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Note on almost additive functions

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P. Erdös [2] raised the following problem: suppose that we are given a function $f:\mathbf{R}\to\mathbf{R}$ such that

$$f(x+y)=f(x)+f(y) \tag{1}$$

holds for almost all pairs $(x, y)\in\mathbf{R}^2$ (an almost additive function). Does there exist an additive function $g:\mathbf{R}\to\mathbf{R}$ such that $f(x)=g(x)$ for almost all $x\in\mathbf{R}$? N. G. de Bruijn [1] (and, independently, W. B. Jurkat [3]) gave, using notation and terminology slightly different from those used here, a positive answer to this question and generalized this result to cummutative groups as follows: Let $(G,+)$ and $(H,+)$ be two abelian groups and let $\mathcal{T}\subset 2^G\setminus\{G\}$ be a nonempty set-family closed under finite unions, hereditary with respect to descending inclusions and such that together with a set U it contains also the family $\{x-U: x\in G\}$. Every such family is said to be a proper linearly invariant set ideal (abbreviated to p.l.i. ideal in the sequel). It turns out that if $\mathcal{T}$ is a p.l.i. ideal of subsets of G, then

$$\Omega(\mathcal{T}):=\left\{M\subset G^2: \bigvee_{U(M)\in\mathcal{T}}\ \bigwedge_{x\in G\setminus U(M)} V_x(M):=\{y\in G: (x, y)\in M\}\in\mathcal{T}\right\}$$

is a p.l.i. ideal in the product group $(G^2, +)$. De Bruijn's result states that for every function $f: G\to H$ such that (1) holds for all $(x, y)\in G^2\setminus M$, $M\in\Omega(\mathcal{T})$, there exists a homomorphism $F: G\to H$ such that the set $\{x\in G: f(x)\neq F(x)\}$ is a member of $\mathcal{T}$.

In the present note we are going to extend de Bruijn's result to the non-abelian case. We shall preserve the symbols used above with the only change that $(G,+)$ and $(H,+)$ are not assumed to be commutative. However, the additive notation will be preserved.

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Remark. Let $\mathcal{T}$ be a p.l.i. ideal in G. If $U \in \mathcal{T}$ and $x \in G$, then $x+U \in \mathcal{T}$ and $U+x \in \mathcal{T}$. In fact, observe that $-U=0-U \in \mathcal{T}$. Hence $x-(-U)=x+U \in \mathcal{T}$. Since $-(U+x)=-x-U \in \mathcal{T}$, we have $U+x=-(-(U+x)) \in \mathcal{T}$.

In the sequel, we shall make use of this remark without explicitly referring to it.

Suppose that a p.l.i. ideal $\mathcal{T}$ in G is given and that a function $f: G \to H$ satifies (1) for all pairs $(x, y) \in G^2 \backslash M$, $M \in \Omega(\mathcal{T})$. Fix an $x \in G$ and choose $w(x) \in G \backslash [U(M) \cup (-U(M)+x)]$. Then $w(x) \notin U(M)$ and $x-w(x) \notin U(M)$. Hence

$$A_x := V_{w(x)} \cup (-w(x)+V_{x-w(x)}) \in \mathcal{T} \qquad \text{for all } x \in G. \tag{2}$$

LEMMA. *For every $x \in G$, the difference $f(x+y)-f(y)$ is constant on G/A_x.*

Proof. Take a $y \in G \backslash A_x$. Then $(w(x), y) \notin M$ as well as $(x-w(x),\ w(x)+y) \notin M$. Hence

$$f(x+y)=f(x-w(x)+w(x)+y)=f(x-w(x))+f(w(x)+y)$$
$$=f(x-w(x))+f(w(x))+f(y),$$

i.e.

$$f(x+y)-f(y)=f(x-w(x))+f(w(x)).$$

This ends the proof, since the right-hand side of the latter equality does not depend on y.

THEOREM. *Suppose that a p.l.i. ideal $\mathcal{T}$ in G is given and that a function $f: G \to H$ satisfies* (1) *for all pairs $(x, y) \in G^2 \backslash M$, $M \in \Omega(\mathcal{T})$. There exists exactly one homomorphism $F: G \to H$ such that the set $T:=\{x \in G : f(x) \neq F(x)\} \in \mathcal{T}$.*

Proof. Define a function $F: G \to H$ by the formula

$$F(x):=f(x+y)-f(y), \qquad x \in G, \qquad y \in G \backslash A_x. \tag{3}$$

The preceding lemma ensures that this definition is correct.

First, we shall show that F is a homomorphism. To this end, fix arbitrarily u and v from G and choose elements $x, s, t \in G$ so that

$$x \notin A_{u+v} \cup U(M) \cup [-(u+v)+U(M)],$$

$$s \notin (-x+A_v) \cup V_x(M) \cup U(M) \cup [-(v+x)+U(M)]$$

and

$$t \notin [-(v+x+s)+A_u] \cup V_{v+x+s}(M) \cup V_s(M) \cup [-s+V_{u+v+x}(M)].$$

This is possible in view of (2), of the fact that $U(M)$, $V_x(M)$, $V_s(M)$, $V_{v+x+s}(M)$ and $V_{u+v+x}(M)$ belong to $\mathcal{T}$ and of the properties of $\mathcal{T}$.
Hence

$$x \notin A_{u+v}, \qquad y:=x+s \notin A_v, \qquad z:=v+x+s+t \notin A_u,$$
$$f(y)=f(x+s)=f(x)+f(s),$$
$$-f(v+x+s)=f(t)-f(v+x+s+t)=f(t)-f(z),$$
$$(s,t) \notin M \quad \text{and} \quad (u+v+x, s+t) \notin M.$$

Consequently, on account of (3),

$$\begin{aligned}
F(u+v)-F(v)-F(u) &= \\
&= f(u+v+x)-f(x)-[f(v+y)-f(y)]-[f(u+z)-f(z)] \\
&= f(u+v+x)-f(x)+f(y)-f(v+y)+f(z)-f(u+z) \\
&= f(u+v+x)+f(s)-f(v+x+s)+f(z)-f(u+z) \\
&= f(u+v+x)+f(s)+f(t)-f(u+z) \\
&= f(u+v+x)+(s+t)-f(u+z) \\
&= f(u+v+x+s+t)-f(u+z)=0.
\end{aligned}$$

Now, take an $x \in G \setminus U(M)$. Choose a $y \in G \setminus [A_x \cup V_x(M)]$. Then

$$F(x)=f(x+y)-(y)=f(x)$$

by (3) and the fact that $(x, y) \notin M$. Thus,

$$T:=\{x \in G: f(x) \neq F(x)\} \subset U(M) \in \mathcal{T},$$

whence $T \in \mathcal{T}$.

To prove the uniqueness, suppose that we are given two homomorphisms F_1, F_2 of G into H such that $F_1(x)=F_2(x)$ for all $x \in G \setminus U(M)$. Take an $s \in G$ and a

$t \in G \setminus [U(M) \cup (-U(M) + s)]$. Then

$$F_1(s) = F_1(s-t) + F_1(t) = F_2(s-t) + F_2(t) = F_2(s),$$

i.e. $F_1 = F_2$, which finishes the proof.

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