Quintuplication of Room Squares

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Abstract

Given a strong starter for a group G of order n, where 3 does not divide n, a construction is given for a strong starter for the direct sum of G and the integers modulo 5. In particular, this gives a Room square of side 5p for all non-Fermat primes p.

A Room square is a $(2n+1)\times(2n+1)$ array of cells each containing either nothing or an unordered pair from a set of $2n+2$ objects. Every object must occur exactly once in every row and column, and every possible pair must occur exactly once. Room squares were first introduced mathematically in [6], where one of side 7 was constructed, and it was pointed out that there is no Room square of side 3 or 5.

A set of pairs $S=\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$ chosen from an abelian group G of $2n+1$ elements is a *starter* if all the non-zero group elements occur exactly once in S, and also occur exactly once in the set of differences of S, $\{\pm(x-y) \mid \{x, y\} \in S\}$. The starter S is said to be *strong* if the set of sums of S, $\{(x+y) \mid \{x, y\} \in S\}$, are all distinct and non-zero. It is known that the existence of a strong starter for a group of order $2n+1$ implies the existence of a Room square of side $2n+1$ [4]. Mullin and Nemeth [4] have constructed strong starters for all finite fields other than those of order $2n+1$.

There are many other constructions for Room squares. Computers have been used to get Room squares of all odd sides between 7 and 55, [9] and [1], including 9 and 17. Also, it was proved in [2] and [3] that if a Room square of side v_1 exists, and if a Room square of side v_2 exists with a subsquare of side v_3, and $v_2-v_3\neq 6$, then a Room square of side $v_1(v_2-v_3)+v_3$ exists. Note that if $v_3=0$, this gives a multiplication theorem for Room squares, as originally proved in [7]. Also, using this theorem, Room squares of side p were found for all Fermat primes p other than 3, 5, 257, and 65537 [3]. Recently, Mullin has used it to obtain a Room square of side 65537 as well. Wallis, in [10], has shown that if there is a Room square of side n with a property called skewness, then there is a Room square of side $2n+1$. Since the Mullin-Nemeth construction gives skew squares, and the recursive construction to give $v_1(v_2-v_3)+v_3$ preserves skewness, this is a useful construction.

THEOREM. *If there exists a strong starter for an abelian group G of order n, and* 3 *does not divide n, then there is a strong starter for* $I_5 \oplus G$, *the direct sum of the integers modulo* 5 *and G.*

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Proof. As G is a finite abelian group, G can be written as the direct sum of cyclic groups, none with order divisible by 2 or 3. Since the integers modulo m form a ring, we may view G as the additive group of the direct sum ring, in which the multiplicative identity 1, and the elements $2=1+1$, $3=1+1+1$, and the respective inverse elements 2^{-1} and 3^{-1} exist.

Let $X=\{\{x_i, y_i\} \mid i=1, 2, \ldots, (n-1)/2\}$ be the strong starter. Then find a non-zero a and b in G such that there is no pair $\{x, y\}$ in X whose sum is a or b. This can be done as there are only $(n-1)/2$ elements in the set of sums of X, while there are $n-1$ non-zero elements of G, and $n \geqslant 7$. Let $h=\frac{1}{4}(b-a)$ and $g=\frac{1}{2}a$. Now partition the non-zero elements of G into two sets P and N, such that x is in P if and only if $-x$ is in N. We add the restrictions that h is in P, and so is $-\frac{1}{3}h$. Now we consider the sets of pairs;

$$
\begin{aligned}
A &= \{\{(0, x), (0, y)\} \mid \{x, y\} \in X\}, \\
B &= \{\{(1, x+g), (2, 2x+g)\} \mid x \in P, x \neq h\}, \\
C &= \{\{(4, x+g), (3, 2x+g)\} \mid x \in P, x \neq h\}, \\
D &= \{\{(1, x+g), (3, 2x+g)\} \mid x \in N\}, \\
E &= \{\{(4, x+g), (2, 2x+g)\} \mid x \in N\}, \\
F &= \{\{1, h+g), (2, g)\}, \{(4, h+g), (3, g)\}, \\
&\quad \{(1, g), (4, g)\}, \{(2, 2h+g), (3, 2h+g)\}.
\end{aligned}
$$

The union of these 6 sets forms a strong starter for $I_5 \oplus G$. All non-zero elements of the form $(0, x)$ appear in some pair of A. All pairs of the form $(1, x)$ appear exactly once in one of B, D, or F, since $x+g$ will run over G as x runs over G. All pairs of the form $(2, x)$ appear in one of B, E, or F, since $2x+g$ will run over G as x runs over G. Similarly, all pairs of the form $(3, x)$ and $(4, x)$ appear exactly once.

All non-zero group elements also occur as the difference of some pair. All elements of the form $(0, x)$ occur as differences of pairs in A. All elements of the form $(1, x)$ occur as differences of pairs in B and C, except for $(1, 0)$, $(1, h)$, and $(1, -h)$. All elements of the form $(4, x)$ occur as differences of pairs in B and C, except for $(4, 0)$, $(4, h)$, and $(4, -h)$. All elements of the form $(2, x)$ and $(3, x)$ occur as differences of pairs in D and E except $(2, 0)$ and $(3, 0)$. The eight elements that have not yet been accounted for are the differences of the four pairs in F.

The sums of all these pairs are also all distinct and non-zero. The sums from A are of the form $(0, x+y)$ where $x+y$ is not zero, and are all distinct; the sums from B are of the form $(3, 3x+2g)$, x in P, and are all distinct, since 3 has a multiplicative inverse; the sums from C are of the form $(2, 3x+2g)$; the sums from D are of the form $(4, 3x+2g)$; the sums from E are of the form $(1, 3x+2g)$. The sums from F are: $(3, h+2g)$, $(2, h+2g)$, $(0, 2g)$, $(0, 4h+2g)$. These are distinct from the other sums. If $h+2g=3x+2g$, then $3x=h$ and $\frac{1}{3}h$ would have to be in P, but $-\frac{1}{3}h$ is in

P. If $2g=x+y$, where $\{x, y\}$ is in X, then $a=x+y$. If $2g+4h=x+y$, where $\{x, y\}$ is in X, then $b=x+y$. Either of these results contradicts the choice of a and b.

Thus, we have constructed a strong starter, and have a Room square of side $5n$.

This theorem is useful in constructing more Room squares. For example, since strong starters are known to exist for groups of order p, where p is an odd non-Fermat prime, it proves the existence of Room squares of side $5p$, for p a non-Fermat prime. This, together with the multiplication theorem and the existence of Room squares for all prime-power sides except 3, 5, and 257, shows that a Room square of side v exists unless v is divisible exactly once by 3, 5, 257, or 5 times a Fermat prime. Other results have also been proved. In [8], a Room square of side $5k$ is shown to exist for all odd $k>1$. In [8], it is also shown that if a Room square of side v does not exist, then v is divisible exactly once by either 3, or 257, but not both. To complete the proof that a Room square of side v exists for all odd v except for $v=3$ or 5, it is sufficient to prove that Room squares of sides $3p$ and 257 exists where p is any prime congruent to 3 modulo 4. The only $v \leqslant 1000$ for which Room squares have not yet been found are: 69, 93, 129, 213, 237, 257, 321, 453, 597, 669, 717, 789, 933.

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