The Functional Equation $f(x+y-xy)+f(xy)=f(x)+f(y)$

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THEOREM. *The equation*

$$f(x+y-xy)+f(xy)=f(x)+f(y) \tag{1}$$

where x, y are real variables is equivalent to Jensen's equation

$$f(x)+f(y)=2f\left(\frac{x+y}{2}\right). \tag{2}$$

The same holds if x, y are complex variables.

Proof. First we show that (2) implies (1).

Putting $y=0$ in (2) and replacing x by $x+y$ we obtain

$$f(x+y)+f(0)=2f\left(\frac{x+y}{2}\right)=f(x)+f(y) \tag{3}$$

and, for $y=-x$,

$$2f(0)=f(x)+f(-x). \tag{4}$$

Furthermore, on account of (3) and (4),

$$\left.\begin{aligned} f(x-y) &= f(x+(-y)) = -f(0)+f(x)+f(-y) \\ &= -f(0)+f(x)+2f(0)-f(y)=f(x)-f(y)+f(0). \end{aligned}\right\} \tag{5}$$

Therefore, by (5) and (3),

$$\begin{aligned} f(x+y-xy)+f(xy) &= f(x+y)-f(xy)+f(0)+f(xy) \\ &= f(x+y)+f(0)=f(x)+f(y). \end{aligned}$$

Thus (2) implies (1).

The inverse implication can be shown as follows.

Put

$$f(x)=\varphi(x)+\frac{x}{4}[\varphi(-2)-\varphi(2)]-\frac{1}{2}[\varphi(-2)+\varphi(2)].$$

Then

$$f(-2)=f(2)=0. \tag{6}$$

If φ satisfies (1), so does f and vice versa, as is readily seen. Also, if f satisfies (2), so does φ and vice versa. Thus it suffices to show that every solution f of (1) with the properties (6) satisfies (2).

We shall need different forms of (1) which are obtained by certain substitutions. Let the symbol $[X, Y]$ mean the substitution by which in (1) x, y are to be replaced

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by X, Y. Then, taking (6) into account, the substitutions

$$[-x,-y], [x,2], [-x,2], [x,-2], [-x,-2], [x+2,y+2],$$
$$[-x+2,-y+2], \left[\frac{x}{2}+1,2\right], \left[-\frac{x}{2}+1,2\right], \left[\frac{x}{2},-1\right], \left[-\frac{x}{2},-1\right]$$

yield the following equations:

$$f(-x-y-xy)+f(xy)=f(-x)+f(-y), \tag{7}$$
$$f(-x+2)+f(2x)=f(x), \tag{8}$$
$$f(x+2)+f(-2x)=f(-x), \tag{9}$$
$$f(3x-2)+f(-2x)=f(x), \tag{10}$$
$$f(-3x-2)+f(2x)=f(-x), \tag{11}$$
$$f(-x-y-xy)+f(2x+2y+xy+4)=f(x+2)+f(y+2), \tag{12}$$
$$f(x+y-xy)+f(-2x-2y+xy+4)=f(-x+2)+f(-y+2), \tag{13}$$
$$f\left(-\frac{x}{2}+1\right)+f(x+2)=f\left(\frac{x}{2}+1\right), \tag{14}$$
$$f\left(\frac{x}{2}+1\right)+f(-x+2)=f\left(-\frac{x}{2}+1\right), \tag{15}$$
$$f(x-1)+f\left(\frac{-x}{2}\right)=f\left(\frac{x}{2}\right)+f(-1), \tag{16}$$
$$f(-x-1)+f\left(\frac{x}{2}\right)=f\left(-\frac{x}{2}\right)+f(-1). \tag{17}$$

Summing (14) and (15) we have

$$f(x+2)+f(-x+2)=0. \tag{18}$$

Summing (8) and (9) and using (18) we get

$$f(2x)+f(-2x)=f(x)+f(-x). \tag{19}$$

By summing (10) and (11) and using (19) it follows that

$$f(3x-2)+f(-3x-2)=0$$

or, writing x for $3x$,

$$f(x-2)+f(-x-2)=0. \tag{20}$$

Now, in (18) replace x by $x+6$ and in (20) replace x by $x+2$. By subtracting we obtain

$$f(x+8)-f(x)=0, \tag{21}$$

i.e. periodicity with the period 8.

Summing (12) and (13) and using (18) we have

$$\left.\begin{array}{l} f(-x-y-xy)+f(x+y-xy)+f(-2x-2y+xy+4)+ \\ \qquad +f(2x+2y+xy+4)=0. \end{array}\right\} \tag{22}$$

By (18)

$$f(-x-y-xy)=f(-x-y-xy-2+2)=-f(x+y+xy+2+2)=$$
$$=-f(x+y+xy+4)$$

and, analogously,

$$f(x+y-xy)=-f(-x-y+xy+4).$$

Thus (22) becomes

$$f(xy+4+x+y)+f(xy+4-x-y)=f(xy+4+2x+2y)+$$
$$+f(xy+4-2x-2y).$$

The substitution

$$xy+4=\xi, \quad x+y=\eta$$

yields

$$f(\xi+\eta)+f(\xi-\eta)=f(\xi+2\eta)+f(\xi-2\eta) \tag{23}$$

where ξ, η are submitted to the restriction

$$\xi \leqslant \frac{\eta^2}{4}+4 \tag{24}$$

if x, y are real.

In order to get rid of this restriction we consider the substitution

$$xy=-\xi, \quad x+y=\eta$$

which yields

$$f(-\xi+\eta+4)+f(-\xi-\eta+4)=f(-\xi+2\eta+4)+f(-\xi-2\eta+4) \tag{25}$$

with the restriction

$$\xi \geqslant -\frac{\eta^2}{4}. \tag{26}$$

But, by (18),

$$f(-\xi+\eta+4)=f(-\xi+\eta+2+2)=-f(\xi-\eta-2+2)=-f(\xi-\eta)$$

and, analogously,

$$f(-\xi-\eta+4)=-f(\xi+\eta),$$
$$f(-\xi+2\eta+4)=-f(\xi-2\eta),$$
$$f(-\xi-2\eta+4)=-f(\xi+2\eta),$$

so that (25) again implies (23). Clearly, at least one of the inequalities (24) and (26) is satisfied, and (23) is valid for any pair (ξ, η) of real numbers.

Replacing η by 2η in (23) one gets immediately

$$f(\xi+\eta)+f(\xi-\eta)=f(\xi+4\eta)+f(\xi-4\eta) \tag{27}$$

and, by iteration,

$$f(\xi+\eta)+f(\xi-\eta)=f(\xi+2^n\eta)+f(\xi-2^n\eta). \tag{28}$$

For $\xi=0$, $\eta=2$, (27) entails, on account of (6) and (21),

$$f(0)=0. \tag{29}$$

For $\xi=-1$, $\eta=1$, $n=3$, we obtain from (28)

$$f(-1)=0, \tag{30}$$

(6), (29) and (21) being taken into account.

Summing (16) and (17) we have, by (30),

$$f(x-1)+f(-x-1)=0. \tag{31}$$

In (20) replace x by $x+2$:

$$f(x)+f(-x-4)=0. \tag{32}$$

In (31) replace x by $x+3$:

$$f(x+2)+f(-x-4)=0. \tag{33}$$

(32) and (33) yield periodicity with the period 2:

$$f(x+2)=f(x).$$

Since, therefore, 4 is also a period, (32) implies

$$f(x)+f(-x)=0, \tag{34}$$

i.e. the function f is odd.

Now we could use a result of H. Światak [2] which says that if $x \to f(x)-f(0)$ is odd, f being a solution of (1), then f satisfies (2). However, for the sake of completeness, we give a somewhat different argument.

Summing (1) and (7) we have, using (34),

$$f(x+y-xy)+f(-x-y-xy)+2f(xy)=0.$$

The substitution

$$x+y-xy=\xi, \qquad -x-y-xy=\eta$$

yields

$$f(\xi)+f(\eta)=-2f\left(-\frac{\xi+\eta}{2}\right)=2f\left(\frac{\xi+\eta}{2}\right) \tag{35}$$

where again (34) was used, and ξ, η are submitted to the restriction

$$(\xi-\eta)^2+8(\xi+\eta)\geqslant 0 \tag{36}$$

if x, y are real.

Consider the slightly different substitution

$$x + y - xy = -\xi, \qquad -x - y - xy = -\eta.$$

We obtain

$$f(-\xi) + f(-\eta) = -2f\left(\frac{\xi+\eta}{2}\right)$$

or, by (34), Jensen's equation (35), the restriction being this time

$$(\xi - \eta)^2 - 8(\xi + \eta) \geqslant 0. \tag{37}$$

Obviously one at least of the restrictions (36) and (37) is satisfied, which completes the proof of our theorem.

If x, y are complex variables, all considerations remain valid, however the restrictions (24), (26), (36), (37) are unnecessary.

The general solution of Jensen's equation is, of course, well known. It is obtained by adding an arbitrary constant to the general solution of Cauchy's equation

$$f(x + y) = f(x) + f(y).$$

Equation (1) was considered first by M. Hosszú who has solved it under a differentiability assumption. It has also been treated by other authors ([1], [2], [3]). They have shown that under some restrictions (integrability, continuity in some points) the solutions are of the form $Ax+B$. As we have already mentioned H. Światak [2] has shown that (1) and $x \to f(x) - f(0)$ odd imply Jensen's equation. Equation (34) was therefore, the main aim of our considerations[1]).

REFERENCES

[1] Daróczy, Z., *Über die Funktionalgleichung* $f(xy)+f(x+y-xy)=f(x)+f(y)$, Publ. Math. Debrecen (to appear).

[2] Światak, H., *On the Functional Equation* $f(x+y-xy)+f(xy)=f(x)+f(y)$, Mat. Vesnik *5* *(20)*, 177–182 (1968).

[3] Światak, H., *Remarks on the Functional Equation* $f(x+y-xy)+f(xy)=f(x)+f(y)$, Aequationes Math. *1*, 239–241 (1968).

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[1]) As one of the referees of the present paper has pointed out, the same result has been reached by Z. Daróczy, independently and at the same time, but by a completely different method of proof. No details of that proof were available at the time of resubmitting this paper.

Remark of the editors: The paper of Z. Daróczy will be published in Aequationes Math.