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On some functional equations of Gołąb–Schinzel type

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Dedicated to the memory of Alexander M. Ostrowski on the occasion of the 100th anniversary of his birth.

Summary. Let E be a real Hausdorff topological vector space. We consider the following binary law $*$ on $\mathbb{R} \times E$:

$$(\alpha, \beta) * (\alpha', \beta') = (\lambda\alpha\alpha', \alpha'^k\beta + \alpha^l\beta') \qquad \text{for } (\alpha, \beta), (\alpha', \beta') \in \mathbb{R} \times E$$

where λ is a nonnegative real number, k and l are integers.

In order to find all subgroupoids of $(\mathbb{R} \times E, *)$ which depend faithfully on a set of parameters, we have to solve the following functional equation:

$$f(f(y)^k x + f(x)^l y) = \lambda f(x) f(y) \qquad (x, y \in E). \tag{1}$$

In this paper, all solutions $f\colon \mathbb{R} \to \mathbb{R}$ of (1) which are in the Baire class I and have the Darboux property are obtained. We obtain also all continuous solutions $f\colon E \to \mathbb{R}$ of (1). The subgroupoids of $(\mathbb{R}^* \times E, *)$ which depend faithfully and continuously on a set of parameters are then determined in different cases. We also deduce from this that the only subsemigroup of L_n^1 of the form $\{(F(x_2, x_3, \ldots, x_n), x_2, x_3, \ldots, x_n);\ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\}$, where the mapping $F\colon \mathbb{R}^{n-1} \to \mathbb{R}^*$ has some regularity property, is $\{1\} \times \mathbb{R}^{n-1}$.

We may notice that the Gołąb–Schinzel functional equation is a particular case of equation (1) ($k = 0, l = 1, \lambda = 1$). So we can say that (1) is of Gołąb–Schinzel type. More generally, when E is a real algebra, we shall say that a functional equation is of Gołąb–Schinzel type if it is of the form:

$$f(f(y)^k x + f(x)^l y) = F(x, y, f(x), f(y), f(xy))$$

where k and l are integers and F is a given function in five variables. In this category of functional equations, we study here the equation:

$$f(f(y)^k x + f(x)^l y) = f(xy) \qquad (x, y \in \mathbb{R}; f\colon \mathbb{R} \to \mathbb{R}). \tag{4}$$

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This paper extends the results obtained by N. Brillouët and J. Dhombres in [3] and completes some results obtained by P. Urban in his Ph.D. thesis [11] (this work has not yet been published).

Introduction

Let E be a real vector space. The functional equation

$$f(f(x)\cdot y + x) = f(x)f(y) \qquad (x, y \in E) \tag{GS}$$

where f is a mapping from E into $\mathbb{R}$, is called the *functional equation of Gołąb–Schinzel*. It has been first considered by Aczél in 1957, and then by Gołąb and Schinzel in 1959. The general solution of (GS) has been described (cf. [1]) and all the continuous solutions of (GS) have been explicitly obtained when E is a real topological vector space (cf [3] and [6]).

We consider now the binary law $*$ defined on $\mathbb{R}\times E$ by

$$(\alpha, \beta) * (\alpha', \beta') = (\lambda\alpha\alpha', \alpha'^k\beta + \alpha^l\beta') \qquad \text{for } (\alpha, \beta), (\alpha', \beta') \in \mathbb{R}\times E$$

where λ is a nonnegative real number, k and l are integers.

Let us recall the following definition (cf. [3]):

Definition 1. *A subset H of $\mathbb{R}\times E$* depends faithfully *on a set F of parameters if there exists a mapping g from F onto H: $g(u) = (\alpha(u), \beta(u))$ for $u \in F$ such that we have either*

(i) $\beta(F) = E$ and $\beta(u) = \beta(u')$ implies $\alpha(u) = \alpha(u')$

or

(ii) $\alpha(F) = \mathbb{R}$ and $\alpha(u) = \alpha(u')$ implies $\beta(u) = \beta(u')$.

We look for the subgroupoids of $(\mathbb{R}\times E, *)$ which depend faithfully on a set F of parameters.

In the case (i), the relation $f(\beta(u)) = \alpha(u)$ $(u \in F)$ defines a function from E into $\mathbb{R}$ which satisfies the following functional equation:

$$f(f(y)^k x + f(x)^l y) = \lambda f(x)f(y) \qquad (x, y \in E). \tag{1}$$

In the case (ii), the relation $f(\alpha(u)) = \beta(u)$ $(u \in F)$ defines a function from $\mathbb{R}$ into E which satisfies the following functional equation:

$$f(\lambda xy) = y^k f(x) + x^l f(y) \qquad (x, y \in \mathbb{R}). \tag{2}$$

The functional equation of Gołąb–Schinzel (GS) is a particular case of equation (1) ($k = 0, l = 1, \lambda = 1$). So we can say that (1) is of Gołąb–Schinzel type.

More generally, when E is a real algebra, we shall say that a functional equation is of *Gołąb–Schinzel type* if it is of the following form:

$$f(f(y)^k x + f(x)^l y) = F(x, y, f(x), f(y), f(xy)), \tag{3}$$

where k and l are integers and F is a given function in five variables.

In this category of functional equations, we shall also study here the following equation:

$$f(f(y)^k x + f(x)^l y) = f(xy) \qquad (x, y \in \mathbb{R}, f\colon \mathbb{R} \to \mathbb{R}). \tag{4}$$

We shall mainly look for the solutions of (1) and (4) which have some regularity property.

Following A. M. Bruckner and J. G. Ceder in [4], we shall denote by $\mathscr{D}\mathscr{B}_1$ *the set of all functions from* $\mathbb{R}$ *into* $\mathbb{R}$ *which are in the Baire class I and possess the Darboux property.*

We shall obtain here explicitly all solutions of (1) and (4) which belong to $\mathscr{D}\mathscr{B}_1$.

For this, we shall use the following property of the functions of $\mathscr{D}\mathscr{B}_1$.

LEMMA 2. *Let f be a function in $\mathscr{D}\mathscr{B}_1$. Let us define the function $\zeta\colon \mathbb{R}^2 \to \mathbb{R}$ by:*

$$\zeta(x, y) = f(y)^k x + f(x)^l y \qquad (x, y \in \mathbb{R}).$$

Then, for every fixed real numbers x and y, the functions $\zeta(\cdot, y)$ and $\zeta(x, \cdot)$ have the Darboux property.

Proof of Lemma 2. If x is a nonzero real number, the graph of the function $xf(\cdot)^k$ is connected since f is in $\mathscr{D}\mathscr{B}_1$ (cf. [4]). Therefore, since the function $(t, s) \to f(x)^l t + s$ is continuous, the function $\zeta(x, \cdot)$ has the Darboux property.

We shall also use the following result:

LEMMA 3. *If $g\colon \mathbb{R} \to \mathbb{R}$ has the Darboux property and satisfies the following functional equation:*

$$g(g(x)) = \alpha g(x) + \beta x \qquad (x \in \mathbb{R}), \tag{5}$$

where α and β are given real numbers and $\beta \neq 0$, then g is continuous.

Proof of Lemma 3. The function g has the Darboux property and, because of the form of (5), g is one-to-one. Therefore, g is continuous (cf. [4]).

Let us notice that in [11] P. Urban has studied the solutions of (1) on a restricted domain in the case where λ is equal to 1, namely the solutions $f: [0, +\infty[\to \mathbb{R}$ of (1) which are in Baire class I, have the Darboux property and satisfy $f(y)^k x + f(x)^l y \geqslant 0$ for every x and y in $[0, +\infty[$. He has also investigated the so-called "trivial solutions" of (1) which are defined on a ring $(X, +, \cdot)$ and take on their values in $\{1, 0, -1\}$.

Finally we mention that W. Benz studied in [2] the cardinality of the set of discontinuous solutions $f: \mathbb{R} \to \mathbb{R}$ of (1).

I. Investigation of functional equation (1)

Let us first study some particular cases.

1. Case $\lambda = 0, k \geqslant 0, l \geqslant 0$

In this case, (1) is just $f(f(y)^k x + f(x)^l y) = 0$ $(x, y \in E)$. For $k = 0$ and $l \geqslant 0$ it is obvious that the unique solution of (1) is $f \equiv 0$.

So we consider now the case where k and l are positive integers. Let us suppose that there exists an element x_0 in E such that $f(x_0) \neq 0$. By taking $x = y = 0$ in (1) we get $f(0) = 0$. Therefore x_0 is different from 0. Let us suppose also that the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(tx_0)$ $(t \in \mathbb{R})$ belongs to $\mathscr{D}\mathscr{B}_1$. By taking $x = x_0$ and $y = tx_0$ $(t \in \mathbb{R})$ in (1), we obtain

$$f(f(tx_0)^k x_0 + f(x_0)^l t x_0) = 0 \qquad \text{for every } t \text{ in } \mathbb{R}.$$

Let us define $\psi(t) = g(t)^k + tf(x_0)^l$ $(t \in \mathbb{R})$. We have $f(\psi(t)x_0) = 0$ for every t in $\mathbb{R}$. Since g is in $\mathscr{D}\mathscr{B}_1$, we may prove, as in Lemma 2, that ψ has the Darboux property.

Therefore $\psi(\mathbb{R})$ is an interval of $\mathbb{R}$ which contains 0 but does not contain 1. So $\psi(\mathbb{R})$ is included in $]-\infty, 1[$. Let us suppose that ψ is bounded below by b. The relation $f(tx_0)^k = \psi(t) - tf(x_0)^l$ $(t \in \mathbb{R})$ shows that $f(\mathbb{R}x_0)^k = \mathbb{R}$. This implies that k is an odd integer and so $f(\mathbb{R}x_0) = \mathbb{R}$. Let c be the unique point of $]0, 1[$ which satisfies $c^k + c^l = 1$. Then there exists a nonzero real number s such that $f(sx_0) = c$. By taking $x = y = sx_0$ in (1), we obtain $f(sx_0) = 0$, which brings a contradiction. Therefore $\psi(\mathbb{R})$ contains $]-\infty, 0]$ and we have $f(tx_0) = 0$ for every nonpositive real number t. Since ψ is bounded above by 1, we deduce first from $\psi(t) = tf(x_0)^l$ $(t \leqslant 0)$ that $f(x_0)^l$ is a positive real number and then that

$g(t)^k = \psi(t) - tf(x_0)^l$ tends to $-\infty$ when t goes to $+\infty$. In view of the Darboux property of g, we deduce that $g([0, +\infty[)^k$ contains $]-\infty, 0]$. By taking now $x = tx_0$, $t < 0$, and $y = rx_0, r > 0$ in (1), we get $f(g(r)^k tx_0) = 0$, and therefore $f(sx_0) = 0$ for every positive real number s. This contradicts $f(x_0) \neq 0$.

PROPOSITION 4. *In the class of functions $f: E \to \mathbb{R}$ which have the property that for every x in E the function defined by $g_x(t) = f(tx)$ $(t \in \mathbb{R})$ belongs to $\mathcal{DB}_1$, the unique solution of (1) in the case $\lambda = 0$ is $f \equiv 0$.*

2. Case $k = l = 0, \lambda > 0$

In this case, (1) is $f(x + y) = \lambda f(x)f(y)$ $(x, y \in E)$. So, λf is a solution of Cauchy's exponential equation. Therefore, all the solutions of (1) are given by

(i) $f \equiv 0$
(ii) $f(x) = (1/\lambda)\, e^{g(x)}$ $(x \in E)$ where $g: E \to \mathbb{R}$ is an arbitrary additive function.

3. Case $k = 0, l > 0, \lambda > 0$

In this case, (1) is $f(x + f(x)^l y) = \lambda f(x)f(y)$ $(x, y \in E)$. We suppose here that E *is a real topological vector space.*

The function $g(x) = f(x)^l$ $(x \in E)$ is a solution of

$$g(x + g(x)y) = \lambda^l g(x)g(y) \qquad (x, y \in E) \tag{6}$$

which is similar to (GS).

By taking $x = y = 0$ in (6), we obtain either $g(0) = 0$ or $g(0) = \lambda^{-l}$.

When $g(0) = 0$, we get $g \equiv 0$ as we can see by taking $y = 0$ in (6).

So we consider now the case where $g(0) = \lambda^{-l}$. By taking $x = 0$ in (6), we get $g(y) = g(\lambda^{-l}y)$ $(y \in E)$ and therefore

$$g(y) = g(\lambda^{-nl}y) \qquad (y \in E) \quad \text{for every positive integer } n. \tag{7}$$

When λ is different from 1, (7) implies $g \equiv g(0) = \lambda^{-l}$ if we suppose f continuous. In this case, f is identically equal to $1/\lambda$.

When λ is equal to 1, (6) is just the functional equation of Gołąb–Schinzel for which we know all the continuous solutions (cf. [3]). We deduce the solutions of (1): either

$$f(x) = \mathrm{Sup}(1 + \langle x, x^* \rangle, 0)^{1/l} \qquad (x \in E)$$

or

$$f(x) = (1 + \langle x, x^* \rangle)^{1/l} \qquad (x \in E) \quad \text{when } l \text{ is an odd integer,}$$

where x^* is an element of the topological dual of E.

So we have the following result.

PROPOSITION 5. *All continuous solutions $f: E \to \mathbb{R}$ of*

$$f(x + f(x)^l y) = \lambda f(x) f(y) \tag{1}$$

are given by

(*i*) $f \equiv 0$

(*ii*) *when* $\lambda > 0, \lambda \neq 1: f \equiv 1/\lambda$

(*iii*) *when* $\lambda = 1$: $f(x) = \operatorname{Sup}(1 + \langle x, x^* \rangle, 0)^{1/l}$ $(x \in E)$

(*iv*) *when* $\lambda = 1$ *and* l *is odd*:

$$f(x) = (1 + \langle x, x^* \rangle)^{1/l} \qquad (x \in E),$$

where x^* is an element of the topological dual of E.

4. So, from now on, we consider only *the case where λ is a positive real number and k, l are positive integers*

In [3] all continuous solutions $f: \mathbb{R} \to \mathbb{R}$ of (1) have been obtained in the case $k = l = 1$. Let us recall the result:

PROPOSITION 6. *When $k = l = 1$, all continuous solutions $f: \mathbb{R} \to \mathbb{R}$ of* (1) *are given by*

$$\textit{if} \quad \lambda \neq 2: \quad f \equiv 0 \quad \textit{and} \quad f \equiv \frac{1}{\lambda}$$

$$\textit{if} \quad \lambda = 2: \quad f \equiv 0, \quad f \equiv \frac{1}{2}, \quad f(x) = \mu x \quad f(x) = \operatorname{Sup}(\mu x, 0)$$

where μ is an arbitrary nonzero real number.

Let us remark that, in the proof of this result, the hypothesis of continuity for f is not necessary. It is enough to suppose that f belongs to $\mathcal{DB}_1$. Namely, let f be a not identically zero solution of (1) in $\mathcal{DB}_1$. There exists $x_0 \neq 0$ in $\mathbb{R}$ such that

$\gamma = f(x_0) \neq 0$. By Lemma 2, the function g defined by $g(y) = x_0 f(y) + \gamma y$ ($y \in \mathbb{R}$) has the Darboux property. Moreover, g satisfies the following functional equation:

$$g(g(y)) = (\lambda + 1)\gamma g(y) - \lambda\gamma^2 y \qquad (y \in \mathbb{R}). \tag{8}$$

Therefore, g is continuous by Lemma 3 and f, obtained from g by

$$f(y) = \frac{1}{x_0}(g(y) - \gamma y),$$

is continuous.

So, *Proposition* 6 *gives all the solutions of* (1) *which are in* $\mathscr{D}\mathscr{B}_1$.

We shall obtain now all the solutions $f: \mathbb{R} \to \mathbb{R}$ of (1) which are in $\mathscr{D}\mathscr{B}_1$, when k and l are arbitrary positive integers.

We give first some conditions under which a solution of (1) is necessarily constant.

We begin with the following Lemma.

LEMMA 7. *If* $f: \mathbb{R} \to \mathbb{R}$ *is a solution of* (1) *in* $\mathscr{D}\mathscr{B}_1$ *which is bounded above on* $\mathbb{R}$, *then* f *is constant.*

Proof of Lemma 7. For an indirect proof, we suppose that f is a solution of (1) in $\mathscr{D}\mathscr{B}_1$ bounded above on $\mathbb{R}$ and that f is non-constant.

Let M be an upper bound of $f(\mathbb{R})$. By taking $x = y$ in (1), we obtain $\lambda f(x)^2 \leqslant M$ for every x in $\mathbb{R}$. Since f is not identically zero, M is a strictly positive real number.

By taking $x = y$ in (1), we get successively:

$$|f(x)| \leqslant \sqrt{\frac{M}{\lambda}} \qquad \text{for every } x \text{ in } \mathbb{R},$$

$$|f(x)| \leqslant \frac{M^{\frac{1}{4}}}{\lambda^{\frac{1}{2}+\frac{1}{4}}} \qquad \text{for every } x \text{ in } \mathbb{R},$$

...

$$|f(x)| \leqslant \frac{M^{\frac{1}{2^n}}}{\lambda^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}}} \qquad \text{for every } x \text{ in } \mathbb{R} \text{ and every positive integer } n.$$

As n goes to $+\infty$, we obtain

$$|f(x)| \leqslant \frac{1}{\lambda} \qquad \text{for every } x \text{ in } \mathbb{R}. \tag{9}$$

Since f is bounded and non identically zero, we have, by the Darboux property of the function $\zeta(\cdot, t)$ (Lemma 2), $\zeta(\mathbb{R}, t) = \mathbb{R}$ for each $t \in \mathbb{R}$ such that $f(t) \neq 0$. Therefore, for every real number x there exists a real number s such that $\zeta(s, t) = x$.

In view of the Darboux property of f, we may choose x and t in $\mathbb{R}$ such that $0 < |f(t)| < |f(x)|$. By using (1) and (9), we obtain

$$0 < |f(t)| < |f(x)| = |f(\zeta(s, t))| = \lambda |f(s)| |f(t)| \leqslant |f(t)|$$

which brings a contradiction.

Therefore, if f is bounded above on $\mathbb{R}$, f is constant.

In [11] P. Urban has proved the following result:

PROPOSITION 8. *If $f: \mathbb{R} \to \mathbb{R}$ is a solution of* (1) *which belongs to* $\mathscr{D}\mathscr{B}_1$, *then*:

(*a*) *if $f(0) = 1/\lambda$, f is identically equal to* $1/\lambda$

(*b*) *if $f(0) = 0$ and if $\lambda \neq 1/c$, where c is the unique point of* $]0, 1[$ *satisfying* $c^k + c^l = 1$, *f is identically zero.*

Proof of Proposition 8. The following is a slight modification of the proof of Theorem 2.1 in [11].

Let us suppose that f is a solution of (1) which belongs to $\mathscr{D}\mathscr{B}_1$. Then, f satisfies either $f(0) = 1/\lambda$ or $f(0) = 0$.

(*a*) *In the case where* $f(0) = 1/\lambda$, let us suppose that f is not identically equal to $1/\lambda$. So, there exists x_0 in $\mathbb{R}$ such that $f(x_0) \neq 1/\lambda$ and we may write $f(x_0) = 1/\lambda + \varepsilon$ where ε is a nonzero real number. By taking $x = y = x_0$ in (1), we get, with $x_1 = x_0(f(x_0)^k + f(x_0)^l)$, $f(x_1) = \lambda(1/\lambda + \varepsilon)^2$.

By taking $x = y = x_1$ in (1), we get with $x_2 = x_1(f(x_1)^k + f(x_1)^l)$: $f(x_2) = \lambda^3(1/\lambda + \varepsilon)^4$. This way we can build a sequence of real numbers x_n such that

$$f(x_n) = \lambda^{2^n - 1}\left(\frac{1}{\lambda} + \varepsilon\right)^{2^n} = \frac{1}{\lambda}(1 + \varepsilon\lambda)^{2^n} \quad \text{for every positive integer } n.$$

If $f(x_0) > 1/\lambda$, ε is a positive real number and the sequence $\{f(x_n)\}_{n \in N}$ tends to $+\infty$. By the Darboux property of f, we deduce $f(\mathbb{R}) \supset [1/\lambda, +\infty[$.

If $f(x_0) < 1/\lambda$, ε is a negative real number and we can assume $-1/\lambda < \varepsilon < 0$.

Therefore, the sequence $\{f(x_n)\}_{n \in N}$ converges to 0. By the Darboux property of f, we deduce: $f(\mathbb{R}) \supset]0, 1/\lambda]$.

We notice that $f(\mathbb{R})$ does not contain 0 since, if there exists x_0 in $\mathbb{R}$ such that $f(x_0) = 0$, we get, by taking $x = y = x_0$ in (1), $f(0) = 0$, which is not the case.

So, by Lemma 7, $f(\mathbb{R})$ satisfies one of the two following conditions:

(i) $f(\mathbb{R}) = [1/\lambda, +\infty[$
(ii) $f(\mathbb{R}) =]0, +\infty[$.
In the case (*i*), there exists a nonzero real number t such that

$$f(t) > \max\left(\frac{1}{\lambda}, \left(\frac{1}{\lambda}\right)^{l/k}\right).$$

We have $\zeta(-t, t) = t(f(-t)^l - f(t)^k)$.

If $f(-t)^l \leqslant f(t)^k$ then $\zeta(-t, t)$ and $\zeta(0, t)$ do not have the same sign. By Lemma 2, $\zeta(\cdot, t)$ has the Darboux property. So there exists a nonzero real number u such that $\zeta(u, t) = 0$. On the other hand, (1) implies $1/\lambda = f(\zeta(u, t)) > f(u) \geqslant 1/\lambda$ which is impossible. Therefore, we have $f(-t)^l > f(t)^k$. It is easy to verify that $\zeta(-t, t)$ and $\zeta(-t, 0)$ do not have the same sign and so, by the Darboux property of $\zeta(-t, \cdot)$ (Lemma 2), there exists a non zero real number u such that $\zeta(-t, u) = 0$. The functional equation (1) implies now

$$f(\zeta(-t, u)) = \frac{1}{\lambda} \geqslant f(-t) > f(t)^{k/l} > \frac{1}{\lambda},$$

which is also impossible.

Therefore, the case (i) cannot occur.

Let us consider now the case (*ii*). Let c be the unique point of $]0, 1[$ satisfying $c^k + c^l = 1$. There exists a real number x_0 such that $f(x_0) = c$. By taking $x = y = x_0$ in (1), we get $\lambda = 1/c$. There also exists a real number y_0 such that $f(y_0) = 1$. By taking first $x = y_0, y = 0$ in (1), we get $f(y_0 c^k) = 1$. Next, by taking $x = 0, y = y_0$ in (1), we get $f(y_0 c^l) = 1$. Setting now $x = y_0 c^k$ and $y = y_0 c^l$ in (1), we obtain $f(y_0 c^k + y_0 c^l) = 1 = \lambda = 1/c$ which brings a contradiction. Therefore, the case (ii) cannot occur.

In conclusion, when $f(0) = 1/\lambda$, f is identically equal to $1/\lambda$.

(*b*) *Let us consider now the case where* $f(0) = 0$ *and* $\lambda \neq 1/c$. If there exists a real number x_0 such that $f(x_0) = c$ then, by taking $x = y = x_0$ in (1), we get as before $\lambda = 1/c$, which is not the case. Therefore, considering the Darboux property of f, we have $f(x) < c$ for every x in $\mathbb{R}$. By Lemma 7, f is constant and is therefore identically zero.

This ends the proof of Proposition 8.

We shall obtain now all the solutions $f\colon \mathbb{R} \to \mathbb{R}$ of (1) which are in $\mathscr{DB}_1$.

For this, we need the following result (cf [5]).

PROPOSITION 9. *The complete set of continuous solutions* $g: \mathbb{R} \to \mathbb{R}$ *of the functional equation*:

$$g(g(x)) = (\gamma + 1)g(x) - \gamma x \qquad (x \in \mathbb{R}) \tag{10}$$

where γ *is a given nonzero real number, is given by*

(*a*) *if* $\gamma > 0, \gamma \neq 1$:

(*i*) $g(x) = \begin{cases} \gamma x + (1-\gamma)a & \text{for } x \leqslant a \\ x & \text{for } a \leqslant x \leqslant b \\ \gamma x + (1-\gamma)b & \text{for } x \geqslant b \end{cases}$

with $-\infty \leqslant a < b \leqslant +\infty$

(*ii*) $g(x) = \gamma x + \delta$ $(x \in \mathbb{R})$ *with* $\delta \in \mathbb{R}$

(*b*) *if* $\gamma = 1$:
$g(x) = x + \delta$ $(x \in \mathbb{R})$ *with* $\delta \in \mathbb{R}$

(*c*) *if* $\gamma < 0, \gamma \neq -1$:
(*i*) $g(x) = \gamma x + \delta$ $(x \in \mathbb{R})$ *with* $\delta \in \mathbb{R}$
(*ii*) $g(x) = x$ $(x \in \mathbb{R})$

(*d*) *if* $\gamma = -1$:
(*i*) $g(x) = x$ $(x \in \mathbb{R})$
(*ii*) $g(x) = \begin{cases} \Phi(x) & \text{for } x \in]-\infty, c] \\ \Phi^{-1}(x) & \text{for } x \in [c, +\infty[, \end{cases}$

where c *is an arbitrary real number and* Φ *is an arbitrary continuous and strictly decreasing function mapping* $]-\infty, c]$ *onto* $[c, +\infty[$

We begin with the following Lemma.

LEMMA 10. *If the functional equation* (1) *has a non constant solution* $f: \mathbb{R} \to \mathbb{R}$ *in* $\mathcal{DB}_1$, *then* $k = l$.

Proof of Lemma 10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non constant solution of (1) in $\mathscr{D}\mathscr{B}_1$. By Proposition 8, it satisfies $f(0) = 0$ and $\lambda = 1/c$. We first prove that there exists a nonzero real number x_0 such that $f(x_0) = c$. For an indirect proof, we suppose that we have $f(x) \neq c$ for every x in $\mathbb{R}$. Then we have $f(x) < c$ for every x in $\mathbb{R}$ since f has the Darboux property and $f(0) = 0$. By Lemma 7, f should be constant, which brings a contradiction.

So, there exists a nonzero real number x_0 such that $f(x_0) = c$. With $y = x_0$ in (1), we get

$$f(xc^k + x_0 f(x)^l) = f(x). \tag{11}$$

With $x = x_0$ in (1) and changing y into x, we get also

$$f(xc^l + x_0 f(x)^k) = f(x). \tag{11'}$$

Let us define $g(x) = xc^k + x_0 f(x)^l$ $(x \in \mathbb{R})$. Then, (11) implies

$$g(g(x)) = (c^k + 1)g(x) - c^k x \qquad (x \in \mathbb{R}). \tag{12}$$

Let us now define $h(x) = xc^l + x_0 f(x)^k$ $(x \in \mathbb{R})$. Then, (11′) implies

$$h(h(x)) = (c^l + 1)h(x) - c^l x \qquad (x \in \mathbb{R}). \tag{12'}$$

Since $g(x) = \zeta(x, x_0)$ and $h(x) = \zeta(x_0, x)$, the functions g and h have the Darboux property. Moreover, by Lemma 3, they are continuous. By using Proposition 9 and the facts that f is a nonconstant solution of (1) and $c^k + c^l = 1$, we get:

from (12),

$$f(x)^l = \begin{cases} c^l \dfrac{a}{x_0} & \text{for } x \leqslant a \\[2ex] c^l \dfrac{x}{x_0} & \text{for } a \leqslant x \leqslant b \quad \text{with } -\infty \leqslant a < b \leqslant +\infty \\[2ex] c^l \dfrac{b}{x_0} & \text{for } x \geqslant b \end{cases}$$

and from (12′),

$$f(x)^k = \begin{cases} c^k \dfrac{\alpha}{x_0} & \text{for } x \leqslant \alpha \\ c^k \dfrac{x}{x_0} & \text{for } \alpha \leqslant x \leqslant \beta \quad \text{with } -\infty \leqslant \alpha < \beta \leqslant +\infty \\ c^k \dfrac{\beta}{x_0} & \text{for } x \geqslant \beta. \end{cases}$$

Since $f(0) = 0$ and $f(x_0) = c$, 0 and x_0 belong to both intervals $[a, b]$ and $[\alpha, \beta]$. Therefore, also $x_0/2$ belongs to these intervals and we have

$$f\left(\frac{x_0}{2}\right)^l = \frac{1}{2} c^l \quad \text{and} \quad f\left(\frac{x_0}{2}\right)^k = \frac{1}{2} c^k.$$

Thus

$$\left|f\left(\frac{x_0}{2}\right)\right| = c\left(\frac{1}{2}\right)^{1/l} = c\left(\frac{1}{2}\right)^{1/k}.$$

This implies $k = l$.

THEOREM 11. *All the solutions* $f: \mathbb{R} \to \mathbb{R}$ *of the functional equation* (1) *which are in* $\mathscr{D}\mathscr{B}_1$, *are given by*

(*a*) *if* $\lambda \neq 1/c$ *or if* $k \neq l$:
 (*i*) $f = 0$ (*ii*) $f = 1/\lambda$

(*b*) *if* $\lambda = 1/c$ *and if* $k = l$ *is even*:
 (*i*) $f = 0$ (*ii*) $f = c = (\tfrac{1}{2})^{1/l}$
 (*iii*) $f(x) = (\operatorname{Sup}(\mu x, 0))^{1/l}$ *where* μ *is an arbitrary real number*

(*c*) *if* $\lambda = 1/c$ *and if* $k = l$ *is odd*:
 (*i*) $f = 0$ (*ii*) $f = c = (\tfrac{1}{2})^{1/l}$
 (*iii*) $f(x) = vx^{1/l}$ (*iv*) $f(x) = \operatorname{Sup}(vx^{1/l}, 0)$ *where* v *is an arbitrary real number*.

Proof of Theorem 11. The constant solutions of (1) are obviously $f = 0$ and $f = 1/\lambda$ since $f(0)$ is either 0 or $1/\lambda$. So we look now for the non constant solutions of (1) which are in $\mathscr{D}\mathscr{B}_1$. If such a solution exists, we have, by Proposition 8 and Lemma 10, $\lambda = 1/c$, $k = l$ and $f(0) = 0$.

Let us define: $\Psi(x) = f(x)^l$ ($x \in \mathbb{R}$). Ψ is a nonconstant solution of

$$\Psi(x\Psi(y) + y\Psi(x)) = 2\Psi(x)\Psi(y) \tag{13}$$

and Ψ is in $\mathscr{DB}_1$. By the remark following Proposition 6, we deduce that

either (i) $\Psi(x) = \mu x$

or (ii) $\Psi(x) = \mathrm{Sup}(\mu x, 0)$,

where μ is a nonzero real number.

If l is even, we have necessarily $\Psi(x) = \mathrm{Sup}(\mu x, 0)$ and therefore $f(x) = \pm(\mathrm{Sup}(\mu x, 0))^{1/l}$. By Lemma 7, we see that the image of f is never contained in $]-\infty, 0]$. Therefore, we get only $f(x) = (\mathrm{Sup}(\mu x, 0))^{1/l}$. It is easy to verify that this is a solution of (1).

If l is odd, the solutions (i) and (ii) of (13) lead to $f(x) = \nu x^{1/l}$ and $f(x) = \mathrm{Sup}(\nu x^{1/l}, 0)$ where ν is an arbitrary nonzero real number. These also are solutions of (1).

We look now for the solutions $f\colon E \to \mathbb{R}$ of (1) when E is a real vector space.

We begin with a generalization of Proposition 8.

PROPOSITION 12. *Let E be a real vector space. If $f\colon E \to \mathbb{R}$ is a solution of* (1) *such that the functions f_x defined by $f_x(t) = f(tx)$ ($t \in \mathbb{R}$) belong to $\mathscr{DB}_1$ for every x in $E - \{0\}$, then*:

(*a*) *if $f(0) = 1/\lambda$, f is identically equal to $1/\lambda$,*

(*b*) *if $f(0) = 0$ and if $\lambda \neq 1/c$ or $k \neq l$, then f is identically equal to* 0.

Proof of Proposition 12. It is easy to verify that, for every x in $E - \{0\}$, the functions $f_x\colon \mathbb{R} \to \mathbb{R}$ are solutions of (1) in $\mathscr{DB}_1$.

(a) If $f(0) = 1/\lambda$, we have $f_x(0) = 1/\lambda$ for every x in $E - \{0\}$. By Proposition 8, f_x is identically equal to $1/\lambda$ for every x in $E - \{0\}$. Therefore f is identically equal to $1/\lambda$.

(b) If $f(0) = 0$ and if $\lambda \neq 1/c$ or $k \neq l$, f_x is identically zero for every x in $E - \{0\}$ by Theorem 11. Therefore, f is identically zero.

REMARK. We may notice that, if the functional equation (1) has a nonconstant solution $f\colon E \to \mathbb{R}$ for which the functions f_x belong to $\mathscr{DB}_1$ for every x in $E - \{0\}$, then there exists $x \in E - \{0\}$ such that f_x is a nonconstant solution of (1) in $\mathscr{DB}_1$. From Lemma 10, we deduce $k = l$.

So, Lemma 10 can be formulated in a more general way as follows:

Let E be a real vector space.

If the functional equation (1) *has a nonconstant solution $f: E \to \mathbb{R}$ such that the functions f_x belong to $\mathcal{DB}_1$ for every x in $E - \{0\}$, then $k = l$.*

We obtain now all continuous solutions $f: E \to \mathbb{R}$ of (1) when E is a real Hausdorff topological vector space.

THEOREM 13. *Let E be a real Hausdorff topological vector space. All the continuous solutions $f: E \to \mathbb{R}$ of functional equation* (1) *are given by*

(*a*) *if $\lambda \neq 1/c$ or if $k \neq l$:*
  (*i*) $f = 0$ (*ii*) $f = 1/\lambda$
(*b*) *if $\lambda = 1/c$ and if $k = l$ is even:*
  (*i*) $f = 0$ (*ii*) $f = c = (\frac{1}{2})^{1/l}$
  (*iii*) $f(x) = (\mathrm{Sup}(\langle x, x^* \rangle, 0))^{1/l}$ *where x^* belongs to the topological dual of E.*
(*c*) *if $\lambda = 1/c$ and if $k = l$ is odd:*
  (*i*) $f = 0$ (*ii*) $f = c = (\frac{1}{2})^{1/l}$
  (*iii*) $f(x) = (\langle x, x^* \rangle)^{1/l}$ (*iv*) $f(x) = \mathrm{Sup}((\langle x, x^* \rangle)^{1/l}, 0)$
  where x^* belongs to the topological dual of E.

Proof of Theorem 13. Let $f: E \to \mathbb{R}$ be a continuous solution of (1). Then the functions f_x defined by $f_x(t) = f(tx)$ ($t \in \mathbb{R}$) for every x in $E - \{0\}$ are continuous solutions of (1).

By Proposition 12, if $f(0) = 1/\lambda$, f is identically equal to $1/\lambda$ and, if $f(0) = 0$ and if $\lambda \neq 1/c$ or $k \neq l$, f is identically zero.

So, we consider now the case where $k = l$, $\lambda = 1/c$ and $f(0) = 0$. In this case, $c = (\frac{1}{2})^{1/l}$. The function $\Psi: E \to \mathbb{R}$ defined by $\Psi(x) = f(x)^l$ ($x \in E$) is a non constant continuous solution of (13). All continuous solutions $\Psi: E \to \mathbb{R}$ of (13) are known and the non constant continuous solutions are given by (cf. [3] Theorem 15)

(i) $\Psi(x) = \langle x, x^* \rangle$

(ii) $\Psi(x) = \mathrm{Sup}(\langle x, x^* \rangle, 0)$,

where x^* is a nonzero element of the topological dual of E. (We note here that, in a private communication, K. Baron observed that Theorem 15 of [3] stated for a real Hausdorff locally convex topological vector space is true for a general real Hausdorff topological vector space.)

As in Theorem 11, we deduce then the nonconstant continuous solutions of (1) given in (b) and (c).

5. Application to finding subgroupoids

(a) We consider first the groupoid $\mathbb{R} \times E$, where E is a real topological vector space and the binary operation is given by

$$(\alpha, \beta) * (\alpha', \beta') = (\lambda\alpha\alpha', \alpha'^k\beta + \alpha^l\beta') \qquad (\alpha, \alpha' \in \mathbb{R}; \beta, \beta' \in E) \tag{14}$$

where λ is a positive real number and k, l are positive integers.

Let us recall the following definition (cf. [3]):

DEFINITION 14. *A subset H of* $\mathbb{R} \times E$ depends faithfully and continuously *upon a set F of parameters if F is a topological space and if the mapping $g: F \to H$ defined in Definition* 1 *satisfies the following property*:

— *in the case* (*i*), *the mapping $\alpha: F \to \mathbb{R}$ is continuous and β admits locally a continuous lifting.*

— *in the case* (*ii*), *the mapping $\beta: F \to E$ is continuous and α admits locally a continuous lifting.*

When we look for the subgroupoids of $(\mathbb{R} \times E, *)$ which depend faithfully and continuously on a topological space F of parameters, we have to find:

in the case (i), all the continuous functions $f: E \to \mathbb{R}$ defined by $f(\beta(u)) = \alpha(u)$ $(u \in F)$ which satisfy the functional equation (1)

in the case (ii), all the continuous functions $f: \mathbb{R} \to E$ defined by $f(\alpha(u)) = \beta(u)$ $(u \in F)$ which satisfy the functional equation (2).

The continuous solutions of (1) are given by Theorem 13 when E is a real Hausdorff topological vector space.

For the functional equation (2), we have the following result which has been proved in the case $k = 1 < l$, $\lambda = 1$, by S. Midura (cf. [7], Theorem 1):

PROPOSITION 15. *Let E be a real vector space. All solutions $f: \mathbb{R} \to E$ of the functional equation*

$$f(\lambda xy) = y^k f(x) + x^l f(y) \qquad (x, y \in \mathbb{R}) \tag{2}$$

are given by

(*a*) $f = 0$

and

(*b*) *if* $k \neq l$ *and* $\lambda = 1$, *by*
$f(x) = (x^l - x^k)v$ $(x \in \mathbb{R})$, *where* v *is an arbitrary nonzero element of* E.

(*c*) *if* $k = l$ *and* $\lambda = 1$, *by*

$$f(x) = \begin{cases} x^l h(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where h *is a homomorphism from* $(\mathbb{R}^*, \cdot)$ *into* $(E, +)$.

(*d*) *if* $k = l$ *and* $\lambda = 2^{1/l}$, *by*
$f(x) = x^l v$ $(x \in \mathbb{R})$, *where* v *is an arbitrary nonzero element of* E.

Proof of Proposition 15. Let $f: \mathbb{R} \to E$ be a not identically zero solution of (2). By inverting x and y in (2), we get

$$f(\lambda xy) = x^k f(y) + y^l f(x) \qquad (x, y \in \mathbb{R}) \tag{2'}$$

(2) and (2′) imply

$$(x^l - x^k)f(y) = (y^l - y^k)f(x) \qquad (x, y \in \mathbb{R}).$$

If $k \neq l$, there exists a nonzero real number y_0 such that $y_0^l \neq y_0^k$. We deduce

$$f(x) = (x^l - x^k)v \qquad (x \in \mathbb{R}), \tag{15}$$

where v is a nonzero element of E. It is easy to check that the function given by (15) is a solution of (2) if, and only if, $\lambda = 1$.

Let us suppose now $k = l$. By taking $x = y = 1/\lambda$ in (2), we get

$$f\left(\frac{1}{\lambda}\right)\left(1 - \frac{2}{\lambda^l}\right) = 0,$$

which implies either $f(1/\lambda) = 0$ or $\lambda = 2^{1/l}$.

Let us suppose that $f(1/\lambda) = 0$. By taking $y = 1/\lambda$ in (2), we obtain

$$f(x)\left(1 - \frac{1}{\lambda^l}\right) = 0$$

for every x in $\mathbb{R}$. Since f is not identically zero, this implies $\lambda = 1$. Let us define

$$g(x) = \frac{f(x)}{x^l} \tag{16}$$

for every nonzero real number x. We see that f is a solution of (2) if, and only if, g is a homomorphism from $(\mathbb{R}^*, \cdot)$ into $(E, +)$, where $\mathbb{R}^*$ is the set of all nonzero real numbers. This gives the solution (c) of (2).

Finally, let us suppose $\lambda = 2^{1/l}$. Now f is a solution of (2) if, and only if, the function g defined by (16) is a solution of

$$g(x) + g(y) = 2g(2^{1/l}xy) \qquad (x, y \in \mathbb{R}^*). \tag{17}$$

Taking $y = 1/2^{1/l}$ in (17), we see that g is a constant function. Therefore, we obtain $f(x) = x^l v$ $(x \in \mathbb{R})$ where v is a nonzero element of E.

REMARK. Notice that (b), (c), (d) of Proposition 15 give the expression of f^{-1} when $f: E \to \mathbb{R}$ is an arbitrary invertible solution of the functional equation (1).

From Proposition 12 and Proposition 15, we get easily the following results when E is a real topological vector space:

COROLLARY 16. *Let λ be a positive real number different from* 1 *and* $2^{1/l}$. *We consider the groupoid* $(\mathbb{R}^* \times E, *)$ *where the binary law* $*$ *is defined by* (14). *All the subgroupoids of* $(\mathbb{R}^* \times E, *)$ *which depend faithfully and continuously on a set of parameters are the groupoid* $\{(1/\lambda, \beta); \beta \in E\}$ *and the groupoid* $\{(\alpha, 0); \alpha \in \mathbb{R}^*\}$.

The following Corollary can be compared with Corollary 1 from [7].

COROLLARY 17. *Let us consider the groupoid* $(\mathbb{R}^* \times E, *)$ *where the binary law* $*$ *is defined by*

$$(\alpha, \beta) * (\alpha', \beta') = (\alpha\alpha', \alpha'^k\beta + \alpha^l\beta') \qquad (\alpha, \alpha' \in \mathbb{R}^*; \beta, \beta' \in E).$$

All the subgroupoids of $(\mathbb{R}^* \times E, *)$ *which depend faithfully and continuously on a set of parameters are the groupoids* $\{(\alpha, 0); \alpha \in \mathbb{R}^*\}$ *and* $\{(1, \beta); \beta \in E\}$ *and, if* $k \neq l$, *the groupoids* $G_v = \{(\alpha, (\alpha^l - \alpha^k)v); \alpha \in \mathbb{R}^*\}$, *where v is an element of E; if $k = l$, the groupoids* $G_v = \{(\alpha, \alpha^l \operatorname{Log}(|\alpha|)v); \alpha \in \mathbb{R}^*\}$, *where v is an element of E.*

(b) Let us apply now the result of Theorem 11 for determining some subsemigroups of L_n^1. In [11] P. Urban describes this example for $n = 3$ and 4 and asks the question for an arbitrary positive integer n. This example is based on the papers [8] and [10].

We recall first the definition of L_n^1 (cf. [7]). We consider a family $\mathcal{J}$ of intervals of $\mathbb{R}$ containing 0 and a family $\mathcal{D}$ of diffeomorphisms of class C^∞, each element of $\mathcal{D}$ being defined on an element of $\mathcal{J}$ and mapping 0 to 0. Let n be a positive integer. We introduce on $\mathcal{D}$ the equivalence relation j^n defined by $(f, g) \in j^n$ $(f, g \in \mathcal{D})$ if, and only if, all the derivatives of $(f - g)$ of order $k \leqslant n$ vanish at 0. On the set $J_n \mathbb{R}$ of all the equivalence classes $j^n f$, we define the binary law

$$(j^n f) \cdot (j^n g) = j^n(f \circ g).$$

With this law, $J_n \mathbb{R}$ is a group which is called L_n^1.

The coordinates of the point $j^n f$ are the coefficients of the nth Taylor's expansion of f. Let $j^n f$ and $j^n g$ be two elements of L_n^1. Let us define

$$\beta_i = f^{(i)}(0), \qquad \alpha_i = g^{(i)}(0) \qquad \text{for } i = 1, 2, \ldots, n,$$

where $f^{(i)}$ is the ith derivative of f. $(\beta_1, \beta_2, \ldots, \beta_n)$ is the set of coordinates of $j^n f$. Therefore, the set of coordinates of $(j^n f) \cdot (j^n g) = j^n(f \circ g)$ is

$$((f \circ g)'(0), (f \circ g)''(0), \ldots, (f \circ g)^{(n)}(0)).$$

We shall look first for the subsemigroups of L_3^1 of the form $L = \{(F(y, z), y, z); y, z \in \mathbb{R}\}$ where F is a mapping from $\mathbb{R}^2$ into $\mathbb{R}^*$.

The following proof has been given by P. Urban in [11]. We can prove that L_3^1 is just $\mathbb{R}^* \times \mathbb{R}^2$ endowed with the following binary law:

$$(\beta_1, \beta_2, \beta_3) \cdot (\alpha_1, \alpha_2, \alpha_3) = (\beta_1 \alpha_1, \beta_1 \alpha_2 + \beta_2 \alpha_1^2, \beta_1 \alpha_3 + 3\beta_2 \alpha_2 \alpha_1 + \beta_3 \alpha_1^3).$$

Then, L is a subsemigroup of L_3^1 if, and only if, F satisfies the following functional equation:

$$F(F(y_1, z_1)y_2 + y_1 F(y_2, z_2)^2, F(y_1, z_1)z_2 + 3y_1 y_2 F(y_2, z_2) + z_1 F(y_2, z_2)^3)$$
$$= F(y_1, z_1)F(y_2, z_2). \tag{18}$$

Taking $y_1 = y_2 = 0$ in (18), we obtain:

$$F(0, F(0, z_1)z_2 + z_1 F(0, z_2)^3) = F(0, z_1)F(0, z_2).$$

Let us define $f(z) = F(0, z)$ $(z \in \mathbb{R})$. This f is a solution of:

$$f(f(z_1)z_2 + f(z_2)^3 z_1) = f(z_1)f(z_2)$$

which is just the functional equation (1) with $\lambda = 1, k = 1, l = 3$.

If we suppose that the function $f: \mathbb{R} \to \mathbb{R}^*$ is in $\mathcal{DB}_1$, Theorem 11 implies that f is identically equal to 1.

Let us take now $y_2 = 0$ in (18). We obtain:

$$F(y_1, F(y_1, z_1)z_2 + z_1) = F(y_1, z_1) \qquad (y_1, z_1, z_2 \in \mathbb{R}). \tag{19}$$

Since $F(y_1, z_1)$ belongs to $\mathbb{R}^*$, the mapping: $z_2 \to F(y_1, z_1)z_2 + z_1$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. Therefore, (19) implies that $F(y_1, z_1)$ does not depend on z_1, and so, is equal to $F(y_1, 0)$. So, we have:

$$F(y_1, z_1) = F(y_1, 0) \qquad (y_1, z_1 \in \mathbb{R}). \tag{20}$$

Let us consider now (18) with $z_1 = z_2 = 0$ and let us define:

$$g(y) = F(y, 0) \qquad (y \in \mathbb{R}).$$

Using (20), we see that g satisfies:

$$g(g(y_1)y_2 + g(y_2)^2 y_1) = g(y_1)g(y_2)$$

which is just the functional equation (1) with $\lambda = 1, k = 1, l = 2$.

If we suppose that $g: \mathbb{R} \to \mathbb{R}^*$ is in $\mathcal{DB}_1$, Theorem 11 implies that g is identically equal to 1. We deduce from (20) that F is identically equal to 1.

So, we obtain the following result.

PROPOSITION 18. *The only subsemigroup of L_3^1 of the form*

$$L = \{(F(y, z), y, z); y, z \in \mathbb{R}\}$$

where the mapping $F: \mathbb{R}^2 \to \mathbb{R}^*$ has the property that the functions $g(y) = F(y, 0)$ $(y \in \mathbb{R})$ and $f(z) = F(0, z)$ $(z \in \mathbb{R})$ are in $\mathcal{DB}_1$, is $\{1\} \times \mathbb{R}^2$.

P. Urban has proved in [11] that the same result holds for L_4^1 with a similar proof. Namely, the only subsemigroup of L_4^1 of the form:

$$L = \{(F(y, z, u), y, z, u); y, z, u \in \mathbb{R}\}$$

where the mapping $F: \mathbb{R}^3 \to \mathbb{R}^*$ has the property that the functions $F(y, 0, 0)$, $F(0, z, 0)$ and $F(0, 0, u)$ are in $\mathcal{DB}_1$, is: $\{1\} \times \mathbb{R}^3$.

Now, it is possible to prove, by using similar arguments, that this result holds for L_n^1 with an arbitrary positive integer n. Namely, we have the following:

THEOREM 19. *The only subsemigroup of L_n^1 of the form*

$$L = \{(F(x_2, x_3, \ldots, x_n), x_2, x_3, \ldots, x_n); (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}\}$$

where the mapping $F: \mathbb{R}^{n-1} \to \mathbb{R}^*$ has the property that the functions $x_i \to F(0, \ldots, 0, x_i, 0, \ldots, 0)$ belong to $\mathcal{DB}_1$, is $\{1\} \times \mathbb{R}^{n-1}$.

Proof of Theorem 19. In [9] S. Midura has proved that L_n^1 is just the set $\mathbb{R}^* \times \mathbb{R}^{n-1}$ endowed with the following binary law:

$$(\beta_1, \beta_2, \ldots, \beta_n) * (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\gamma_1, \gamma_2, \ldots, \gamma_n)$$

where

$$\gamma_p = \sum_{j=1}^{p} \sum_{\substack{k_1 + k_2 + \cdots + k_p = j \\ k_1 + 2k_2 + \cdots + pk_p = p \\ k_i \in N}} \beta_j \frac{p! \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_p^{k_p}}{k_1! \cdots k_p! (1!)^{k_1} (2!)^{k_2} \cdots (p!)^{k_p}}$$

for $p = 1, 2, \ldots, n$.

Using this characterization of L_n^1, we see that the set

$$L = \{(F(x_2, x_3, \ldots, x_n), x_2, x_3, \ldots, x_n); (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}\},$$

where the mapping $F: \mathbb{R}^{n-1} \to \mathbb{R}^*$ has the required property, is a subsemigroup of L_n^1 if, and only if, F satisfies the following functional equation:

$$F\Bigg(F(x_2, x_3, \ldots, x_n) y_2 + x_2 F(y_2, y_3, \ldots, y_n)^2, \ldots, F(x_2, x_3, \ldots, x_n) y_p + \sum_{j=2}^{p} \sum_{\substack{k_1 + k_2 + \cdots + k_p = j \\ k_1 + 2k_2 + \cdots + pk_p = p \\ k_i \in N}} x_j \frac{p! (F(y_2, \ldots, y_n))^{k_1} y_2^{k_2} \cdots y_p^{k_p}}{k_1! \cdots k_p! (1!)^{k_1} \cdots (p!)^{k_p}}, \ldots\Bigg)$$

$$= F(x_2, x_3, \ldots, x_n) F(y_2, y_3, \ldots, y_n). \tag{21}$$

By taking $x_2 = x_3 = \cdots = x_{n-1} = y_2 = y_3 = \cdots = y_{n-1} = 0$ in (21) and by setting $f(x) = F(0, 0, \ldots, x)$ $(x \in \mathbb{R})$, we get

$$f(f(x_n)y_n + f(y_n)^n x_n) = f(x_n)f(y_n).$$

Therefore, f is a solution of functional equation (1) with $\lambda = 1, l = 1, k = n$. Since f is in $\mathscr{D}\mathscr{B}_1$ and does not vanish, f is identically equal to 1 by Theorem 11. So, we have $F(0, 0, \ldots, x_n) = 1$ for every x_n in $\mathbb{R}$.

By taking now $x_2 = \cdots = x_{n-1} = 0$ in (21), we obtain

$$F(y_2, y_3, \ldots, y_n + x_n(F(y_2, \ldots, y_n))^n) = F(y_2, y_3, \ldots, y_n). \tag{22}$$

Since F does not vanish, the mapping $x_n \to y_n + x_n(F(y_2, \ldots, y_n))^n$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. We deduce from (22) that $F(y_2, y_3, \ldots, y_n)$ does not depend on y_n and is equal to $F(y_2, y_3, \ldots, y_{n-1}, 0)$. Therefore, $F(x_2, x_3, \ldots, x_n) = F(x_2, \ldots, x_{n-1}, 0)$ for every $(x_2, x_3, \ldots, x_n)$ in $\mathbb{R}^{n-1}$.

With the same arguments, it is easy to prove by induction that we have

$$\text{for } k = 0, 1, \ldots, n-3: \quad \begin{cases} F(0, \ldots, 0, x_{n-k}, \ldots, x_n) = 1 \\ F(x_2, \ldots, x_n) = F(x_2, \ldots, x_{n-k-1}, 0, \ldots, 0) \end{cases}$$

for every $(x_2, \ldots, x_n)$ in $\mathbb{R}^{n-1}$.

Considering this assertion for $k = n-3$, we get

$$F(x_2, \ldots, x_n) = F(x_2, 0, \ldots, 0) \quad \text{for every } (x_2, \ldots, x_n) \text{ in } \mathbb{R}^{n-1}. \tag{23}$$

Setting $g(x) = F(x, 0, \ldots, 0)$ $(x \in \mathbb{R})$, we see that (21) is equivalent to the following:

$$g(g(x_2)y_2 + x_2 g(y_2)^2) = g(x_2)g(y_2).$$

Therefore, g is a solution of functional equation (1) with $\lambda = 1, l = 1, k = 2$. Since g is in $\mathscr{D}\mathscr{B}_1$ and does not vanish, g is identically equal to 1 by Theorem 11. By (23), we deduce that F is identically equal to 1.

II. Investigation of functional equation (4)

For this section, let us recall the following notations used for the intervals of $\mathbb{R}$. An interval of $\mathbb{R}$ is denoted by

(a, b) if it is open, half-open, half-closed or closed

$[a, b)$ if a belongs to the interval and b may or may not belong to it

$(a, b]$ if b belongs to the interval and a may or may not belong to it

$[a, b[$ if a belongs to the interval and b does not belong to it.

In the case where $k = 0, l \geqslant 0$, it is easy to see by taking $y = 0$ in (4) that the only solutions of (4) are the constant functions.

So, we suppose now that k and l are positive integers. By taking $y = 0$ in (4), we obtain:

$$f(xf(0)^k) = f(0) \qquad \text{for every } x \text{ in } \mathbb{R}.$$

So, if $f(0) \neq 0$, f is a constant function.

Therefore, we have to look for the solutions $f: \mathbb{R} \to \mathbb{R}$ of (4) which satisfy $f(0) = 0$. More precisely, *we shall find now all solutions of* (4) *which belong to* $\mathscr{D}\mathscr{B}_1$ *and satisfy* $f(0) = 0$.

Let us define $\gamma = f(1)^k$. By taking $y = 1$ in (4), we get

$$f(f(x)^l + \gamma x) = f(x) \qquad \text{for every } x \text{ in } \mathbb{R}. \tag{24}$$

We define $g(x) = f(x)^l + \gamma x$ $(x \in \mathbb{R})$. (24) implies that g is a solution of functional equation (10). Since f belongs to $\mathscr{D}\mathscr{B}_1$, g has the Darboux property by Lemma 2. If $f(1) \neq 0$, g is continuous by Lemma 3 and is therefore given by Proposition 9. So, we shall consider the two cases: $f(1) = 0$ and $f(1) \neq 0$.

1. $f(1) = 0$

In this case, (10) is the functional equation of indempotence:

$$g(g(x)) = g(x) \qquad (x \in \mathbb{R}). \tag{25}$$

Considering the Darboux property of g, we obtain, if f is not identically zero,

$$g(x) = f(x)^l = \begin{cases} x & \text{if } x \in (a, b) \\ \in (a, b) & \text{if } x \notin (a, b) \end{cases} \qquad \text{with } -\infty \leqslant a < b \leqslant +\infty. \tag{26}$$

$f(0) = 0$ and $f(1) = 0$ imply that (a, b) contains 0, but does not contain 1. So, we have $a \leqslant 0 \leqslant b \leqslant 1$.

Furthermore, by taking $x = 1$ in (4), we see that the function $f(x)^k$ is also a solution of (25) which possesses the Darboux property. So, f satisfies

$$f(x)^k = \begin{cases} x & \text{if } x \in (\alpha, \beta) \\ \in (\alpha, \beta) & \text{if } x \notin (\alpha, \beta) \end{cases} \quad \begin{matrix} \text{with } -\infty \leqslant \alpha < \beta \leqslant +\infty \\ (\alpha \leqslant 0 \leqslant \beta \leqslant 1). \end{matrix} \tag{27}$$

It is easily seen that $(a, b) \cap (\alpha, \beta)$ contains 0.

Let us suppose that $(a, b) \cap (\alpha, \beta) = \{0\}$. We have then two possibilities:

(i) $(a, b) = [0, b)$ and $(\alpha, \beta) = (\alpha, 0]$

(ii) $(a, b) = (a, 0]$ and $(\alpha, \beta) = [0, \beta)$.

Let us consider the first case. By taking x in $]0, b[$ and y in $]\alpha, 0[$ in the functional equation (4), we obtain $f(2xy) = f(xy)$. We may choose x and y sufficiently close to 0 such that $2xy$ belongs to $]\alpha, 0[$. We get then, by using (27), $2xy = xy$, which is impossible. So, the first case cannot occur.

We may prove similarly that the second case cannot occur either.

Therefore, $(a, b) \cap (\alpha, \beta)$ is an interval (η, δ) which contains 0. For every x in (η, δ), we have, by (26) and (27), $f(x)^k = f(x)^l = x$. This implies $k = l$. Consequently, we have

$$\alpha = \eta = a \quad \text{and} \quad \beta = \delta = b.$$

We have: $0 \leqslant b \leqslant 1$. If b is strictly positive we have, by taking $0 < x < b$ and $y = x/2$ in (4),

$$f(x^2) = f\left(\frac{x^2}{2}\right).$$

Since

$$0 < \frac{x^2}{2} < x^2 < x < b,$$

this implies, by (26), $x^2 = x^2/2$, which is impossible. We conclude $b = 0$.

We notice that, since $f(\mathbb{R})^l$ is contained in $]-\infty, 0]$, l is necessarily an odd integer.

We shall first determine f on $[0, 1]$. If we take $a < y < 0$ and $0 < x \leqslant 1$, we have $a < xy < 0$ and the functional equation (4) implies $f((x + f(x)^l)y)^l = xy$. Let us define $h(x) = x + f(x)^l$ ($x \in \mathbb{R}$). So, we have

$$f(yh(x))^l = xy \tag{28}$$

for $a < y < 0$ and $0 < x \leqslant 1$.

Let us suppose that h vanishes at x_0 on $]0, 1]$. Then (28) implies $x_0 y = 0$, which is impossible. Therefore, h is a Darboux function which satisfies $h(1) = 1$ and which does not vanish on $]0, 1]$. This implies, with $f(x)^l \leqslant 0$ ($x \in \mathbb{R}$), that $h(]0, 1])$ is a subset of $]0, 1]$. So $yh(x)$ belongs to $]a, 0[$. (26) and (28) imply $h(x) = x$, or $f(x) = 0$. Therefore, f is identically zero on $[0, 1]$.

Let us consider now (4) with x in $]1, +\infty[$ and $y = 1/x$. We get

$$f\left(\frac{1}{x} f(x)^l\right) = f(1) = 0.$$

Since $f(x)^l$ belongs to $(a, 0]$, we obtain, by (26),

$$\frac{1}{x} f(x)^l = 0 \quad \text{or} \quad f(x) = 0.$$

So, f is identically zero on $]1, +\infty[$, and therefore on $[0, +\infty[$.

Let us suppose that a is a finite real number. The functional equation (4) with $x < a$ and $0 < y < a/x$ gives, by (26),

$$f(yf(x)^l)^l = xy. \tag{29}$$

Therefore, $f(x)$ is different from zero for every $x < a$. Let us suppose that y satisfies

$$0 < y < \operatorname{Inf}\left(\frac{a}{x}, \frac{a}{f(x)^l}\right).$$

Equations (26) and (29) imply $f(x)^l = x$, which is impossible since $x < a$. We deduce $a = -\infty$. So, we obtain

$$f(x)^l = \begin{cases} x & \text{if } x \leqslant 0 \\ 0 & \text{if } x \geqslant 0. \end{cases}$$

This implies: $f(x) = \operatorname{Inf}(x^{1/l}, 0)$ $(x \in \mathbb{R})$ since l is an odd integer. It is easy to verify that f is a solution of (4).

2. $f(1) \neq 0$

In this case, g is given by Proposition 9.

(α) In the case where γ is different from -1, the solutions of (10) of the form $g(x) = \gamma x + \delta$ $(x \in \mathbb{R})$ correspond to constant functions f. Since $f(0) = 0$, this gives only the identically zero solution of (4).

(β) In the case where γ is strictly negative, the solution of (10) of the form $g(x) = x$ $(x \in \mathbb{R})$ implies:

$$f(x)^l = (1-\gamma)x \qquad (x \in \mathbb{R}). \tag{30}$$

Therefore, l is necessarily an odd integer and, setting $x = 1$ in (30), we get $\gamma = (1-\gamma)^{k/l}$, which is impossible. So, this solution of (10) does not give a solution of (4).

(γ) In the case where γ is a positive real number different from 1, the solution (a) (i) of (10) implies

$$f(x)^l = \begin{cases} (1-\gamma)a & \text{for } x \leqslant a \\ (1-\gamma)x & \text{for } a \leqslant x \leqslant b \\ (1-\gamma)b & \text{for } x \geqslant b \end{cases} \quad \text{with } -\infty \leqslant a < b \leqslant +\infty.$$

With the hypothesis $f(0) = 0$, we deduce $a \leqslant 0 \leqslant b$.

If we had $b = 0$, we would have $f(1) = 0$, which is not the case. Therefore, b is strictly positive. If we had $a = 0$, we would have $f(-1) = 0$, which is not the case as we can see by taking $x = y = -1$ in (4). Therefore, a is negative. From the inequality $a < 0 < b$, we deduce immediately that l is an odd integer. So, the expression of f is the following:

$$f(x) = \begin{cases} (1-\gamma)^{1/l}a^{1/l} & \text{for } x \leqslant a \\ (1-\gamma)^{1/l}x^{1/l} & \text{for } a \leqslant x \leqslant b \\ (1-\gamma)^{1/l}b^{1/l} & \text{for } x \geqslant b \end{cases} \quad \text{with } -\infty \leqslant a < 0 < b \leqslant +\infty. \tag{31}$$

Setting $x = y$ in (4), we get

$$f(x(f(x)^k + f(x)^l)) = f(x^2). \tag{32}$$

Since the function f, defined by (31), is continuous at 0, there exists a positive real number η, $\eta < \mathrm{Inf}(b, \sqrt{b})$, such that $x(f(x)^k + f(x)^l)$ belongs to the interval $]a, b[$ for every x in $]0, \eta[$. From (31) and (32), we obtain for every x in $]0, \eta[$

$$(1-\gamma)^{k/l+1}x^{k/l+1} + (1-\gamma)^2x^2 = (1-\gamma)x^2$$

or

$$x^{(l-k)/k} = \frac{(1-\gamma)^{k/l}}{\gamma}.$$

This implies $l = k$ and $\gamma = \frac{1}{2}$. We deduce that $1 \leqslant b$, since, if had $b < 1$, we would have, by setting $x = 1$ in the expression of $f(x)^l$,

$$\gamma = \frac{1}{2} = \frac{1}{2}b.$$

Let us suppose that b is a finite real number. By taking $x = \frac{1}{2}$ and $2b < y < 3b$ in (4) and using (31), we get

$$\left(\frac{1}{2}\right)^{1/l}\left(\frac{b+y}{4}\right)^{1/l} = \left(\frac{1}{2}\right)^{1/l} b^{1/l}.$$

This implies $y = 3b$, which gives a contradiction. Hence $b = +\infty$.

Let us suppose now that a is a finite real number. By taking $x = y < a$ in (4) and using (31), we get

$$\left(\frac{1}{2}\right)^{1/l}(ax)^{1/l} = \left(\frac{1}{2}\right)^{1/l} x^{2/l}$$

which is impossible. Hence $a = -\infty$.

We obtain therefore $f(x) = (\frac{1}{2})^{1/l}x^{1/l}$ $(x \in \mathbb{R})$ and it is easy to verify that this is a solution of the functional equation (4) in the case where $k = l$ is an odd integer.

(δ) We shall study now the case where $\gamma = f(1)^k = -1$. This corresponds obviously to the case where k is an odd integer and $f(1) = -1$.

Let us suppose that l is an even integer. By taking $x = y = 1$ in (4), we obtain $f(0) = f(1)$, which is impossible. So, l is an odd integer in this case.

Since the solution (d) (i) of (10) does not give any solution of (4), we consider the solutions (d) (ii) of (10). The function: $x \mapsto g(x) - x$ is continuous and strictly decreasing on $\mathbb{R}$. Therefore it vanishes at most at one place. From $g(c) = c$ and

$g(0) = 0$, we deduce $c = 0$. So, by using also the fact that l is an odd integer, we see that the expression of the solutions of (4) which correspond to the solutions (d) (ii) of (10), is

$$f(x) = \begin{cases} (\Phi(x) + x)^{1/l} & \text{for } x \leqslant 0 \\ (\Phi^{-1}(x) + x)^{1/l} & \text{for } x \geqslant 0, \end{cases} \tag{33}$$

where Φ is a continuous strictly decreasing function from $]-\infty, 0]$ onto $[0, +\infty[$.

We shall study the subset $\mathscr{P}$ of $\mathbb{R}$ defined by $\mathscr{P} = \{x \in \mathbb{R}: f(x) = -1\}$. $\mathscr{P}$ contains 1, but does not contain 0. Equation (24) implies $f(g(x)) = f(x)$ for every x in $\mathbb{R}$. By taking $x = 1$, we see that $\mathscr{P}$ contains also -2. We shall prove that there exists no interval containing either 1 or -2 and included in $\mathscr{P}$. Since g is a bijection which transforms any interval containing 1 into an interval containing -2, it suffices to show that $\mathscr{P}$ cannot include an interval containing -2.

Let us suppose for contradiction that there exists an interval $[a, b]$ containing -2 and included in $\mathscr{P}$. By taking x in $\mathscr{P}$ and $y = g(x) = -x - 1$ in (4), we obtain

$$f(-x(x+1)) = -1 \qquad \text{for every } x \text{ in } \mathscr{P}. \tag{34}$$

From (34), we may notice that $\mathscr{P}$ does not contain -1 and we have necessarily $a \leqslant -2 \leqslant b < -1$.

Let us define $h(x) = -x(x+1)$ $(x \in \mathbb{R})$. (34) implies $f(h(x)) = -1$ for every x in $[a, b]$. Since h is strictly increasing on $]-\infty, -\frac{1}{2}[$, we have $h([a, b]) = [h(a), h(b)]$, and, since $h(x) < x$ for $x \in]-\infty, -2[$ and $h(x) > x$ for $x \in]-2, -1[$, this interval includes $[a, b]$. Moreover, in view of (34), the interval $[h(a), h(b)]$ is included in $\mathscr{P}$. So, it does not contain -1. Therefore, we have

$$h(a) \leqslant a \leqslant -2 \leqslant b \leqslant h(b) < -1.$$

Let h^n denote the nth iterate of h:

$$h^n(x) = h(h^{n-1}(x)) \qquad (x \in \mathbb{R}),$$

where n is a positive integer. It is easy to prove by induction that, for every positive integer n, the interval $[h^n(a), h^n(b)]$ is included in $\mathscr{P}$ and

$$h^n(a) \leqslant a \leqslant -2 \leqslant b \leqslant h^n(b) < -1.$$

Let us suppose $b > -2$. The sequence $\{h^n(b)\}_{n \in N}$ is strictly increasing and bounded from above by -1. Therefore it converges to a limit l which satisfies $h(l) = l$

and $-2 < l \leqslant -1$. This is impossible. So, we have necessarily $b = -2$ and the sequence $\{h^n(b)\}_{n \in N}$ is constant and equal to -2. Therefore we have $a < -2$. The sequence $\{h^n(a)\}_{n \in N}$ is strictly decreasing and converges to $-\infty$. We deduce $f(x) = -1$ for every x in $]-\infty, -2]$. Since $g(]-\infty, -2]) = [1, +\infty[$, we have, by (24), $f(x) = -1$ for every x in $[1, +\infty[$. By taking $x = -2$ and $y = 2$ in the functional equation (4), we get $f(2-2) = f(0) = 0 = f(-4) = -1$, which brings a contradiction. We deduce that $\mathscr{P}$ includes no interval containing either -2 or 1.

Let us consider the functional equation (4) with $x \neq 0$ and $y = 1/x$. We obtain

$$f\left(xf\left(\frac{1}{x}\right)^k + \frac{1}{x}f(x)^l\right) = -1 \qquad \text{for every } x \neq 0. \tag{35}$$

Let us define

$$F(x) = xf\left(\frac{1}{x}\right)^k + \frac{1}{x}f(x)^l \qquad (x \in \mathbb{R}^*).$$

By (33), f is continuous on $\mathbb{R}$. Therefore, F is continuous on $\mathbb{R}^*$ and, by (35), F takes its values in $\mathscr{P}$. Now, $F(1) = -2$ implies that $F(]0, +\infty[)$ is an interval of $\mathbb{R}$ containing -2 and included in $\mathscr{P}$. From the previous result, we deduce $F(]0, +\infty[) = \{-2\}$. Therefore we have

$$xf\left(\frac{1}{x}\right)^k + \frac{1}{x}f(x)^l = -2 \qquad \text{for every } x \text{ in }]0, +\infty[. \tag{36}$$

Let us now consider the functional equation (4) with $x \neq 0$ and $y = -2/x$. We obtain

$$f\left(xf\left(-\frac{2}{x}\right)^k - \frac{2}{x}f(x)^l\right) = -1 \qquad \text{for every } x \neq 0. \tag{37}$$

Let us define

$$G(x) = xf\left(-\frac{2}{x}\right)^k - \frac{2}{x}f(x)^l \qquad \text{for } x \text{ in } \mathbb{R}^*.$$

In the same way as for F, we may prove that $G(]-\infty, 0[) = \{1\}$. Therefore we have

$$xf\left(-\frac{2}{x}\right)^k - \frac{2}{x}f(x)^l = 1 \qquad \text{for every } x \text{ in }]-\infty, 0[. \tag{38}$$

Changing x into $-2x$, we obtain

$$-2xf\left(\frac{1}{x}\right)^k + \frac{1}{x} f(-2x)^l = 1 \qquad \text{for every } x \text{ in }]0, +\infty[. \tag{39}$$

From (36) and (39), we deduce

$$f(x)^l = -\frac{3}{2}x - \frac{1}{2} f(-2x)^l \qquad \text{for every } x \text{ in }]0, +\infty[. \tag{40}$$

Equations (40) and (33) imply

$$\Phi^{-1}(x) = -\frac{3}{2}x - \frac{1}{2}\Phi(-2x) \qquad \text{for every } x \text{ in }]0, +\infty[.$$

Since $\Phi(-2x) > 0$, we have

$$\Phi^{-1}(x) < -\frac{3}{2}x$$

and so $f(x)^l < -\frac{1}{2}x$ for every x in $]0, +\infty[$. Therefore f cannot vanish on $]0, +\infty[$. The functional equation (4) with $x = y < 0$ shows that f cannot vanish on $]-\infty, 0[$ either. Since f is continuous and satisfies $f(-2) = -1$, $f(]-\infty, 0[)$ is an interval of $\mathbb{R}$ included in $]-\infty, 0[$. We deduce

$$-\frac{3}{2}x < f(x)^l < -\frac{1}{2}x \qquad \text{for every } x \text{ in }]0, +\infty[. \tag{41}$$

Let us define

$$\zeta(x) = \frac{f(x)^l}{x} \qquad \text{for } x \text{ in } \mathbb{R}^*.$$

By (24), (40) and (41), ζ is a continuous function which satisfies

$$\left.\begin{array}{ll} -\frac{3}{2} < \zeta(x) < -\frac{1}{2} & (42) \\ \zeta(x) = -\frac{3}{2} + \zeta(-2x) & (43) \\ \zeta(x(\zeta(x) - 1)) = \frac{\zeta(x)}{\zeta(x) - 1} & (44) \end{array}\right\} \text{for every } x \text{ in }]0, +\infty[.$$

This implies

$$\zeta\left(\frac{x}{2}(1-\zeta(x))\right)=\frac{\zeta(x)}{\zeta(x)-1}-\frac{3}{2} \qquad \text{for every } x \text{ in }]0,+\infty[. \tag{45}$$

For an arbitrary fixed real number x in $]0, +\infty[$, we consider the following sequence of positive real numbers:

$$x_0 = x \quad \text{and} \quad x_n = \frac{x_{n-1}}{2}(1-\zeta(x_{n-1})) \qquad \text{for } n \geqslant 1.$$

By (45), we have

$$\zeta(x_n)+1 = -\frac{1}{2}+\frac{\zeta(x_{n-1})}{\zeta(x_{n-1})-1}=\frac{\zeta(x_{n-1})+1}{2(\zeta(x_{n-1})-1)} \qquad \text{for } n \geqslant 1.$$

Equation (42) implies now

$$|\zeta(x_n)+1| < \frac{1}{3}|\zeta(x_{n-1})+1|.$$

and therefore

$$|\zeta(x_n)+1| < \frac{1}{3^n}|\zeta(x)+1| \qquad \text{for } n \geqslant 1.$$

From this, we deduce that the sequence $\{\zeta(x_n)\}_{n\in N}$ converges to -1.
Let us study now the sequence $\{x_n\}_{n\in N}$. We have

$$x_n = \frac{x}{2^n}\prod_{p=0}^{n-1}(1-\zeta(x_p)) \qquad \text{for } n \geqslant 1.$$

By using (45), it is easy to prove by induction the following relation for $1 \leqslant k \leqslant n$:

$$\prod_{i=1}^{k}(1-\zeta(x_{n-i})) = \alpha_k(1-\zeta(x_{n-k}))+\beta_k, \tag{46}$$

where the real numbers α_k and β_k satisfy

$$\alpha_1 = 1, \qquad \beta_1 = 0, \qquad \alpha_{k+1}=\frac{3}{2}\alpha_k+\beta_k, \qquad \beta_{k+1}=\alpha_k.$$

We have therefore, for $k \geqslant 2$.

$$\alpha_k = \frac{1}{5}\left(2^{k+1} + \left(-\frac{1}{2}\right)^{k-1}\right)$$

$$\beta_k = \frac{1}{5}\left(2^k + \left(-\frac{1}{2}\right)^{k-2}\right).$$

Writing (46) with $k = n$, we obtain:

$$x_n = \frac{x}{5}\left(3 - 2\zeta(x) + \frac{(-1)^n}{2^{2n-1}}(1 + \zeta(x))\right).$$

Therefore, the sequence $\{x_n\}_{n\in N}$ converges to $(x/5)(3 - 2\zeta(x))$. Using the continuity of ζ on $]0, +\infty[$, we obtain:

$$\zeta\left(\frac{x}{5}(3 - 2\zeta(x))\right) = -1 \qquad \text{for every } x \text{ in }]0, +\infty[$$

and then, by using (42), $\zeta(x) = -1$ for every x in $]0, +\infty[$. Equation (43) implies now $\zeta(x) = \frac{1}{2}$ for every x in $]-\infty, 0[$. We deduce

$$f(x) = \begin{cases} -x^{1/l} & \text{for } x \geqslant 0 \\ (\frac{1}{2})^{1/l}x^{1/l} & \text{for } x \leqslant 0 \end{cases} \tag{47}$$

By changing x into $1/x$ in (36), we see that: $f(x)^k = -x$ for $x \geqslant 0$ and, therefore, we have in this case $k = l$.

It is now easy to show that (47) is a solution of (4).

Therefore we have the following result.

THEOREM 20. *All the solutions of the functional equation* (4) *in the class of functions* $\mathscr{D}\mathscr{B}_1$ *are: the constant functions and, in the case where* $k = l$ *is an odd integer,*

(*i*) $f(x) = (\frac{1}{2})^{1/l}x^{1/l}$ $(x \in \mathbb{R})$

(*ii*) $f(x) = \mathrm{Inf}(x^{1/l}, 0)$ $(x \in \mathbb{R})$

(*iii*) $f(x) = \mathrm{Inf}(-x^{1/l}, (\frac{1}{2})^{1/l}x^{1/l})$ $(x \in \mathbb{R})$.

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