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On τ_T semigroups of probability distribution functions II

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0. Introduction

In this paper we continue the study of τ_T semigroups begun in [8]. Let Δ be the space of one-dimensional probability distribution functions and let T be a *t-norm* (i.e., a suitable semigroup on [0, 1]). Then, for any F, $G \in \Delta$ and any real x, we define

$$\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v)). \tag{0.1}$$

The space Δ under the operation τ_T is then a semigroup, called a τ_T *semigroup.*

This paper consists of three sections. In the first section we study iseomorphisms among τ_T semigroups. Since, for any *t-norm* T, ([0, 1], T) is a semigroup, we can relate properties of the τ_T semigroups to properties of the *t-norms* on [0, 1]. We first show that if T_1, T_2 are left-continuous *t-norms* such that ([0, 1], T_1) and ([0, 1], T_2) are iseomorphic then the corresponding τ_T semigroups are iseomorphic in the topology of weak convergence. Additional results establish a partial converse to this theorem. For any Archimedean *t-norm* T, we show that τ_T can be represented in terms of τ_{Prod}. If T is strict, then τ_T and τ_{Prod} are iseomorphic. If T is Archimedean, but not strict, then τ_T and τ_{T_m} are iseomorphic where $T_m(x, y) = \max\{x+y-1, 0\}$. For distribution functions concentrated on $[0, \infty)$ we show that no τ_T semigroup is isomorphic to the convolution semigroup.

In Section 2 we solve equations in τ_T semigroups. Given $F, H \in \Delta$ and a left-continuous *t-norm* T, we construct the maximal solution G to the inequality $\tau_T(F, G) \leq H$. Hence, if the corresponding equation, $\tau_T(F, G) = H$, has a solution then the G we construct is a solution. In addition, we exhibit a necessary

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condition on F, H for the existence of a solution to the above equation and show that, in some cases, this condition is also sufficient.

In the final section we consider a class of operations on the unit interval introduced by A. Sklar [15] and named *copulas* by him. These functions are used to relate joint distribution functions and their marginal distributions. We show that a *t-norm* T is a copula if and only if it satisfies the simple Lipschitz condition $T(a, c) - T(b, c) \leq a - b$ for all a, b, c in $[0, 1]$ with $a \geq b$.

To keep this paper reasonably self-contained we state some definitions and known facts:

DEFINITION 0.1. The spaces of probability distribution functions which we will consider are:

$$\Delta = \{F: \mathbb{R} \to [0, 1] \mid F \text{ is left-continuous and non-decreasing}\},$$
$$\Delta^+ = \{F \in \Delta \mid F(0) = 0\},$$
$$\mathcal{D} = \{F \in \Delta \mid \inf_x F(x) = 0 \quad \text{and} \quad \sup_x F(x) = 1\},$$
$$\mathcal{D}^+ = \Delta^+ \cap \mathcal{D}.$$

DEFINITION 0.2. A *t-norm* is any two-place function $T: [0, 1] \times [0, 1] \to [0, 1]$ satisfying

(a) $T(a, 1) = a$,
(b) $T(c, d) \geq T(a, b)$, for $c \geq a$, $d \geq b$,
(c) $T(a, b) = T(b, a)$,
(d) $T(T(a, b), c) = T(a, T(b, c))$.

We say that a *t-norm* T is *left-continuous* if it is left-continuous as a two-place function. We then have ([11], [13]):

THEOREM 0.1. *Let T be a left-continuous t-norm and, for any F, G in Δ, let the operation τ_T be defined as in* (0.1). *Then (Δ, τ_T), (Δ^+, τ_T), $(\mathcal{D}, \tau_T)$, and $(\mathcal{D}^+, \tau_T)$ are all semigroups, called τ_T semigroups.*

The *modified Lévy metric*, $\mathcal{L}$, induces the topology of weak convergence on Δ (and hence also on any of its subsets). To be precise, in any of the spaces $\Delta, \Delta^+, \mathcal{D}, \mathcal{D}^+$ we will say that a sequence $\{F_n\}$ *converges weakly* to F if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point x of the limit function F. In this topology $(\Delta, \mathcal{L})$ and $(\Delta^+, \mathcal{L})$ are compact, hence complete, metric spaces, whereas $(\mathcal{D}, \mathcal{L})$ and $(\mathcal{D}^+, \mathcal{L})$ are not compact [14]. In addition, if the *t-norm* T is continuous, then $(\Delta^+, \tau_T, \mathcal{L})$, $(\mathcal{D}, \tau_T, \mathcal{L})$ and $(\mathcal{D}^+, \tau_T, \mathcal{L})$ are topological semigroups, whereas $(\Delta, \tau_T, \mathcal{L})$ is not [11].

In this setting, an *iseomorphism* between two semigroups is an algebraic isomorphism which is also a homeomorphism.

The following distribution functions will be useful in the sequel. For any real a, we define

$$\varepsilon_a(x) = \begin{cases} 0, & x \le a, \\ 1, & a < x; \end{cases} \quad \text{and } \varepsilon_\infty(x) = 0 \text{ for all } x. \tag{0.2}$$

Note that ε_0 is the unique unit in any τ_T semigroup and ε_∞ is the unique null element in (Δ, τ_T) and (Δ^+, τ_T).

It will also be useful to characterize *t-norms*. Commonly used *t-norms* are Product, Min, and T_m, where $T_m(a, b) = \max\{a + b - 1, 0\}$. From now on we will let I denote the unit interval $[0, 1]$.

DEFINITION 0.3. A *t-norm* T is (a) *Archimedean* if T is continuous on $I \times I$ and satisfies $T(a, a) < a$ for all $a \in (0, 1)$; and (b) *strict* if T is continuous on $I \times I$ and strictly increasing in each place on $(0, 1] \times (0, 1]$.

It is easily seen that every strict t-norm is Archimedean. Note that T_m is Archimedean but not strict, Product is strict, and Min is continuous but non-Archimedean. Archimedean t-norms form an important class and the next result provides a useful tool for studying them.

DEFINITION 0.4. If $h: I \to I$ is a continuous and increasing function with $h(1) = 1$ then the *pseudo-inverse* of h, denoted $h^{[-1]}$, is given by

$$h^{[-1]}(x) = \begin{cases} 0, & 0 \le x \le h(0), \\ h^{-1}(x), & h(0) \le x \le 1, \end{cases}$$

where h^{-1} is the usual inverse of h on $[h(0), 1]$. Note $h^{[-1]}$ is uniformly continuous and non-decreasing.

Combining results from [1], [5], and [7] we then have:

THEOREM 0.2. *Let T be an Archimedean t-norm. Then there exists a* *continuous and increasing function* $h : I \to I$ *with* $h(1) = 1$ *such that* T *is representable in the form*

$$T(x, y) = h^{[-1]}(h(x) \cdot h(y)), \tag{0.3}$$

where $h^{[-1]}$ *is the pseudo-inverse of* h.

If T is an Archimedean t-norm and h is a function derived from Theorem 0.2 so that (0.3) holds, then h is called a *multiplicative generator* of T. Every multiplicative generator of T has the form h^{λ} for some $\lambda > 0$. Also if $h : I \to I$ is a continuous and increasing function with $h(1) = 1$ and we define T by (0.3) then it is easily shown that T is an Archimedean t-norm. Thus we have a simple procedure for generating Archimedean t-norms.

If h is a multiplicative generator of an Archimedean t-norm T with pseudo-inverse $h^{[-1]}$, then, using Definition 0.4, $h^{[-1]}h : I \to I$ and $hh^{[-1]} : I \to I$ are given by

$$h^{[-1]}h(x) = x, \quad \text{for all } x,$$

and

$$hh^{[-1]}(x) = \max\{h(0), x\} = \begin{cases} h(0), & 0 \le x \le h(0), \\ x, & h(0) \le x \le 1. \end{cases} \tag{0.4}$$

For a multiplicative generator h, if $h(0) = 0$ then by Definition 0.4 we have $h^{[-1]} = h^{-1}$, whence $h^{[-1]}$ is an increasing function on I and we have:

COROLLARY 0.1. *If* h *is a multiplicative generator of an Archimedean t-norm* T, *then* T *is strict if and only if* $h(0) = 0$, *i.e., if and only if* $h^{[-1]} = h^{-1}$.

Multiplicative generators can also be looked at as iseomorphisms, via:

THEOREM 0.3. *Let* T *be an Archmedean t-norm with multiplicative generator* h. (1) *If* T *is strict then* h *is an iseomorphism between* (I, T) *and* $(I, Product)$. *Hence, any two strict t-norms are iseomorphic and any t-norm iseomorphic to a strict t-norm is strict.* (2) *If* T *is non-strict then the function* $g(x) = 1 - \log h(x)/\log h(0)$ *is an iseomorphism between* (I, T) *and* (I, T_m). *Hence, any two Archimedean non-strict t-norms are iseomorphic and any t-norm that is iseomorphic to an Archimdean non-strict t-norm is Archimedean non-strict.* (3) *The only t-norm iseomorphic to* Min *is* Min.

Remark. Archimedean t-norms are distinguished among the continuous t-norms by the fact that they have no interior idempotents. Strict t-norms are distinguished from Archimedean non-strict t-norms by having no nilpotents. Min is the only t-norm for which every real in I is idempotent.

1. Iseomorphisms among τ_T semigroups

In this section we will establish iseomorphism relationships and distinctions among the τ_T semigroups and also between τ_T semigroups and convolution. A key step in this direction is:

THEOREM 1.1. *Let T_1, T_2 be left-continuous t-norms. If the semigroup (I, T_1) is iseomorphic to (I, T_2) then $(\Delta, \tau_{T_1}, \mathscr{L})$ is iseomorphic to $(\Delta, \tau_{T_2}, \mathscr{L})$. Moreover, the same conclusion holds when Δ is replaced by any of Δ^+, $\mathscr{D}$, or $\mathscr{D}^+$.*

Hence if the t-norms T_1 and T_2 are both strict, or both Archimedean but not strict, then the corresponding τ_T semigroups are iseomorphic.

Proof. Let $\phi:(I, T_1)\to(I, T_2)$ be an iseomorphism. By Definition 0.2, 0 is the unique null element and 1 is the unique identity for any t-norm. Thus we must have $\phi(0)=0$ and $\phi(1)=1$. But ϕ is necessarily one-to-one and continuous. Hence ϕ is an increasing function.

Consider the map $\phi^*:(\Delta, \tau_{T_1}, \mathscr{L})\to(\Delta, \tau_{T_2}, \mathscr{L})$ defined for any $F\in\Delta$ by

$$\phi^*(F)(x)=\phi\circ F(x)=\phi(F(x)) \text{ for all real } x. \tag{1.1}$$

Since $\phi: I\to I$ is continuous and increasing, it is clear that $\phi^*(F)\in\Delta$ for all $F\in\Delta$. Also, since ϕ is one-to-one and onto, it easily follows that ϕ^* is one-to-one and onto.

Next, using (1.1), the fact that ϕ is a continuous increasing isomorphism, and (0.1), for any F, G in Δ and real number x we have that

$$\phi^*(\tau_{T_1}(F, G))(x)=\phi(\sup_{u+v=x} T_1(F(u), G(v)))$$

$$=\sup_{u+v=x}\phi(T_1(F(u), G(v)))=\sup_{u+v=x} T_2(\phi(F(u)), \phi(G(v)))$$

$$=\tau_{T_2}(\phi^*(F), \phi^*(G))(x).$$

Hence $\phi^*(\tau_{T_1}(F, G)) = \tau_{T_2}(\phi^*(F), \phi^*(G))$ for all F, G in Δ and ϕ^* is an algebraic isomorphism.

It remains to show that ϕ^* is a homeomorphism. Thus suppose $\{F_n\}$ is a sequence in Δ such that $F_n \xrightarrow{w} F \in \Delta$. Then at every continuity point x of F we have $F_n(x) \to F(x)$, whence $\phi(F_n(x)) \to \phi(F(x))$, since ϕ is continuous. Furthermore, since ϕ is strictly increasing, $\phi^*(F)$ is continuous at x if and only if F is continuous at x. Thus $\phi^*(F_n) \xrightarrow{w} \phi^*(F)$. Hence ϕ^* is continuous on $(\Delta, \mathscr{L})$ and, since $(\Delta, \mathscr{L})$ is a compact Hausdorff space, we have that ϕ^* is a homeomorphism [16, Th. 17.14].

Since, for any left-continuous t-norm T, (Δ^+, τ_T), $(\mathscr{D}, \tau_T)$ and $(\mathscr{D}^+, \tau_T)$ are sub-semigroups of (Δ, τ_T) which are closed under ϕ^*, i.e., $\phi^*(\Delta^+) = \Delta^+$, $\phi^*(\mathscr{D}) = \mathscr{D}$ and $\phi^*(\mathscr{D}^+) = \mathscr{D}^+$, the conclusion also holds for them.

Finally, the last statement follows from the fact that, in each case, (I, T_1) and (I, T_2) are iseomorphic [7].

Our next theorem is useful in distinguishing τ_T semigroups and establishes a partial converse to Theorem 1.1. It is a consequence of the following:

LEMMA 1.1 *Let T be a continuous t-norm. Then:*

(a) *(Δ^+, τ_T) contains (non-trivial) idempotent elements if and only if T is non-Archimedean.*

(b) *(Δ^+, τ_T) contains neither (non-trivial) idempotent nor (non-trivial) nilpotent elements if and only if T is strict.*

Proof. First note that in any τ_T semigroup ε_0 is the unique unit and ε_∞ is the unique null element. Let T be continuous and non-Archimedean. Then there is a point $c \in (0, 1)$ such that $T(c, c) = c$. If $F \in \Delta^+$ is given by

$$F(x) = \begin{cases} 0, & x \le 0, \\ c, & 0 < x; \end{cases}$$

then it is easily seen from (0.1) that $\tau_T(F, F) = F$. Hence F is a non-trivial idempotent in (Δ^+, τ_T).

In the other direction, suppose T_0 is Archimedean. Let $G \in \Delta^+$ with $G \neq \varepsilon_0$ and $G \neq \varepsilon_\infty$. Then either

$$0 < G(y) < 1 \text{ for some } y > 0 \tag{1.2}$$

or $G=\varepsilon_a$ for some $a>0$. If (1.2) holds then, since we are in Δ^+, we have

$$\tau_{T_0}(G,G)(y)=\sup_{\substack{u+v=y\\u,v>0}} T_0(G(u),G(v))$$
$$\leq T_0(G(y),G(y))<G(y);$$

and if $G=\varepsilon_a$ then $\tau_{T_0}(G,G)=\varepsilon_{2a}$. In other words, (Δ^+,τ_{T_0}) contains no non-trivial idempotent elements. This establishes (a).

For (b), first assume that T is an Archimedean non-strict t-norm with multiplicative generator h. Then by Corollary 0.1 we have that $0<h(0)<1$. Thus $h(0)<\sqrt{h(0)}<1$, so that, since h is continuous and increasing, there is a point $b\in(0,1)$ where $h(b)=\sqrt{h(0)}$. Define $E\in\Delta^+$ by

$$E(x)=\begin{cases}0, & x\leq 0,\\ b, & 0<x;\end{cases} \tag{1.3}$$

so that $E\neq\varepsilon_\infty$. But, using (0.3) and the fact that $h^{[-1]}$ is non-decreasing, we have, for any x, that

$$\tau_T(E,E)(x)=\sup_{u+v=x} h^{[-1]}(h(E(u))\cdot h(E(v)))$$

$$\leq h^{[-1]}(h(b)\cdot h(b))=h^{[-1]}(h(0))=0.$$

Hence $\tau_T(E,E)=\varepsilon_\infty$, i.e., E is nilpotent in (Δ^+,τ_T). Note G is also nilpotent in (Δ,τ_T).

Next assume T is a strict t-norm and let $F\in\Delta^+$ with $F\neq\varepsilon_\infty$. Then, for some real w, we have $F(w)>0$. Since T is strict it then follows that

$$\tau_T(F,F)(2w)\geq T(F(w),F(w))>0,$$

whence $\tau_T(F,F)\neq\varepsilon_\infty$. Note that the same argument holds for any $F\in\Delta$ with $F\neq\varepsilon_\infty$. Thus neither (Δ^+,τ_T) nor (Δ,τ_T) contain non-trivial nilpotent elements.

This completes our proof.

THEOREM 1.2. (a) *If the continuous t-norms T_1, T_2, T_3, are, respectively, strict, Archimedean but not strict, and non-Archimedean, then no two of the semigroups (Δ^+,τ_{T_1}), (Δ^+,τ_{T_2}), (Δ^+,τ_{T_3}) are isomorphic. In addition, (Δ,τ_{T_1}) and (Δ,τ_{T_2}) are not isomorphic.* (*b*) *There is no continuous t-norm T, other than*

T = Min, *such that* $(\mathcal{D}, \tau_T)$ *is isomorphic to* $(\mathcal{D}, \tau_{\mathrm{Min}})$. *The same statement holds with* $\mathcal{D}$ *replaced by* $\mathcal{D}^+$.

Proof. (a) follows from the preceding Lemma and its proof, where we showed, in particular, that (Δ, τ_{T_2}) contains non-trivial nilpotent elements whereas (Δ, τ_{T_1}) does not.

For (b), we demonstrated in [8] that, for any continuous t-norm T, the cancellation law holds in $(\mathcal{D}, \tau_T)$ or in $(\mathcal{D}^+, \tau_T)$ if and only if $T = \mathrm{Min}$. This fact completes the proof.

Combining Theorem 1.2 with Theorem 0.3 yields:

COROLLARY 1.1. *Let* T_1 *be an Archimedean t-norm. Then, for any continuous t-norm* T_2, *if* (Δ^+, τ_{T_2}) *is isomorphic to* (Δ^+, τ_{T_1}), *then* (I, T_2) *is iseomorphic to* (I, T_1).

The preceding results are helpful in classifying τ_T semigroups but provide no means for going from one τ_T semigroup to another. In this respect the following theorem is very useful.

THEOREM 1.3. *Let* T *be an Archimedean t-norm with multiplicative generator* h. *Then*:

(a) *For any* $F, G \in \Delta$,

$$\tau_T(F, G) = h^{[-1]}(\tau_{\mathrm{Prod}}(hF, hG)), \tag{1.4}$$

where $h^{[-1]}$ *is the pseudo-inverse of* h.

(b) *If* T *is strict, then* h *induces an iseomorphism between* τ_T *and* τ_{Prod}, *i.e., for* $S = \Delta, \Delta^+, \mathcal{D}$ *or* $\mathcal{D}^+$, *the map* $h^*: (S, \tau_T, \mathcal{L}) \to (S, \tau_{\mathrm{Prod}}, \mathcal{L})$, *defined for any* $F \in S$ *and any* x *by* $h^*(F)(x) = h(F(x))$, *is an iseomorphism.*

(c) *If* T *is not strict then the function* g *defined on* I *by*

$$g(x) = 1 - \log h(x)/\log h(0) \tag{1.5}$$

induces an iseomorphism between τ_T *and* τ_{T_m} *on each of the spaces given above, where* $T_m(a, b) = \max\{a + b - 1, 0\}$.

Proof. By Theorem 0.2 and Definition 0.4, h is a continuous increasing function of I and $h^{[-1]}$ is continuous and non-decreasing on I. In particular note that $hF \in \Delta$ for any $F \in \Delta$. Thus, using (0.1) and (0.3), for any x

$$\tau_T(F, G)(x) = \sup_{u+v=x} h^{[-1]}(hF(u) \cdot hG(v))$$

$$= h^{[-1]}(\sup_{u+v=x} hF(u) \cdot hG(v)) = h^{[-1]}(\tau_{\mathrm{Prod}}(hF, hG)(x)),$$

establishing (a).

Parts (b) and (c) follow by combining Theorem 0.3 with the proof of Theorem 1.1.

Finally we would like to distinguish the τ_T semigroups from the convolution semigroup. That the τ_T operations are fundamentally different from the operation of convolution of distribution functions has been demonstrated in [12]. We further characterize the distinction in the following result:

THEOREM 1.5. *For any continuous t-norm, T, the semigroup (Δ^+, τ_T) is not isomorphic to the semigroup $(\Delta^+, *)$ where $*$ is convolution. The same result holds with Δ^+ replaced by $\mathcal{D}^+$.*

Proof. In [8, Th. 3.1] we showed that the cancellation law fails in the semigroup (Δ, τ_T). The same proof also shows that the cancellation law fails in (Δ^+, τ_T). We will show that the cancellation law does hold in the semigroup $(\Delta^+, *)$ which will yield our desired result.

The validity of the cancellation law in $(\Delta^+, *)$ is easily established by combining some theorems given in Feller [2]. Let F, G_1, $G_2 \in \Delta^+$ with $F \neq \varepsilon_\infty$ and suppose $F*G_1 = F*G_2$. Then $F*G_1 \in \Delta^+$, thus using [2, p. 411] if ϕ denotes the Laplace transform then

$$\phi(F) \cdot \phi(G_1) = \phi(F*G_1) = \phi(F*G_2) = \phi(F) \cdot \phi(G_2). \tag{1.6}$$

Now, as is easily seen from the definition of the Laplace transform [2, p . 407], since $F \neq \varepsilon_\infty$, we have $\phi(F)(\lambda) > 0$ for all $\lambda \in [0, \infty)$. Hence (1.6) yields that $\phi(G_1) = \phi(G_2)$. But the Laplace transform is one-to-one on Δ^+ [2, Th. 1, p. 408]. Therefore $G_1 = G_2$ and the cancellation law holds.

If $T \neq \mathrm{Min}$ then the same argument works for $\mathcal{D}^+$. However, in $(\mathcal{D}^+, \tau_{\mathrm{Min}})$ the cancellation law holds. But in [8] we showed that any distribution function F has a square root under τ_{Min} in $\mathcal{D}^+$. This fact does not obtain for $(\mathcal{D}^+, *)$ [6], completing the proof.

2. Solving equations in τ_T semigroups

In [8] we studied the algebra of τ_T semigroups and showed that, for every continuous t-norm T, the cancellation law fails in the semigroup (Δ, τ_T) and that, for every strict T, (Δ, τ_T) contains indecomposable elements. In this section we study the question of when, given $F, H \in \Delta$, there exists a $G \in \Delta$ such that $\tau_T(F, G) = H$. The existence of indecomposable elements implies that such a solution G need not exist and the failure of the cancellation law suggests that a solution, if it exists, may not be unique. Nevertheless, we will show that, for any given $F, H \in \Delta$ and any left-continuous t-norm T, the inequality $\tau_T(F, G) \leq H$ has a maximal solution G. This solution G is constructed explicitly and, when the corresponding equation has a solution, it is the unique maximal solution. This answers the question in theory. In practice, the method has the drawback that, in order to determine whether or not the equation $\tau_T(F, G) = H$ has a solution, one must construct the function G and evaluate $\tau_T(F, G)$. Thus we also exhibit a necessary condition on $F, H \in \Delta$ for a solution to exist. For certain restricted F, H this condition is sufficient.

In the sequel we shall occasionally encounter distribution functions which are not left-continuous. Any such distribution function will be distinguished by an asterisk, e.g., G^*; and its left-continuous version will be denoted by G, so that

$$G(x) = \lim_{y \to x-} G^*(y). \tag{2.1}$$

Thus, removing the asterisk normalizes G^* to be left-continuous, so that $G \in \Delta$. Clearly,

$$G(x) \leq G^*(x), \text{ for all } x. \tag{2.2}$$

THEOREM 2.1. *Let T be a left-continuous t-norm and let $F, H \in \Delta$.* For any *x, y define*

$$w(x, y) = \sup\{a \in I \mid T(F(y), a) \leq H(x + y)\} \tag{2.3}$$

and let

$$G^*(x) = \inf_y \{w(x, y)\}. \tag{2.4}$$

Then we have:

(a) *G is the unique maximal solution in Δ to the inequality*

$$\tau_T(F, G) \leq H, \tag{2.5}$$

i.e., for any $E \in \Delta$, *if* $\tau_T(F, E) \leq H$ *then* $E \leq G$.

(b) *There is an* $E \in \Delta$ *such that*

$$\tau_T(F, E) = H$$

if and only if

$$\tau_T(F, G) = H. \tag{2.6}$$

(c) *If* G_0 *is defined by*

$$G_0(x) = \begin{cases} 0, & x \leq 0, \\ G(x), & 0 < x; \end{cases} \tag{2.7}$$

then the above results hold with Δ *replaced either by* Δ^+ *or by* $\mathcal{D}^+$ *and* G *replaced by* G_0.

Proof. Note first that, for any fixed y, $w(x, y)$ is a non-decreasing function of x. Hence G^* is non-decreasing and, since $0 \leq G^*(x) \leq 1$ for any x, $G \in \Delta$. Now from (2.2), (2.3), (2.4) and the fact that T is left-continuous and non-decreasing we have, for any u, v, that

$$\begin{aligned} T(F(u), G(v)) &\leq T(F(u), G^*(v)) \\ &\leq T(F(u), w(v, u)) \leq H(u+v), \end{aligned}$$

whence, by (0.1), for any x, $\tau_T(F, G)(x) \leq H(x)$.

Now assume that $\tau_T(F, E) \leq H$ for some $E \in \Delta$. Then for any x, y we have by (0.1) that $T(F(y), E(x)) \leq H(x+y)$. Thus from (2.3) it follows that $E(x) \leq w(x, y)$, whence

$$E(x) \leq \inf_y \{w(x, y)\} = G^*(x).$$

Since E is left-continuous we then have that

$$E(x) = \lim_{y \to x-} E(y) \leq \lim_{y \to x-} G^*(y) = G(x).$$

This proves (a).

Next, (b) is an easy consequence of the maximality of G and the non-decreasing character of τ_T.

Finally, (c) can be established by checking some minor details, completing the proof.

Remark. It is interesting to note that Theorem 2.1 does not hold with Δ replaced by $\mathcal{D}$, i.e., the maximal solution G to (2.5) need not lie in $\mathcal{D}$. To see this, let T = Product and let $F, H \in \mathcal{D}$ be given by

$$F(x)=\begin{cases}-1/x, & x \leq -1,\\ 1, & -1 \leq x;\end{cases} \quad \text{and} \quad H(x)=\begin{cases}1/x^2, & x \leq -1,\\ 1, & -1 \leq x.\end{cases}$$

If G satisfies (2.5) then, by (0.1), for all x, y, we must have $F(y) \cdot G(x) \leq H(x+y)$, whence, for any fixed x,

$$G(x) \leq \lim_{y \to -\infty} H(x+y)/F(y) = \lim_{y \to -\infty} -y/(x+y)^2 = 0.$$

Thus $G = \varepsilon_\infty \notin \mathcal{D}$.

Theorem 2.1 thus shows that in Δ, Δ^+ and $\mathcal{D}^+$ we can solve simple equations and inequalities of the form (2.5) and (2.6). However, in order to find out whether a solution to the equation (2.6) exists, one must compute G via (2.4) and (2.1) and then determine whether (2.6) does indeed hold. This is inconvenient and therefore easily verifiable necessary and/or sufficient conditions on F, H for the existence of a G satisfying (2.6) are desirable. Our next result is directed toward this end. First, for notation, we need:

DEFINITION 2.1. For any function F and any x, let $D_L F(x)$ denote the left-hand derivative of F at x and let $D_R F(x)$ denote the right-hand derivative of F at x (when they exist).

THEOREM 2.2. *Let T be a left-continuous t-norm and let $F, H \in \Delta$. Suppose that $t_1 = \inf\{x \mid F(x)=1\}$ and $t_2 = \inf\{x \mid H(x)=1\}$ both exist and are finite. Then a necessary condition for the existence of a $G \in \Delta$ satisfying*

$$\tau_T(F, G) = H \tag{2.8}$$

is:

$$F(t_1-\delta) \leq H(t_2-\delta) \text{ for all } \delta > 0. \tag{2.9}$$

If, in addition, F, H are continuous at t_1, t_2, respectively, if $D_LF(t_1)$ and $D_LH(t_2)$ both exist, and if (2.8) holds for some $G \in \Delta$, then necessarily,

$$D_LF(t_1) \geq D_LH(t_2). \tag{2.10}$$

Proof. Assume there is a $G \in \Delta$ so that (2.8) holds. We then claim that

$$G((t_2 - t_1)^+) = 1. \tag{2.11}$$

Otherwise, for some $\varepsilon > 0$,

$$G(t_2 - t_1 + \varepsilon) < 1. \tag{2.12}$$

Thus for any points u, v with $u + v = t_2 + \varepsilon/2$ either $u \leq t_1 - \varepsilon/2$, so that

$$T(F(u), G(v)) \leq T(F(u), 1) = F(u) \leq F(t_1 - \varepsilon/2),$$

or $v \leq t_2 - t_1 + \varepsilon$, so that

$$T(F(u), G(v)) \leq T(1, G(v)) = G(v) \leq G(t_2 - t_1 + \varepsilon),$$

whence, by (0.1), (2.12) and the definition of t_1,

$$\begin{aligned} H(t_2 + \varepsilon/2) &= \tau_T(F, G)(t_2 + \varepsilon/2) \\ &\leq \max\{F(t_1 - \varepsilon/2), G(t_2 - t_1 + \varepsilon)\} < 1, \end{aligned}$$

contradicting the definition of t_2.

Next, for any $\delta > 0$ and any $\varepsilon > 0$, using (2.8) and (2.11) we have

$$H(t_2 - \delta) \geq T(F(t_1 - \delta - \varepsilon), G(t_2 - t_1 + \varepsilon)) = F(t_1 - \delta - \varepsilon),$$

whence, since F is left-continuous and $\varepsilon > 0$ arbitrary, (2.9) follows. If F, H are continuous at t_1, t_2, respectively, then clearly $F(t_1) = H(t_2) = 1$. Thus if $D_LF(t_1)$ and $D_LH(t_2)$ exist and (2.8) holds for some $G \in \Delta$, then (2.10) easily follows from (2.9), completing the proof.

In some cases the necessary condition (2.10) of Theorem 2.2 is also sufficient. More precisely, we have

THEOREM 2.3. *Let T be an Archimedean t-norm with multiplicative* *generator h and let $F, H \in \mathcal{D}$. Suppose, for some numbers $s_1 < t_1$ and $s_2 < t_2$, we have:*

(I) $F(s_1) = 0$, $F(t_1) = 1$, *and hF is concave and strictly increasing on* $(s_1, t_1]$; *and*

(II) $H(s_2) = H(s_2^+) = 0$, $H(t_2) = 1$, *and hH is convex and strictly increasing on* $(s_2, t_2]$.

Then there is an $E \in \mathcal{D}$ satisfying

$$\tau_T(F, E) = H \tag{2.13}$$

if and only if

$$D_L(hF)(t_1) \geq D_L(hH)(t_2). \tag{2.14}$$

Proof. Note by (I), (II) that, in fact, hF is concave on (s_1, ∞) and hH is convex on $(-\infty, t_2]$. We will use basic facts concerning the continuity and differentiability of convex (concave) functions as given in [9, p. 42] and [4, pp. 1–5]. In particular, for convex (concave) functions, one-sided derivatives exist at each point and are themselves non-decreasing (non-increasing) functions.

Assume that (2.13) holds. Then by Theorem 1.3 (a) and (0.4) we have

$$hH(x) = h(\tau_T(F, E)(x)) = \tau_{\text{Prod}}(hF, hE)(x),$$

whenever $H(x) > 0$. The proof of Theorem 2.2 then yields (2.14).

Now assume that (2.14) holds and, for our given T, F, H, let G^* be defined by (2.4) of Theorem 2.1. Choose any x_0 with $s_2 < x_0 < t_2$ and let

$$c = \sup \{\{s_1\} \cup \{x \mid x > s_1 \text{ and } D_L(hF)(x)/hF(x) \geq D_L(hH)(x_0)/hH(x_0)\}\}. \tag{2.15}$$

Note by condition (II) that

$$0 < D_L(hH)(x_0)/hH(x_0) < \infty,$$

so that $s_1 \leq c \leq t_1$, since $D_L(hF)(x) = 0$ for $x > t_1$. Now by (2.15)

$$D_L(hF)(x)/hF(x) < D_L(hH)(x_0)/hH(x_0) \text{ for all } x > c. \tag{2.16}$$

But, using (I), (II) and (2.14), for any x with $c < x \le t_1$, we have

$$D_L(hF)(x) \ge D_L(hF)(t_1) \ge D_L(hH)(t_2) \ge D_L(hH)(x_0),$$

or, using (2.16), $hF(x) > hH(x_o)$ for all $x > c$, whence $hF(c^+) \ge hH(x_0)$. So let

$$k = hH(x_0)/hF(c^+). \tag{2.17}$$

Since $x_0 > s_2$ we have $H(x_0) > 0$ and clearly $hF(c^+) \le 1$, so that

$$h(0) < k \le 1. \tag{2.18}$$

Thus, using (2.16), (2.17), the concavity of hF on (s_1, ∞) and the convexity of hH on $(s_2, t_2]$, we have for any y with $0 < y \le t_2 - x_0$ that

$$\begin{aligned}
&[(hF(c+y) - hF(c^+))/y] \cdot k \\
&= \lim_{\delta \to 0+} [(hF(c+y) - hF(c+\delta))/(y-\delta)] \cdot k \\
&\le \lim_{\delta \to 0+} [D_R(hF)(c+\delta)] \cdot k \le \lim_{\delta \to 0+} [D_L(hF)(c+\delta)] \cdot k \\
&\le \lim_{\delta \to 0+} [hF(c+\delta) \cdot D_L(hH)(x_0)/hH(x_0)] \cdot k \\
&= D_L(hH)(x_0) \le D_R(hH)(x_0) \le (hH(x_0+y) - hH(x_0))/y.
\end{aligned}$$

Hence, using (2.17) again, the above yields that

$$hF(c+y) \cdot k \le hH(x_0+y). \tag{2.19}$$

Note if $y > t_2 - x_0$ then, by (II), $hH(x_0+y) = h(1) = 1$ so that (2.19) holds for all $y > 0$. Similarly, if $y \le s_1 - c$ then, by (I), $F(c+y) = 0$ so that (2.19) holds in this case also. In particular, if $c = s_1$ then we have shown via (2.19), that

$$hF(u) \cdot k \le hH(x_0 + u - c) \text{ for all } u. \tag{2.20}$$

To establish (2.20) in general assume that $s_1 < c \le t_1$. Then, since from [4], [9] $D_L(hF)(x)$ is left-continuous and $hF(x)$ is continuous on (s_1, ∞), it follows from (2,15) that

$$D_L(hF)(c)/hF(c) \ge D_L(hH)(x_0)/hH(x_0). \tag{2.21}$$

Note, since we are assuming $c > s_1$, that $hF(c^+) = hF(c)$ here. Thus, if $0 < y < c - s_1$, then, using (I), (II), (2.17), (2.21) and the fact that hH is convex on $(-\infty, t_2]$, we have

$$\begin{aligned}[(hF(c) - hF(c-y))/y] \cdot k &\leq [D_L(hF)(c)] \cdot k \\ &= [D_L(hF)(c)/hF(c)] \cdot hH(x_0) \geq D_L(hH)(x_0) \\ &\geq (hH(x_0) - hH(x_0 - y))/y,\end{aligned}$$

whence, using (2.17),

$$hF(c-y) \cdot k \leq hH(x_0 - y). \tag{2.22}$$

Note here if $y \geq c - s_1$ then by (I), $F(c-y) = 0$, whence (2.22) holds for all $y \geq 0$. Combined with the fact that (2.19) holds for all $y > 0$, this establishes (2.20).

Hence, using (0.3), (0.4), (2.18), (2.20), and the fact that $h^{[-1]}$ is non-decreasing, we have, for any u, that

$$\begin{aligned}T(F(u), h^{[-1]}(k)) &= h^{[-1]}(hF(u) \cdot hh^{[-1]}(k)) \\ &= h^{[-1]}(hF(u) \cdot k) \leq h^{[-1]}(hH(x_0 + u - c)) \\ &= H(x_0 + u - c),\end{aligned}$$

whence, using the notation of Theorem 2.1, $w(x_0 - c, u) \geq h^{[-1]}(k)$ for all u, so that, by (2.4), $G^*(x_0 - c) \geq h^{[-1]}(k)$. Therefore, by (2.1) and (2.17), we have

$$\begin{aligned}\tau_T(F, G)(x_0^+) &\geq \lim_{\delta \to 0+} T(F(c + \delta/2), G(x_0 - c + \delta/2)) \\ &\geq T(F(c^+), G^*(x_0 - c)) \geq h^{[-1]}(hF(c^+) \cdot hh^{[-1]}(k)) \\ &= h^{[-1]}(hH(x_0)) = H(x_0)\end{aligned}$$

for all $x_0 \in (s_2, t_2)$. Since both H and $\tau_T(F, G)$ are left-continuous and non-decreasing, this implies, in view of (II), that $\tau_T(F, G) \geq H$. Combined with Theorem 2.1, this yields $\tau_T(F, G) = H$. It is trivial to show that G is actually in $\mathcal{D}$, completing the proof.

Remark. Assume in Theorem 2.3 that $D_L h(1)$ exists. If (2.14) holds then by (I), (II) both $D_L(hF)(t_1)$ and $D_L(hH)(t_2)$ must be positive and finite so that one can use a "chain rule" for left-handed derivatives to show that $D_L F(t_1)$ and $D_L H(t_2)$ necessarily exist. Moreover, in this case, it follows that

$$D_L F(t_1) \geq D_L H(t_2). \tag{2.23}$$

Conversely, if (2.23) holds and one of $D_L h(1)$, $D_L F(t_1)$, $D_L H(t_2)$ is positive and finite, then the chain rule yields that (2.14) holds. Thus, in most cases, we can use (2.23) instead of (2.14) in applying Theorem 2.3.

A simple argument, which we omit, extends Theorem 2.3 to $\mathscr{D}^+$, at least for strict t-norms. To be precise, we have:

THEOREM 2.4. *Let T be a strict t-norm with multiplicative generator h and let $F, H \in \mathscr{D}^+$. Assume, for some numbers $0 \leq s_1 < t_1$ and $0 \leq s_2 < t_2$, that F, H satisfy (I), (II) of Theorem 2.3. Then there is a $G \in \mathscr{D}^+$ so that $\tau_T(F, G) = H$ if and only if $s_1 \leq s_2$ and $D_L(hF)(t_1) \geq D_L(hH)(t_2)$.*

To illustrate the use of Theorem 2.3 and the method given by Theorem 2.1, we have:

EXAMPLE 2.1. Let $T = \text{Product}$ (which has the identity as its multiplicative generator). Consider the functions $F_{pq} \in \mathscr{D}^+$ defined for $p, q > 0$ by

$$F_{pq}(x) = \begin{cases} 0, & x \leq 0, \\ (px)^q, & 0 \leq x \leq 1/p, \\ 1, & 1/p \leq x. \end{cases} \tag{2.24}$$

Given F_{pq}, F_{rs} we want to determine when there is a $G \in \mathscr{D}^+$ so that $\tau_{\text{Prod}}(F_{pq}, G) = F_{rs}$ and then to calculate G. First, to satisfy (I), (II) of Theorem 2.3 we must have

$$q \leq 1 \quad \text{and} \quad s \geq 1. \tag{2.25}$$

If (2.25) holds then by Theorem 2.4 a solution $G \in \mathscr{D}^+$ exists if and only if

$$D_L F_{pq}(1/p) = pq \geq D_L F_{rs}(1/r) = rs. \tag{2.26}$$

Note (2.25) and (2.26) imply that $p \geq r$, or $1/p \leq 1/r$.

Now if G is a solution then by (2.11) we have $G(x) = 1$ for $x > 1/r - 1/p$. Thus choose x so that $0 < x < 1/r - 1/p$. Then from Theorem 2.1 we obtain

$$w(x, y) = \begin{cases} \min\{1, (r(x+y))^s/(py)^q\}, & 0 < y \leq 1/p, \\ \min\{1, (r(x+y))^s\}, & 1/p \leq y; \end{cases}$$

which, using simple calculus minimization techniques, yields by (2.4) that

$$G(x)=\begin{cases}0, & x\le o,\\ \left(\dfrac{rs}{s-q}\right)^s\left(\dfrac{s-q}{pq}\right)^q x^{s-q}, & 0\le x\le (s-q)/pq,\\ (rx+r/p)^s, & (s-q)/pq\le x\le 1/r-1/p,\\ 1, & 1/r-1/p\le x.\end{cases} \tag{2.27}$$

If $s=q$ (so that, by (2.25), $s=q=1$), then

$$G(x)=\begin{cases}0, & x\le 0,\\ rx+r/p, & 0<x\le 1/r-1/p\\ 1, & 1/r-1/p\le x.\end{cases} \tag{2.28}$$

In this case note that the solution G is discontinuous at 0.

Even without using Theorem 2.3, we could use Theorem 2.1 and direct calculation to discover that whenever $s\ge q$ and $pq\ge rs$ then the distribution function G, given by (2.27), is a solution to $\tau_{\text{Prod}}(F_{pq}, G)=F_{rs}$. Otherwise, no solution exists.

3. Copulas.

In [15] A. Sklar introduced a class of operations on the unit interval which he called copulas. The importance of copulas lies in the fact that they describe the functional relationships between joint distribution functions and their marginal distributions. Thus, if X and Y are real-valued random variables defined on a common probability space, with distribution functions F_X and F_Y, respectively, and joint distribution function H_{XY}, then there exists a copula C_{XY} such that

$$H_{XY}(u, v)=C_{XY}(F_X(u), F_Y(v))$$

for all u, v on the extended real line [12]. A corresponding result holds for any n-tuple of random variables [15].

In this final section we will study the relationship between t-norms and (two-dimensional) copulas. The latter are defined as follows:

DEFINITION 3.1. A function C is a *copula* if
(1) $C: I\times I\to I$,
(2) $C(a, 0)=C(0, a)=0$, for all $a\in I$,

(3) $C(a, 1) = C(1, a) = a$, for all $a \in I$,

(4) $C(a, b) - C(a, d) - C(c, b) + C(c, d) \geq 0$,

whenever $a, b, c, d \in I$ and $a \leq c, b \leq d$.

It follows from properties (2), (3) and (4) of the above definition that every copula is continuous and non-decreasing in each place. Thus every associative copula is a topological semigroup on I and hence [see e.g., 10, Cor. p. 85 and Th. 2.5.6, p. 87] a continuous t-norm. In the other direction, it is clear from Definition 0.2 that a t-norm is a copula if and only if it satisfies property (4). This latter condition can be weakened. For we have:

THEOREM 3.1. *A t-norm T is a copula if and only if it satisfies the Lipschitz condition*

$$T(c, b) - T(a, b) \leq c - a, \tag{3.1}$$

for all a, b, c in I with $a \leq c$.

Proof. If T is a copula then letting $d = 1$ in (4) of Definition 3.1 yields (3.1). In the other direction, assume T satisfies (3.1) and choose $s, t, u, v \in I$ satisfying $s \leq t$ and $u \leq v$. Note by (3.1) that T is continuous. Hence, since $T(0, v) = 0$ and $T(1, v) = v$, there exists $c \in I$ so that $T(c, v) = u$. Thus, since T is associative and commutative, we have that

$$\begin{aligned} T(t, u) - T(s, u) &= T(t, T(c, v)) - T(s, T(c, v)) \\ &= T(T(t, v), c - T(T(s, v), c) \\ &\leq T(t, v) - T(s, v), \end{aligned}$$

which is equivalent to (4) of Definition 3.1. Thus T is a copula, completing the proof.

Note that Theorem 3.1 states that, for t-norms, property (4) of Definition 3.1 and (3.1) are equivalent.

There is a somewhat more general version of Theorem 3.1 that applies to a wider class of binary operations on I. To state it we need the following:

DEFINITION 3.2. For any binary operation T on I, denote by T^* the binary operation on I defined by

$$T^*(x, y) = 1 - T(1 - x, 1 - y).$$

When T is a t-norm, T^* is called the *t-conorm* of T. We then have:

THEOREM 3.2. *Let T be a semigroup with unit on I. If*

$$0 \leq T(c, d) - T(a, b) \leq (c-a) + (d-b) \tag{3.2}$$

whenever $0 \leq a \leq c \leq 1$ and $0 \leq b \leq d \leq 1$, then either T or T^* is a copula and also a t-norm.

Proof. Assume T satisfies the hypotheses above with e as unit. By (3.2) T is continuous, and hence T is a topological semigroup on I. Since I is compact and connected [3, p. 169, item 17] then implies that either $e = 0$ or $e = 1$.

CASE 1. Suppose $e = 1$. Using (3.2) we then have that, for any $a \in I$, $T(a, 0) \leq T(1, 0) = 0$ and $T(0, a) \leq T(0, 1) = 0$ or $T(a, 0) = T(0, a) = 0$ for all $a \in I$. Hence 0 is a zero element of the semigroup. By [10, Th. 2.5.6] T is then abelian. But by (3.2) T is non-decreasing in each place, whence T is a t-norm. Applying Theorem 3.1 then yields that T is a copula.

CASE 2. Suppose $e = 0$. Note by Definition 3.2 that T^* is continuous if T is continuous. Also, the associativity of T^* follows easily from that of T. Thus T^* is a topological semigroup on I. It is also easily checked that $T^*(x, 1) = T^*(1, x) = x$ for any $x \in I$ so that 1 is the unit for T^*. Now choose a, b, c, d in I with $a \leq c$ and $b \leq d$. Then, using Definition 3.2 and (3.2),

$$\begin{aligned} T^*(c, d) - T^*(a, b) &= (1 - T(1-c, 1-d)) - (1 - T(1-a, 1-b)) \\ &= T(1-a, 1-b) - T(1-c, 1-d) \\ &\leq (c-a) + (d-b), \end{aligned}$$

since $1 - c \leq 1 - a$ and $1 - d \leq 1 - b$. Thus T^* satisfies (3.2). But then applying the argument of Case 1 to T^* yields that T^* is a t-norm and a copula, completing the proof.

It is apparent then that (3.2) is a very strong condition on semigroups on I. One would expect, therefore, that τ_T semigroups associated with t-norms which are copulas have distinctive properties. One such property is given in:

THEOREM 3.3. *Let the t-norm T be a copula. Suppose $F \in \Delta$ satisfies the Lipschitz condition*

$$|F(x) - F(y)| \leq M \cdot |x - y|^\alpha \text{ for all } x, y, \tag{3.3}$$

for constants $M, \alpha > 0$. *Then, for any* $G \in \Delta$, $H = \tau_T(F, G)$ *satisfies the same Lipschitz condition.*

Proof. Let F, G, H be as in the hypotheses above. Let $\varepsilon > 0$ and x, y be arbitrary. If $x = y$ there is nothing to show. So, without loss of generality, assume $x > y$. Then by (0.3) there exist points u, v with $u + v = x$ such that

$$H(x) \geq T(F(u), G(v)) \geq H(x) - \varepsilon|x-y|^\alpha. \tag{3.4}$$

Combining (3.3) and (3.4) with Theorem 3.1 then yields that

$$\begin{aligned} 0 &\leq H(x) - H(y) \leq T(F(u), G(v)) + \varepsilon\,|x-y|^\alpha - T(F(u-(x-y)), G(v)) \\ &\leq F(u) - F(u-(x-y)) + \varepsilon \cdot |x-y|^\alpha \\ &\leq (M + \varepsilon) \cdot |x-y|^\alpha. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves our desired result.

A slight variation of the above proof yields:

COROLLARY 3.1. *If the t-norm T satisfies the Lipschitz condition*

$$|T(a, c) - T(b, c)| \leq N \cdot |a-b|^\beta \text{ for } a, b, c \in I$$

for constants N, $\beta > 0$ *and if* $F \in \Delta$ *satisfies the Lipschitz condition*

$$|F(x) - F(y)| \leq M \cdot |x-y|^\alpha \text{ for all real } x, y,$$

where M, $\alpha > 0$ *are constants, then, for any* $G \in \Delta$, $H = \tau_T(F, G)$ *satisfies the Lipschitz condition*

$$|H(x) - H(y)| \leq N \cdot M^\beta \cdot |x-y|^{\alpha\beta} \text{ for all real } x, y.$$

CONJECTURE. It would be of interest to know when the absolute continuity of F and G implies that of $\tau_T(F, G)$. For real functions it is of course known that if f is absolutely continuous and g satisfies a simple Lipschitz condition then the composite function $g \circ f$ is absolutely continuous [9, Vol. 1]. In view of Theorem 3.1 then, it seems reasonable to expect that if F and G are absolutely continuous and if T is a copula then $\tau_T(F, G)$ is absolutely continuous. But this is an open question. Note that if this question were answered in the affirmative then τ_T would induce a binary operation on density functions.

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