The Disk-Packing Constant

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Let U be an open subset of the Euclidean plane, which has finite area. A simple osculatory packing C_0 of U is a sequence of disjoint disks $\{D_n\}$ each contained in U, and such that D_n has the largest radius of disks contained in $U \backslash (D_1 \cup D_2 \cup \cdots \cup D_{n-1})$, for $n=1, 2, \ldots$. The exponent of such a packing is the number

$$e(C_0, U) = \sup\left\{t : \sum_{n=1}^{\infty} r_n^t = \infty\right\} = \inf\left\{t : \sum_{n=1}^{\infty} r_n^t < \infty\right\},$$

where r_n is the radius of D_n.

In this paper, we shall consider simple osculatory packings of a curvilinear triangle T which has mutually tangent circular sides. The exponent in this case does not depend on the radii of the sides (see Wilker [8]), and will be denoted by S. It was shown by Melzak [5], that

$$1.035 < S < 1.999971. \tag{1}$$

The lower bound was subsequently improved by Wilker [8] to 1.059, and by the author [2] to 1.28467. An improved upper bound of $1.5403\ldots = (9+\sqrt{41})/10$ was proved in [3], but the arguments there, although they apply to sphere packings in higher dimensions, are too general in nature to yield a significant improvement of this bound.

In this paper, we present a method of attack which gives both upper and lower bounds. In fact, for any integer κ, we obtain bounds $\lambda(\kappa) < S < \mu(\kappa)$ such that $\mu(\kappa) - \lambda(\kappa) < 1/\log_{10}\kappa$. Thus, in principle, S can be determined to arbitrary accuracy by this method. However, the amount of computation needed to determine $\lambda(\kappa)$ and $\mu(\kappa)$ increases quite rapidly, and the convergence to S is quite slow. The numbers $\lambda(\delta_m^2)$ and $\mu(\delta_m^2)$, where $\delta_0 = 2$, $\delta_1 = 5$ and $\delta_m = 2\delta_{m-1} + \delta_{m-2}$, are of special significance as we shall see in § 2, and we give the results of the computation of these for $m = 0, 1, 2, 3$ in § 3. This gives

$$\lambda(841) = 1.272441 < S < 1.357603 = \mu(841).$$

Note that the upper bound is a significant improvement over the previously known bounds, but the lower bound is somewhat less than our previous bound of 1.28467. Although $\mu(\delta_4^2)$ could not be computed in the time we allowed, we did show that $\mu(\delta_4^2) < 1.3500$. Thus we have improved (1) to

$$1.28467 < S < 1.35000. \tag{2}$$

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Our results are consistent with the heuristic result $S \approx 1.306951$ which Melzak [6] obtained by fitting a power law $r_n \approx An^{-\alpha}$ to the radii of the first 19600 disks in a packing. We can use the four values of $\lambda(\kappa)$ and $\mu(\kappa)$ to obtain alternate heuristic estimates by approximating $\lambda(\kappa) \approx A_1 (\log \kappa)^{-1} + S_1$ and $\mu(\kappa) \approx A_2 (\log \kappa)^{-1} + S_2$. The least squares fit gives $S_1 = 1.29179$ and $S_2 = 1.29764$, suggesting a value for S near 1.3. (The fact that $S_2 - S_1 > 5 \times 10^{-3}$ prevents us from attaching too much significance to these heuristic results.)

One further point we may observe is that since the Hausdorff dimension $d(C_0, T)$ of the residual set $T \setminus \bigcap (D_n : n \geq 1)$ is dominated by $e(C_0, R)$ (see [1]), our upper bound (2) improves the bound $1.43113 \geq d(C_0, T)$ obtained by Hirst [4].

Our main result is Theorem 1 which appears in § 2. The details of the computation are in § 3.

(Added in proof: We have recently improved our method somewhat, and can now show that $1.300197 < S < 1.314534$.)

1. Preliminary Results

Let $T(a, b, c)$ be the region bounded by three mutually externally tangent circles of curvatures a, b, c, where $0 \leq a \leq b \leq c$, and $b > 0$. We observe that $T(a, b, c)$ has finite area even if $a = 0$, so we may, (and do), allow $a = 0$. Let t be real, and let

$$M(a, b, c; t) = \sum_{n=1}^{\infty} r_n^t, \quad \text{(possibly } \infty), \tag{3}$$

where the r_n are the radii of the disks in a simple osculatory packing of $T(a, b, c)$. (Note the notational difference between (3) and the paper [8] where a, b, c are the radii of the bounding circles.)

The results of [8] show that $M(a, b, c; t)$ is a decreasing function of the variables a, b and c, (strictly decreasing, if finite), and, if $\alpha > 0$,

$$M(\alpha a, \alpha b, \alpha c; t) = \alpha^{-t} M(a, b, c; t). \tag{4}$$

Also, by the results of [8], (or by Lemma 1 of this paper) $M(a, b, c; t) < \infty$ if and only if $M(0, 1, 1; t) < \infty$. Hence, we wish to determine

$$S = \sup\{t : M(0, 1, 1; t) = \infty\} = \inf\{t : M(0, 1, 1; t) < \infty\}. \tag{5}$$

The following Lemma, though basically very simple, is crucial to our method.

LEMMA 1. *Let a, b, c be real numbers with $0 \leq a \leq b \leq c$, and $0 < b$, and let M be defined by* (3). *Then*

$$(a, b, c; t) \leq b^{-t} M(0, 1, 1; t) \tag{7a}$$

$$(a + c)^{-t} M(0, 1, 1; t) \leq M(a, b, c; t). \tag{7b}$$

Equality holds in (7a) *and* (7b) *for finite values of $M(a, b, c; t)$ and $M(0, 1, 1; t)$ if and only if $a = 0$ and $b = c$.*

Proof. Using (4) and the fact that M decreases in all variables, we have

$$M(a, b, c; t) = b^{-t} M(ab^{-1}, 1, cb^{-1}; t) \leqslant b^{-t} M(0, 1, 1; t),$$

since $ab^{-1} \geqslant 0$ and $cb^{-1} \geqslant 1$. This proves (7a), with strict inequality for finite M, unless $ab^{-1}=0$ and $cb^{-1}=1$.

The proof of (7b) uses inversion. We first note that

$$M(a, b, c; t) = c^{-t} M(ac^{-1}, bc^{-1}, 1; t) \geqslant c^{-t} M(ac^{-1}, 1, 1; t),$$

with equality only if $bc^{-1}=1$. This proves (7b) for $a=0$. It remains to show that if $0<r<\infty$, then

$$M(r^{-1}, 1, 1; t) > (r/(r+1))^t M(0, 1, 1; t).$$

For notational convenience, let (r) mean 'circle of radius r'. Choose coordinates for $T(r^{-1}, 1, 1)$ so that the (r) has its centre at the origin and the two (1)'s have their point of tangency on the negative real axis. For an inversion to map (r) into an (∞), the centre of inversion must be on (r), and in order to preserve the radii of the two (1)'s, the circle of inversion must be orthogonal to each of these. In addition, we wish to map the interior of $T(r^{-1}, 1, 1)$ to the interior of $T(0, 1, 1)$. Hence, the centre of inversion is the intersection of (r) with the positive real axis. The radius of the inverting circle is $\gamma = r + \sqrt{(r+1)^2 - 1}$, and the distance from the centre of inversion to $T(r^{-1}, 1, 1)$ is $\delta = 2r\cos\theta$, where $\cos 2\theta = (\gamma - r)/(r+1)$. Thus,

$$\delta^2\gamma^{-2} = 2r^2(\cos 2\theta + 1)\gamma^{-2} = r/(r+1),$$

after some algebraic manipulation. Hence, by the Corollary to Lemma 3.6 of [8], each of the disks of radius ϱ in a packing of $T(r^{-1}, 1, 1)$ inverts into a disk in the packing of $T(0, 1, 1)$ with radius ϱ', where $\varrho > \delta^2\gamma^{-2}\varrho'$. Thus,

$$M(r^{-1}, 1, 1; t) > \delta^{2t}\gamma^{-2t} M(0, 1, 1; t) = (r/(r+1))^t M(0, 1, 1; t),$$

which proves (7b), substituting $r = ca^{-1}$.

LEMMA 2 (Melzak). *Let A, B, and C be three pairwise externally tangent circles with curvatures a, b, and c. Let $\{C_n\}$ be the sequence of disks in which C_1 is the smaller of the disks tangent to A, B, and C, and C_n is the smaller of the disks tangent to A, B, and C_{n-1} for $n=2, 3, \ldots$. Let c_n be the curvature of C_n. Then, for $n=1, 2, \ldots$,*

$$c_n = g_n(a, b, c) = (a+b)n^2 + 2(ab+bc+ca)^{1/2} n + c. \tag{8}$$

Proof. See Lemma 5 of [5]. The restriction $a \leqslant b \leqslant c$ stated there is not used if a, b and c are non-negative.

Analogously we define disks A_n and B_n with curvatures

$$a_n = g_n(b, c, a) \quad \text{and} \quad b_n = g_n(c, a, b). \tag{9}$$

To motivate some of the definitions to follow, we note that we may write $M(a,b,c;t)$ in the following way, suppressing the variable t, and using the notation of Lemma 2:

$$\left.\begin{aligned} M(a, b, c) = {} & a_1^{-t} + \sum_{n=2}^{\infty} (a_n^{-t} + b_n^{-t} + c_n^{-t}) \\ & + \sum_{n=1}^{\infty} \{M(a, c_n, c_{n+1}) + M(b, c_n, c_{n+1}) + M(a, b_n, b_{n+1}) \\ & + M(c, b_n, b_{n+1}) + M(b, a_n, a_{n+1}) + M(c, a_n, a_{n+1})\} \end{aligned}\right\} \tag{10}$$

To see (10), note that if we remove the disks A_n, $(n \geqslant 1)$, B_n, $(n \geqslant 2)$ and C_n, $(n \geqslant 2)$ from $T(a, b, c)$, we are left with triangles $T(a, c_n, c_{n+1})$, $(n \geqslant 1)$, etc.

Our basic idea will be to iterate (10) starting with $(a, b, c) = (0, 1, 1)$, and then to apply Lemma 1. For example, from (10), applying Lemma 1 to $M(0, c_n, c_{n+1})$ etc., we have

$$M(0, 1, 1) \leqslant a_1^{-t} + \sum_{n=2}^{\infty} (a_n^{-t} + b_n^{-t} + c_n^{-t}) + 2M(0, 1, 1) \sum_{n=1}^{\infty} (a_n^{-t} + b_n^{-t} + c_n^{-t}), \tag{11}$$

where $b_n = c_n = (n+1)^2$ and $a_n = 2n^2 + 2n$. If we write

$$f(t) = 2 \sum_{n=1}^{\infty} (a_n^{-t} + b_n^{-t} + c_n^{-t}),$$

and

$$h(t) = a_1^{-t} + \sum_{n=2}^{\infty} (a_n^{-t} + b_n^{-t} + c_n^{-t}),$$

we will have

$$M(0, 1, 1) \leqslant h(t) + M(0, 1, 1) f(t). \tag{12}$$

This would seem to imply that if $1 > f(t)$, then

$$M(0, 1, 1) \leqslant h(t)/(1 - f(t)). \tag{13}$$

However, we must be more careful since, possibly $M(0, 1, 1) = \infty$ and, for example, the true inequality $2\infty \leqslant 1 + \infty$ does not imply that $\infty \leqslant 1$. Lemma 3 will give the technical machinery needed to avoid this fallacy.

Because (10) is rather cumbersome, and since such sums appear often in this paper, we adopt the following notational device: If $F(a, b, c)$ is a function of a, b and c, then

$$\left.\begin{aligned} \sum_{S_3} F(a, b, c) = {} & F(a, b, c) + F(b, c, a) + F(c, a, b) \\ & + F(a, c, b) + F(c, b, a) + F(b, a, c). \end{aligned}\right\} \tag{14}$$

A similar convention will be used for unions of sets and for minima and maxima taken over sets.

LEMMA 3. *Let a, b, c be real numbers with $0\leqslant a\leqslant b\leqslant c$ and $0<b$. Let a_n, b_n and c_n be as in* (8) *and* (9). *For any real t and integer i define $M_i(a, b, c; t)$ by the following recursion:*

$$M_1(a, b, c; t) = a_1^{-t} \tag{15}$$

$$\left.\begin{aligned} M_i(a, b, c; t) &= a_1^{-t} + \sum_{n=2}^{i} (a_n^{-t} + b_n^{-t} + c_n^{-t}) \\ &\quad + \sum_{S_3}\sum_{n=1}^{i-1} M_{i-1}(a, c_n, c_{n+1}; t). \end{aligned}\right\} \tag{16}$$

Then $M_i(a, b, c; t)$ is an increasing sequence with limit $M(a, b, c; t)$.

Proof. Let $\mathscr{P}=\mathscr{P}(a, b, c)$ denote the set of disks in the simple osculatory packing of $T(a, b, c)$ and for $D\in\mathscr{P}$, let $k(D)$ denote the curvature of D. We claim that there is a nest of finite subsets $\mathscr{P}_i=\mathscr{P}_i(a, b, c)\subset\mathscr{P}(a, b, c)$ such that

$$M_i(a, b, c; t) = \sum \{k(d)^{-t}: D\in\mathscr{P}_i\}.$$

The proof follows by inductively checking

$$\mathscr{P}_i(a, b, c) = \{A_1\} \cup \left(\bigcup_{n=2}^{i} \{A_n, B_n, C_n\}\right) \cup \left(\bigcup_{S_3}\bigcup_{n=1}^{i-1} \mathscr{P}_{i-1}(a, c_n, c_{n+1})\right). \tag{17}$$

To show that M_i increases to M we must prove that $\mathscr{P}_i$ increases to $\mathscr{P}$. To do this we will show that $k_i(a, b, c)$, the minimum curvature for disks in $\mathscr{P}\backslash\mathscr{P}_i$, tends to infinity. Using $0\leqslant a\leqslant b\leqslant c$, we have

$$\left.\begin{aligned} k_1(a, b, c) = c_2 > c_1 &= a + b + c + 2(ab + bc + ca)^{1/2} \\ &\geqslant 2b + 2\sqrt{b^2} \\ &= 4b. \end{aligned}\right\} \tag{18}$$

We shall prove by induction that $k_i(a, b, c)\geqslant i^2 b$. For, if this is true for $i-1$, then

$$\left.\begin{aligned} k_i(a, b, c) &= \min_{S_3} \{a_{i+1}, k_{i-1}(a, c_1, c_2)\} \\ &\geqslant \min \{c_{i+1}, (i-1)^2 c_1\} \\ &\geqslant \min \{(i+1)^2 b, (i-1)^2 4b\} \\ &\geqslant i^2 b. \end{aligned}\right\} \tag{19}$$

2. The Main Result

In the statement of Theorem 1, certain functions will appear which we will define and discuss now. Let $0\leqslant a\leqslant b\leqslant c$, and a_n, b_n, c_n be given by (8) and (9). Let $t>\frac{1}{2}$, let $\kappa>0$ be any positive number, and m be any non-negative integer. Define functions f, g and h recursively by

$$f_0(\kappa; a, b, c; t) = \sum_{S_3} \sum_{n=1}^{\infty} c_n^{-t}, \tag{20}$$

$$g_0(\kappa; a, b, c; t) = \sum_{S_3} \sum_{n=1}^{\infty} (a + c_{n+1})^{-t}, \tag{21}$$

$$\left.\begin{aligned} h_0(\kappa; a, b, c; t) &= a_1^{-t} + \sum_{n=2}^{\infty} (a_n^{-t} + b_n^{-t} + c_n^{-t}) \\ &= \tfrac{1}{2} f_0(\kappa; a, b, c; t) - 2a_1^{-t}. \end{aligned}\right\} \tag{22}$$

And, for $m \geqslant 1$,

$$f_m(\kappa; a, b, c; t) = \sum_{S_3} \{\sum [f_{m-1}(\kappa; a, c_n, c_{n+1}, t) : c_n < \kappa] + \sum [c_n^{-t} : c_n \geqslant \kappa]\} \tag{23}$$

$$\left.\begin{aligned} g_m(\kappa; a, b, c; t) = \sum_{S_3} \{\sum [g_{m-1}(\kappa; a, c_n, c_{n+1}; t) : c_n < \kappa] \\ + \sum [(a + c_{n+1})^{-t} : c_n \geqslant \kappa]\} \end{aligned}\right\} \tag{24}$$

$$h_m(\kappa; a, b, c; t) = h_0(\kappa; a, b, c; t) + \sum_{S_3} \sum [h_{m-1}(\kappa; a, c_n, c_{n+1}; t) : c_n < \kappa]. \tag{25}$$

We observe that for fixed m, κ, a, b, c, the functions f, g and h are defined for $t>\frac{1}{2}$ (since $c_n \sim (a+b)\, n^2$), and, if $c_1>1$, are non-negative strictly decreasing, continuous functions of t, which tend to zero as $t \to \infty$ and to ∞ as $t \to \frac{1}{2}+$. Note that the triples (a, c_n, c_{n+1}), (and those obtained by permuting a, b, c) satisfy $a \leqslant c_n \leqslant c_{n+1}$ and $0<c_n$. The following simple result is set aside as a lemma.

LEMMA 4. *Let f, g and h be defined as in the above paragraph. Then, for $\kappa \leqslant 4^{p+1} b$, we have*

$$f_m(\kappa; a, b, c; t) = f_p(\kappa; a, b, c; t) \quad \text{for all } m \geqslant p.$$

Similarly for g_m and h_m.

In case $(a, b, c) = (0, 1, 1)$, let δ_m be defined by $\delta_0 = 2$, $\delta_1 = 5$, $\delta_m = 2\delta_{m-1} + \delta_{m-2}$ for $m \geqslant 3$. Suppose $\kappa \leqslant \delta_p^2$, then

$$f_m(\kappa; 0, 1, 1; t) = f_p(\kappa; 0, 1, 1; t) \quad \text{for all } m \geqslant p.$$

Similarly for g_m and h_m.

Proof. Let τ be the set valued mapping which maps (a, b, c) into the set $\mathcal{S}_1 = \{(a, c_n, c_{n+1}) : n = 1, 2, \ldots, (a, b, c) \text{ permuted in all possible ways}\}$. Let $\mathcal{S}_m = \tau^m(a, b, c)$. The recursive definition of f_m implies that $f_m = f_{m-1}$ provided $\min\{\beta : (\alpha, \beta, \gamma) \in \mathcal{S}_m\} \geqslant \kappa$. But, this is true provided $4^m b \geqslant \kappa$, by methods analogous to those used in Lemma 2 to prove that $\mathcal{P}_i$ increases to $\mathcal{P}$.

In the special case that $(a, b, c)=(0, 1, 1)$ the minimum β is explicitly δ_m^2 which arises from the triple $(0, \delta_m^2, (\delta_m+\delta_{m-1})^2)\in\mathcal{S}_m$. The induction begins with $\delta_{-1}^2=b=1$ and $\delta_0^2=a+b+c+2\sqrt{ab+bc+ca}=4$ and shows that $\delta_m=2\delta_m+\delta_{m-1}$.

Remark. Note that $\delta_m^2/\delta_{m-1}^2\to 1+\sqrt{2}<4$, so that the statement of Lemma 4 for $(0, 1, 1)$ is stronger than for general (a, b, c).

THEOREM 1. *Let S be defined by* (5). *Let $f_m(\kappa; 0, 1, 1; t)$ and $g_m(\kappa; 0, 1, 1; t)$ be defined for $\kappa>0$, $t>\frac{1}{2}$, $m=0, 1,\ldots$ by* (20) *through* (25). *Let $\mu_m(\kappa)$ and $\lambda_m(\kappa)$ be the unique solutions of $f_m(\kappa; 0, 1, 1; \mu_m(\kappa))=1$ and $g_m(\kappa; 0, 1, 1; \lambda_m(\kappa))=1$ respectively. Then*

$$\lambda_m(\kappa)\leqslant S\leqslant \mu_m(\kappa) \quad \text{for all } \kappa>0,\ m=0, 1, \ldots. \tag{26}$$

Furthermore, let δ_m be as in Lemma 4; then for $\delta_m^2\geqslant\kappa$, $\lambda_m(\kappa)=\lambda_{m+1}(\kappa)=\ldots$ and $\mu_m(\kappa)=\mu_{m+1}(\kappa)=\ldots$. For such m, define $\lambda(\kappa)=\lambda_m(\kappa)$ and $\mu(\kappa)=\mu_m(\kappa)$. Then,

$$0<\mu(\kappa)-\lambda(\kappa)<(\log 10)/(\log\kappa). \tag{27}$$

Proof. We first observe that $\lambda_m(\kappa)$ and $\mu_m(\kappa)$ are uniquely defined since f_m and g_m are strictly decreasing and continuous and tend to ∞ and 0, at $\frac{1}{2}$ and ∞ respectively, since $c_1=4>1$.

It will suffice to show that for $t>\lambda_m(\kappa)$, one has

$$M(0, 1, 1; t)<h_m(\kappa; 0, 1, 1; t)/(1-f_m(\kappa; 0, 1, 1; t)) \tag{28}$$

and for $t>\mu_m(\kappa)$, that

$$M(0, 1, 1; t)>h_m(\kappa; 0, 1, 1, t)/(1-g_m(\kappa; 0, 1, 1; t)). \tag{29}$$

We shall treat (28) first, since the proof requires more care. We shall show by induction on m that if M_i is as in Lemma 3, then

$$M_i(a, b, c; t)\leqslant h_m(\kappa; a, b, c; t)+f_m(\kappa; a, b, c; t)\,M_i(0, 1, 1; t). \tag{30}$$

For convenience, we shall suppress the variables κ and t, regarding these as fixed. For $m=0$, from (16) and (22), we have, using Lemma 1 and $M_{i-1}<M_i$, that

$$\left.\begin{aligned} M_i(a, b, c)&<h_0(a, b, c)+\sum_{S_3}\sum_{n=1}^{i-1} M_{i-1}(a, c_n, c_{n+1})\\ &\leqslant h_0(a, b, c)+\sum_{S_3}\sum_{n=1}^{i-1} c_n^{-t}M_{i-1}(0, 1, 1)\\ &<h_0(a, b, c)+f_0(a, b, c)\,M_i(0, 1, 1).\end{aligned}\right\} \tag{31}$$

Now, using induction on m, assume (30) for $1, 2, \ldots, m-1$. Then using Lemma 1,

$$\left.\begin{aligned} M_i(a,b,c) &\leqslant h_0(a,b,c) + \sum_{S_3} \sum [M_{i-1}(a, c_n, c_{n+1}) : c_n < \kappa] \\ &\quad + \sum_{S_3} \sum [M_{i-1}(a, c_n, c_{n+1}) : c_n \geqslant \kappa, n \leqslant i-1] \\ &< h_0(a,b,c) + \sum_{S_3} \sum [h_{m-1}(a, c_n, c_{n+1}) \\ &\quad + f_{m-1}(a, c_n, c_{n+1}) M_{i-1}(0,1,1) : c_n < \kappa] \\ &\quad + \sum_{S_3} \sum [c_n^{-t} M_{i-1}(0,1,1) : c_n \geqslant \kappa]. \end{aligned}\right\} \tag{32}$$

This completes the induction, using the definitions of f_m and h_m. Now set $(a, b, c) = (0, 1, 1)$ in (30) and assume $t > \mu_m(\kappa)$, so that $f_m(\kappa; 0, 1, 1; t) < 1$. We can then transpose the second term of (30) to the left member, divide by $1 - f_m$ and let $i \to \infty$ to obtain (28). This shows $M(0, 1, 1; t) < \infty$ for $t > \mu_m(\kappa)$, hence that $S \leqslant \mu_m(\kappa)$.

The inequality (29) is proved in a similar way. Here we may assume that $M(0, 1, 1; t) < \infty$, since (29) is obvious otherwise. Thus, instead of (16) we may use

$$M(a,b,c) = h_0(a,b,c) + \sum_{S_3} \sum_{n=1}^{\infty} M(a, c_n, c_{n+1}), \tag{33}$$

which is valid for all t. Once (29) is established for $t > \lambda_m(\kappa)$ we let $t \to \lambda_m(\kappa)$ to see that $M(0, 1, 1; \lambda_m(\kappa)) = \infty$ so that $S \geqslant \lambda_m(\kappa)$.

The fact that $\lambda_m(\kappa) = \lambda_{m+1}(\kappa) = \ldots$ and $\mu_m(\kappa) = \mu_{m+1}(\kappa) = \ldots$ for $\delta_m^2 \geqslant \kappa$ follows directly from Lemma 4.

Finally, we must demonstrate (27). We claim that $f_m(\kappa; a, b, c; t)$ is a sum of the form

$$f_m(\kappa; a, b, c; t) = \sum \{w(D)\, k(D)^{-t} : D \in \mathscr{P}_m(\kappa; a, b, c)\}, \tag{34}$$

where $\mathscr{P}_m(\kappa; a, b, c)$ is a subset of the disks in the packing $\mathscr{P}(a, b, c)$ of $T(a, b, c)$, where $k(D)$ denotes the curvature of D, and where $w(D)$ is an integer in the set $\{1, 2, \ldots, 6\}$. This follows easily by induction from the definition of f_m. Further, if $\kappa \leqslant \delta_m^2$, then we claim that

$$k_m(0,1,1) = \min\{k(D) : D \in \mathscr{P}_m(\kappa; 0, 1, 1)\} \leqslant \kappa. \tag{35}$$

To do this, we use the sets $\mathscr{S}_m$ defined in the proof of Lemma 4, and observe that

$$\begin{aligned} k_m(0,1,1) &\leqslant \min(\kappa, k_{m-1}(0,4,9)) \\ &= \min(\kappa, k_{m-2}(0,25,49)) \\ &= \cdots = \min(\kappa, k_0(0, \delta_{m-1}^2, (\delta_{m-1} + \delta_{m-1})^2)) = \min(\kappa, \delta_m^2), \end{aligned}$$

which shows that $k_m(0, 1, 1) \leqslant \kappa$ provided $\delta_m^2 \geqslant \kappa$.

Furthermore, we claim that (independent of κ and m),

$$g_m(\kappa; a, b, c; t) \geqslant 5^{-t} f_m(\kappa; a, b, c; t). \tag{36}$$

To prove (36), we again use induction. First, for $m=0$, we use (20) and (21) and compare sums of terms of the form c_n^{-t} and $(a+c_{n+1})^{-t}$. It is easy to see that if $d=(ab+bc+ca)^{1/2}$, then

$$\left.\begin{aligned} a + c_{n+1} &= a + (a+b)(n+1)^2 + 2d(n+1) + c \\ &\leqslant 5c_n = 5\big((a+b)\,n^2 + 2dn + c\big), \end{aligned}\right\} \tag{37}$$

where we do *not* use $a \leqslant b \leqslant c$. This shows (36) for $m=0$ and the induction to general m is straightforward. Now, using the representation (34) and (35), we see that for $\delta>0$, if $\delta_m^2 \geqslant \kappa$,

$$\left.\begin{aligned} f_m(\kappa, 0, 1, 1\,; t) &= \sum w(D)\, k(D)^{-t} \\ &> \kappa^{\delta} \sum w(D)\, k(D)^{-t-\delta} = \kappa^{\delta} f_m(\kappa\,; 0, 1, 1, t+\delta). \end{aligned}\right\} \tag{38}$$

As part of our computation we shall see that $S<1.4$ so that $\lambda_m(\kappa)<1.4$ for all m and κ, and thus $5^{-\lambda_m(\kappa)}>10^{-1}$. Now, using (36) and (38), we obtain

$$\left.\begin{aligned} 1 &= g_m(\kappa\,; 0, 1, 1\,; \lambda_m(\kappa)) \\ &\geqslant 5^{-\lambda_m(\kappa)} f_m(\kappa\,; 0, 1, 1\,; \lambda_m(\kappa)) \\ &> 10^{-1} \kappa^{\mu_m(\kappa)-\lambda_m(\kappa)} f_m(\kappa\,; 0, 1, 1\,; \mu_m(\kappa)) \\ &= 10^{-1} \exp\big((\mu_m(\kappa) - \lambda_m(\kappa)) \log \kappa\big) \cdot 1\,. \end{aligned}\right\} \tag{39}$$

This proves (27) and the Theorem.

3. Computations Based on Theorem 1

In order to apply Theorem 1, we must compute the solutions of $f_m(\delta_m^2; 0, 1, 1\,; t)=1$ and $g_m(\delta_m^2; 0, 1, 1\,; t)=1$. To compute f_m from (23) and (20), we must first decide how to compute

$$\sum [c_n^{-t} : c_n \geqslant \kappa], \tag{40}$$

where $c_n=(a+b)\,n^2+2dn+c$, $d=(ab+bc+ca)^{1/2}$. We consider the computation of

$$\sum_{n=N}^{\infty} ((a+b)\,n^2 + 2dn + c)^{-t}, \tag{41}$$

where N is to be chosen. We may write

$$(a+b)\,n^2 + 2dn + c = (a+b)\,((n+\alpha)^2 - \beta), \tag{42}$$

where

$$\alpha = d/(a+b), \quad \text{and} \quad \beta = ab/(a+b)^2. \tag{43}$$

Observe that $\beta \leqslant \frac{1}{4}$. Thus

$$\left.\begin{aligned} \sum_{n=N}^{\infty} ((a+b)\,n^2 + 2dn + c)^{-t} &= \sum_{n=N}^{\infty} (a+b)^{-t}((n+\alpha)^2 - \beta)^{-t} \\ &= (a+b)^{-t} \sum_{k=0}^{\infty} \binom{-t}{k} (-\beta)^k \sum_{n=N}^{\infty} (n+\alpha)^{-2t-2k}. \end{aligned}\right\} \tag{44}$$

We now use the well-known Euler-Maclaurin formula for the latter sum (see [7], p. 128), for which the first few terms are

$$\left.\begin{aligned}\sum_{n=N}^{\infty}(n+\alpha)^{-s} \approx \tfrac{1}{2}(N+\alpha)^{-s} + (N+\alpha)^{-s+1}/(s-1) + s(N+\alpha)^{-s-1}/12 \\ - s(s+1)(s+2)(N+\alpha)^{-s-3}/720.\end{aligned}\right\} \tag{45}$$

For g_m, we need series of the form

$$\sum[(a+c_{n+1})^{-t} : c_n \geqslant \kappa]. \tag{46}$$

In this case, we may write

$$(a+b)n^2 + 2dn + (c+a) = (a+b)((n+\alpha)^2+\gamma), \tag{47}$$

where

$$\alpha = d/(a+b), \qquad \gamma = a^2/(a+b)^2. \tag{48}$$

It is natural to compute the series $\sum(a+c_n)^{-t}$ and $\sum(b+c_n)^{-t}$ together. Then

$$\sum_{n=N}^{\infty}\{(a+c_n)^{-t}+(b+c_n)^{-t}\} = (a+b)^{-t}\sum_{k=0}^{\infty}\binom{-t}{k}(\gamma_1^k+\gamma_2^k)\sum_{n=N}^{\infty}(n+\alpha)^{-2t-2k}, \tag{49}$$

where $\gamma_1 = a^2/(a+b)^2$ and $\gamma_2 = b^2/(a+b)^2$, so that $\gamma_1^k + \gamma_2^k \leqslant 1$.

We can always insure a choice of $N \geqslant 20$ by summing a few initial terms if necessary, and since we will find that $2t \geqslant 2.4$, we can compute the series in (44) and (49) with a relative accuracy better than 10^{-9} by using only $k=0$, 1, and the terms of the expansion shown in (45). For example, we will have

$$\sum_{n=N}^{\infty} c_n^{-t} = (a+b)^{-t}(\zeta(2t, N+\alpha) + t\beta\zeta(2t+2, N+\alpha) + \varepsilon) \tag{50}$$

where $|\varepsilon| < 10^{-9}$, and

$$\zeta(s, N+\alpha) = \sum_{n=N}^{\infty}(n+\alpha)^{-s} \tag{51}$$

is computed as the right member of (45).

We can avoid doing multiplications and taking square roots in the computation of $c_n = (a+b)n^2 + 2(ab+bc+ca)^{1/2}n + c$ by introducing a new variable which we denote by u. Given a, b, c with $0 \leqslant a \leqslant b \leqslant c$, u will be the curvature of the largest circle tangent to the circles which bound $T(a, b, c)$. Thus, by the formula of Descartes,

$$u = a + b + c - 2(ab+bc+ca)^{1/2}. \tag{52}$$

Then

$$c_0 = c \tag{53}$$

$$c_1 = 2(a+b+c) - u \tag{54}$$

$$c_n = 2(a+b+c_{n-1}) - c_{n-2}, \quad \text{for} \quad n \geqslant 2. \tag{55}$$

Regarding κ and t as fixed we write $f_m(\kappa; a, b, c; t) = F_m(u, a, b, c)$. Now formula (23) becomes

$$F_m(u, a, b, c) = \sum_{S_3} \{\sum [F_{m-1}(b, a, c_n, c_{n+1}): c_n < \kappa] + \sum [c_n^{-t}: c_n \geqslant \kappa]\}, \tag{56}$$

because the circle of curvature b is the larger of the two circles tangent to the circles which bound $T(a, c_n, c_{n+1})$. For fixed κ and m, we wish to compute t so that $F_m(0, 0, 1, 1) = 1$. Note that the initial values of (u, a, b, c) are integers, and the 4-tuples needed in (56) are computed by the linear recursion (53)–(55), so that the variables u, a, b, c are always integers.

One can estimate quite simply the time needed to compute $f_m(\kappa, a, b, c; t)$, by using the formula (56). One finds, using induction that this time $T_m(\kappa; a, b, c) \approx A(3 \log \kappa)^m \sqrt{\kappa/b}$, for some constant A. Since we want $\kappa = \delta_m^2 \approx (1+\sqrt{2})^{2m}$, it is apparent that the amount of computing increases very rapidly with m.

The above ideas were incorporated into a computer program written in Fortran, using double precision, which was run on the 360/75 at the California Institute of Technology. The equations $f=1$ and $g=1$ were solved by the method of regula falsi using a starting value 1.35 for upper bounds and 1.26 for lower bounds. The second value was computed using the approximate slope $-(\log \kappa)$. The results are given in table 1. The heading 'iter' gives the number of values of t for which f and g were computed to obtain the indicated accuracy for $f-1$ and $g-1$.

Table 1

m	κ	$\lambda_m(\kappa)$	$g-1$	iter	$\mu_m(\kappa)$	$f-1$	iter
0	4	1.191561	2×10^{-5}	6	1.571658	-1×10^{-5}	7
1	25	1.246116	-5×10^{-5}	5	1.410266	-1×10^{-5}	6
2	144	1.263876	2×10^{-6}	5	1.373234	-3×10^{-7}	6
3	841	1.272441	7×10^{-5}	5	1.357603	4×10^{-6}	5
4	4900				< 1.35	< 0	1

Computing time was 128 seconds for $\mu_0, \mu_1, \mu_2, \mu_3$ and 10 minutes for the first iteration for μ_4 which showed that $\mu_4(4900) < 1.35$. For λ_0, λ_1 and λ_2, computing time was 36 seconds, while λ_3 took 314 seconds. The shorter time needed for the upper bounds is due to an extra symmetry present in f_0 over g_0.

A least squares fit of $\lambda_0, \ldots, \lambda_3$ and $\mu_0, \ldots, \mu_3$ to curves of the form $A(\log \kappa)^{-1} + B$ gives

$$\left.\begin{aligned} \lambda(\kappa) &\doteqdot 1.29179 - .06990(\log \kappa)^{-1} \\ \mu(\kappa) &\doteqdot 1.29764 + .18897(\log \kappa)^{-1}. \end{aligned}\right\} \tag{57}$$

The R.M.S. deviation in each case was less than 3×10^{-3}. The results (57) would suggest that the constant $\log(10)$ in (27) is somewhat pessimistic, but that the order of the error $\mu(\kappa) - \lambda(\kappa)$ is, in fact, approximately $(\log \kappa)^{-1}$, as suggested by (27).

4. Concluding Remarks

1. The methods developed here can be modified in many ways. For example, rather than using a parameter κ, one could instead use a sequence of integers $(k_1, k_2, \ldots)$ and define f_m by

$$f_m(a, b, c; t) = \sum_{S_3} \left\{ \sum_{n=1}^{k_m} f_{m-1}(a, c_n, c_{n+1}) + \sum_{n>k_m} c_n^{-t} \right\}, \tag{58}$$

with a similar expression for g. If $k_m \geqslant 2^m$, for example, one can show that the solutions of $f_m = 1$ do converge to S. The disadvantage of (58) over (23) is that an excessive amount of computation is done with very small quantities of the order $c_{k_m}^{-t}$, which contribute mainly to round-off error. Also, one does not have the property that eventually $f_m = f_{m+1} = \cdots$. With (23), the smallest quantities which appear in the computation are, generally speaking, of the order κ^{-t}, which would seem to be an advantage.

2. One can combine the method of this paper with that of [2] for obtaining lower bounds. That is, one considers, instead of g_m which is a sum of terms of the form s^{-t}, a weighed average $\sum w \cdot s$, and then uses Holder's inequality to determine conditions on the weights w so that $\sum s^{-t} \geqslant 1$. This is quite easy to formalize, using the sort of g_m mentioned in the previous paragraph, but will not be pursued here.

3. A possible way to proceed further would be to use the heuristic formulas (57) to predict values for $\lambda(\delta_m^2)$ and $\mu(\delta_m^2)$ for $m \geqslant 4$, and then to simply compute g_m and f_m for these predicted values, without iterating. Since it is known that $|g_m'/g_m| \geqslant \log \kappa$ and $|f_m'/f_m| \geqslant \log \kappa$, the values of λ and μ could then be corrected to insure that $g_m \geqslant 1$ and $f_m \leqslant 1$.

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