = f(r)f(s)^{-1}f(t),$$
$$f(z)f(u)f(v)^{-1} = f(y)f(a)f(b)^{-1}f(c).$$

From these two equations we get

$$f(x)f(y)^{-1}f(z)f(u)f(v)^{-1}f(w) = f(x)f(a)f(b)^{-1}f(c)f(w)$$
$$= f(r)f(s)^{-1}f(t).$$

So, with (23) and (24) we have

$$g(p)\,g(q) = g(pq) \quad \text{for all} \quad p, q \in G,$$

as claimed. The converse is obvious. Finally, g is a unique extension, since for all homomorphisms h of G, satisfying (3), we have

$$h(xy^{-1}z) = h(x)\,h(y)^{-1}\,h(z) = f(x)f(y)^{-1}f(z)$$
$$= g(xy^{-1}z) \quad \text{for all} \quad x, y, z \in S.$$

This concludes the proof of Theorem 4.

5. THEOREM 5. *Let G, H be groups, S a subsemigroup of G such that*

$$G = S\cdot S^{-1}\cdot S\cdot S^{-1} = \{xy^{-1}uv^{-1} \mid x, y, u, v \in S\} \tag{25}$$

and let f be a homomorphism of S into H. The map f can be extended to a homomorphism g of G into H iff

$$x, y, z, u, v, w \in S,\ xy^{-1}z = uv^{-1}w \Rightarrow f(x)f(y)^{-1}f(z) = f(u)f(v)^{-1}f(w). \tag{20}$$

When the extension exists, then it is unique.

Again, condition (20), which in 3 and 4 we were able to deduce from (13) resp. from (18) and (19), had to be supposed here.

Proof of Theorem 5. First we prove

$$\begin{gathered} x, y, z, w, s, t, u, v \in S,\ xy^{-1}zw^{-1} = st^{-1}uv^{-1} \Rightarrow \\ f(x)f(y)^{-1}f(z)f(w)^{-1} = f(s)f(t)^{-1}f(u)f(v)^{-1}. \end{gathered} \tag{26}$$

In fact, by (25) there exist $a, b, c, d \in S$ such that

$$y^{-1}zw^{-1}v = ab^{-1}cd^{-1} \tag{27}$$

and the hypothesis of (26) implies that

$$xab^{-1}cd^{-1} = xy^{-1}zw^{-1}v = st^{-1}u, \quad \text{or} \quad xab^{-1}c = st^{-1}ud.$$

Also, by (27),

$$zw^{-1}vd = yab^{-1}c.$$

Thus, by (20) and (2), we get

$$f(x)f(a)f(b)^{-1}f(c) = f(s)f(t)^{-1}f(u)f(d)$$

and

$$f(z)f(w)^{-1}f(v)f(d) = f(y)f(a)f(b)^{-1}f(c),$$

or

$$\begin{aligned} f(x)f(y)^{-1}f(z)f(w)^{-1}f(v)f(d) &= f(x)f(a)f(b)^{-1}f(c) \\ &= f(s)f(t)^{-1}f(u)f(d). \end{aligned}$$

This gives exactly

$$f(x)f(y)^{-1}f(z)f(w)^{-1} = f(s)f(t)^{-1}f(u)f(v)^{-1},$$

as asserted in (26). Thus the following mapping g is well defined on G.

$$g(xy^{-1}zw^{-1}) = f(x)f(y)^{-1}f(z)f(w)^{-1} \quad (x, y, z, w \in S). \tag{28}$$

Putting $z=yy$, $w=y$, we see that g is an extension of f:

$$g(x) = f(x)f(y)^{-1}f(y)f(y)f(y)^{-1} = f(x) \quad \text{for all} \quad x \in S.$$

Now we prove that g is a homomorphism of G. For arbitrary $p, q \in G$, there exist, by (25) $x, y, z, w, s, t, u, v, k, m, n, r, a, b, c, d \in S$ so that

$$p = xy^{-1}zw^{-1}, \quad q = st^{-1}uv^{-1}, \quad pq = xy^{-1}zw^{-1}st^{-1}uv^{-1} = km^{-1}nr^{-1} \tag{29}$$

and

$$y^{-1}zw^{-1}st^{-1}u = ab^{-1}cd^{-1}.$$

Thus

$$xab^{-1}cd^{-1}v^{-1} = km^{-1}nr^{-1} \quad \text{and} \quad zw^{-1}st^{-1} = yab^{-1}cd^{-1}u^{-1}.$$

By (26) and (2),

$$\begin{aligned} f(x)f(a)f(b)^{-1}f(c)f(d)^{-1}f(v)^{-1} &= f(xa)f(b)^{-1}f(c)f(vd)^{-1} \\ &= f(k)f(m)^{-1}f(n)f(r)^{-1} \end{aligned} \tag{30}$$

and

$$\begin{aligned} f(z)f(w)^{-1}f(s)f(t)^{-1} &= f(ya)f(b)^{-1}f(c)f(ud)^{-1} \\ &= f(y)f(a)f(b)^{-1}f(c)f(d)^{-1}f(u)^{-1}. \end{aligned} \tag{31}$$

So we get

$$f(x)f(y)^{-1}f(z)f(w)^{-1}f(s)f(t)^{-1}f(u)f(v)^{-1}$$
$$\overset{(31)}{=} f(x)f(a)f(b)^{-1}f(c)f(d)^{-1}f(v)^{-1} \overset{(30)}{=} f(k)f(m)^{-1}f(n)f(r)^{-1}.$$

In view of (28) and (29), this is exactly

$$g(p)g(q) = g(pq) \quad \text{for all} \quad p, q \in G$$

which had to be proved. The converse is again obvious. The proof of the uniqueness of the extension g of f is quite analogous to the uniqueness proofs in Theorems 2–4. This concludes the proof of Theorem 5.

Of course, one could continue, with only the technical difficulties increasing, on the path opened by Theorems 1, 4, and 5. But to find *necessary and* sufficient conditions for the extendability of a subsemigroup-homomorphism to a homomorphism of the whole nonabelian group seems to require new ideas.

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