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Research Papers

Triangles II: Complex triangle coordinates

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Summary. This paper is the second in a series of three examining Euclidean triangle geometry via complex cross ratios. In the first paper of the series, we examined triangle shapes. In this paper, we coordinatize the Euclidean plane using cross ratios, and use these triangle coordinates to prove theorems about triangles. We develop a complex version of Ceva's theorem, and apply it to proofs of several new theorems. The remaining paper of this series will deal with complex triangle functions.

1. Introduction

This paper is the second of three examining triangle geometry in terms of complex cross ratios. The first paper of the series [4] developed the notion of triangle shapes. We now use cross ratios to coordinatize the complex plane relative to a given triangle. After some preliminaries, the coordinates are defined in §2, and some of their properties developed. In §3, we examine the coordinate map and its uses, and follow this in §4 by a discussion of isogonal conjugates. The remaining sections (§5, 6) deal with complex versions of the theorems of Ceva and Menelaus and their applications. The last paper of this series [5] will discuss complex triangle functions.

The main advantage of complex triangle coordinates over the more usual trilinear/projective formulations is that, with complex triangles coordinates, the calculations and proofs tend to be simpler and often provide more information. With one coordinate (vs. a triple), checks for things like collinear or concyclic points involve a single cross ratio [Theorem 3.1, (c), (d)] instead of a 3×3 determinant, and furthermore, as an automatic part of the same calculation, give the ratio of distances between the collinear points or the order of the concyclic points. Simultaneous calculation of the ratio of lengths of two vectors and the angle

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between them [Theorem 3.1, (a)] becomes trivial, automatically keeping track of the correct angle orientation. Proofs involving extensive knowledge of trigonometric identities (by either human or computer; see [2]) become unnecessary: trigonometry becomes submerged in complex algebra, just as the trigonometric identities for $\cos(n\theta)$ and $\sin(n\theta)$ are submerged in the algebraic de Moivre's formula $(e^{i\theta})^n = e^{in\theta}$.

We outline some notational and mathematical preliminaries; further details may be found in [4]. Identify the Euclidean plane with the complex numbers $\mathbb{C}$, and set $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The *cycle notation* and its properties are given by

$$z' := \frac{1}{1-z} \qquad z'' = \frac{z-1}{z} = 1 - \frac{1}{z}, \qquad z''' = z, \qquad zz'z'' = -1$$

for any $z \in \mathbb{C}_\infty$. The numbers $\omega := e^{\pi i/3} = \frac{1}{2}(1 + \sqrt{3}i)$ and $\bar{\omega}$ are the only solutions of $z = z' = z''$.

The cross ratio of any $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ in $\mathbb{C}_\infty$ with at most two alike is the number

$$[\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}] := \frac{(\mathbf{a}-\mathbf{c})(\mathbf{b}-\mathbf{d})}{(\mathbf{a}-\mathbf{d})(\mathbf{b}-\mathbf{c})}$$

("cancel" any terms involving ∞). Cross ratios have the symmetry properties

$$[\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}]^{-1} = [\mathbf{a}, \mathbf{b}; \mathbf{d}, \mathbf{c}] = [\mathbf{b}, \mathbf{a}; \mathbf{c}, \mathbf{d}], \qquad [\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}]' = [\mathbf{a}, \mathbf{c}; \mathbf{d}, \mathbf{b}].$$

We may prove cross ratios equal with the following theorem from [4].

EQUAL CROSS RATIO THEOREM (ECRT). *Let* $\mathbf{r}_1, \mathbf{s}_1, \mathbf{t}_1, \mathbf{u}_1$ *and* $\mathbf{r}_2, \mathbf{s}_2, \mathbf{t}_2, \mathbf{u}_2$ *be arbitrary quadruples of points in* $\mathbb{C}_\infty$.

(*a*) *Assume that both quotients*

$$\frac{[\mathbf{r}_1, \mathbf{s}_1; \mathbf{t}_1, \mathbf{u}_1]'}{[\mathbf{r}_2, \mathbf{s}_2; \mathbf{t}_2, \mathbf{u}_2]'} \quad \text{and} \quad \frac{[\mathbf{r}_1, \mathbf{s}_1; \mathbf{t}_1, \mathbf{u}_1]''}{[\mathbf{r}_2, \mathbf{s}_2; \mathbf{t}_2, \mathbf{u}_2]''}$$

are real, but that neither cross ratio is real. Then $[\mathbf{r}_1, \mathbf{s}_1; \mathbf{t}_1, \mathbf{u}_1] = [\mathbf{r}_2, \mathbf{s}_2; \mathbf{t}_2, \mathbf{u}_2]$.

(*b*) *Assume that both products*

$$[\mathbf{r}_1, \mathbf{s}_1, \mathbf{t}_1, \mathbf{u}_1]'[\mathbf{r}_2, \mathbf{s}_2; \mathbf{t}_2, \mathbf{u}_2]' \quad \text{and} \quad [\mathbf{r}_1, \mathbf{s}_1; \mathbf{t}_1, \mathbf{u}_1]''[\mathbf{r}_2, \mathbf{s}_2; \mathbf{t}_2, \mathbf{u}_2]''$$

are real, but that neither cross ratio is real. Then $[\mathbf{r}_1, \mathbf{s}_1; \mathbf{t}_1, \mathbf{u}_1] = \overline{[\mathbf{r}_2, \mathbf{s}_2; \mathbf{t}_2, \mathbf{u}_2]}$.

Linear fractional transformations or conjugate linear fractional transformations of the form

$$z \to \frac{az+b}{cz+d} \quad \text{or} \quad z \to \frac{a\bar{z}+b}{c\bar{z}+d} \qquad (\text{for } ad-bc \neq 0),$$

respectively preserve or conjugate cross ratios. They are similarities or anti-similarities respectively whenever they fix ∞, i.e. whenever $c = 0$.

The following geometric properties of cross ratios may be found in [10] or [11].

- For distinct $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in $\mathbb{C}$,

  $\measuredangle\,\mathbf{qpr} \equiv \arg[\infty, \mathbf{p}; \mathbf{q}, \mathbf{r}]$.

- Points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in $\mathbb{C}$ are collinear whenever $[\infty, \mathbf{p}; \mathbf{q}, \mathbf{r}]$ is real. In this case, $\mathbf{p}$ divides segment $\mathbf{qr}$ in the signed ratio $-[\infty, \mathbf{p}; \mathbf{q}, \mathbf{r}]$, so $\mathbf{p}$ is between $\mathbf{q}$ and $\mathbf{r}$ whenever $[\infty, \mathbf{p}, \mathbf{q}, \mathbf{r}]$ is negative, and is the mid-point of $\mathbf{qr}$ whenever $[\infty, \mathbf{p}; \mathbf{q}, \mathbf{r}] = -1$.
- Points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ are concyclic or collinear whenever $[\mathbf{p}, \mathbf{q}; \mathbf{r}, \mathbf{s}]$ is real. In this case, the pairs $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, $\mathbf{s}$ separate each other whenever $[\mathbf{p}, \mathbf{q}; \mathbf{r}, \mathbf{s}]$ is negative, and $\mathbf{p}$, $\mathbf{q}$ are harmonic conjugates with respect to $\mathbf{r}$, $\mathbf{s}$ whenever $[\mathbf{p}, \mathbf{q}; \mathbf{r}, \mathbf{s}] = -1$.
- For distinct points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, the mapping $\mathbf{z} \to \mathbf{w}$ given by

  $[\mathbf{w}, \mathbf{p}; \mathbf{q}, \mathbf{r}] = \overline{[\mathbf{z}, \mathbf{p}; \mathbf{q}, \mathbf{r}]}$

  is the reflection in the line containing the points if they are collinear, or the inversion in the circle through them otherwise.

The *shape* of any triangle $\triangle\mathbf{abc}$ is the number $\triangle_{\mathbf{abc}} := [\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}]$. Equilateral triangles have shape $\omega := e^{\pi i/3}$ or $\bar{\omega}$. Two triangles are similar whenever they have the same shape and anti-similar whenever they have conjugate shapes. The angles of a triangle and its shape thus determine each other: for $A := \measuredangle\,\mathbf{bac}$, $B := \measuredangle\,\mathbf{cba}$ and $C := \measuredangle\,\mathbf{acb}$, we have [4]

$$\triangle_{\mathbf{abc}} = \frac{1-e^{-2iB}}{1-e^{2iC}} = (e^{2iB})''(e^{2iC})' \quad \textbf{(angle-shape formula)}$$

together with

$$e^{iA} = \frac{\triangle_{\mathbf{abc}}}{|\triangle_{\mathbf{abc}}|}, \qquad e^{iB} = \frac{\triangle'_{\mathbf{abc}}}{|\triangle'_{\mathbf{abc}}|}, \qquad e^{iC} = \frac{\triangle''_{\mathbf{abc}}}{|\triangle'''_{\mathbf{abc}}|}$$

and

$$e^{2iA} = \frac{\triangle_{\mathbf{abc}}}{\overline{\triangle_{\mathbf{abc}}}}, \qquad e^{2iB} = \frac{\triangle'_{\mathbf{abc}}}{\overline{\triangle'_{\mathbf{abc}}}}, \qquad e^{2iC} = \frac{\triangle''_{\mathbf{abc}}}{\overline{\triangle''_{\mathbf{abc}}}}.$$

2. Triangle coordinates

Complex triangle coordinates are made possible by the following theorem.

COORDINATE THEOREM. *For distinct* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *in* $\mathbb{C}_\infty$, *the mapping* $\mathbf{z} \to [\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$ *is linear fractional, and any linear fractional transformation has this form. The mapping is a similarity if and only if* $\mathbf{c} = \infty$.

Proof. Written out in full, the mapping is

$$\mathbf{z} \to \frac{(\mathbf{a}-\mathbf{c})\mathbf{z}-(\mathbf{a}-\mathbf{c})\mathbf{b}}{(\mathbf{a}-\mathbf{b})\mathbf{z}-(\mathbf{a}-\mathbf{b})\mathbf{c}},$$

so since $(\mathbf{a}-\mathbf{c})\{-(\mathbf{a}-\mathbf{b})\mathbf{c}\} - \{-(\mathbf{a}-\mathbf{c})\mathbf{b}\}(\mathbf{a}-\mathbf{b}) = (\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{c})(\mathbf{b}-\mathbf{c}) \neq 0$, the mapping is linear fractional. Conversely, any linear fractional transformation $T: \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by

$$T\mathbf{z} = \frac{\mathbf{p}\mathbf{z}+\mathbf{q}}{\mathbf{r}\mathbf{z}+\mathbf{s}}, \qquad \mathbf{p}\mathbf{s}-\mathbf{q}\mathbf{r} \neq 0$$

has an inverse

$$T\mathbf{z} = [T\mathbf{z}, 1; 0, \infty] = [\mathbf{z}, T^{-1}1; T^{-1}0; T^{-1}\infty] = \left[\mathbf{z}, -\frac{\mathbf{s}-\mathbf{q}}{\mathbf{r}-\mathbf{p}}; -\frac{\mathbf{q}}{\mathbf{p}}, -\frac{\mathbf{s}}{\mathbf{r}}\right].$$

The condition $\mathbf{p}\mathbf{s}-\mathbf{q}\mathbf{r} \neq 0$ implies that $\mathbf{a} := -(\mathbf{s}-\mathbf{q})/(\mathbf{r}-\mathbf{p})$, $\mathbf{b} := -\mathbf{q}/\mathbf{p}$ and $\mathbf{c} := -\mathbf{s}/\mathbf{r}$ are all distinct.

The mapping $\mathbf{z} \to [\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$ is a similarity if and only if it fixes ∞, i.e. if and only if $[\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}] = \infty$. This holds if and only if $\mathbf{c} = \infty$. □

To define complex triangle coordinates, first fix an arbitrary, non-degenerate *base triangle* $\triangle\mathbf{abc}$ and denote its shape by $\triangle := \triangle_{\mathbf{abc}}$. Now, since the mapping $\mathbf{z} \to [\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$ is linear fractional, it is bijective on $\mathbb{C}_\infty$, and so may be used to coordinatize $\mathbb{C}_\infty$.

DEFINITION. For any point $\mathbf{z} \in \mathbb{C}_\infty$, the *triangle coordinate* of $\mathbf{z}$ with respect to the base triangle $\triangle\mathbf{abc}$ is the number

$$\mathbf{z}_\triangle := [\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}] \in \mathbb{C}_\infty.$$

Direct calculation shows immediately that

$$\mathbf{a}_\triangle = 1, \qquad \mathbf{b}_\triangle = 0, \qquad \mathbf{c}_\triangle = \infty \quad \text{and} \quad \infty_\triangle = \triangle.$$

We shall use the subscripts $\triangle'$ and $\triangle''$ to refer to the cycled triangles $\triangle\mathbf{bca}$ and $\triangle\mathbf{cab}$ respectively, so

$$\mathbf{z}_{\triangle'} := [\mathbf{z}, \mathbf{b}; \mathbf{c}, \mathbf{a}] = (\mathbf{z}_\triangle)' = \frac{1}{1-\mathbf{z}_\triangle} \quad \text{and} \quad \mathbf{z}_{\triangle''} := [\mathbf{z}, \mathbf{c}; \mathbf{a}, \mathbf{b}] = (\mathbf{z}_\triangle)'' = \frac{\mathbf{z}_\triangle - 1}{\mathbf{z}_\triangle}.$$

We then have that

$$\mathbf{z}_\triangle \mathbf{z}_{\triangle'} \mathbf{z}_{\triangle''} = -1.$$

We now look at some elementary geometry in triangle coordinates. In the following, we will always denote the angles of the base triangle $\triangle\mathbf{abc}$ by $A := \measuredangle\,\mathbf{bac}$, $B := \measuredangle\,\mathbf{cba}$ and $C := \measuredangle\,\mathbf{acb}$ (so the angle shape formula for $\triangle\mathbf{abc}$ is $\triangle = (e^{2iB})''(e^{2iC})'$). The first thing to notice is that triangle coordinates are the product of two shapes: for $\lambda := \triangle_{\mathbf{zcb}}$, we have $\mathbf{z}_\triangle = [\infty, \mathbf{z}; \mathbf{c}, \mathbf{b}][\infty, \mathbf{a}, \mathbf{b}; \mathbf{c}] = \lambda\,\triangle$. This relation, in conjunction with the angle-shape formula, can be used to find the coordinates of various special triangle points. (See [3] for a catalogue of special points.)

Example 2.1: Incentres and excentres. The angle bisectors of $\triangle\mathbf{abc}$ meet at its incentre $\mathbf{i}$, so $\measuredangle\,\mathbf{bci} = -\frac{1}{2}C$ and $\measuredangle\,\mathbf{ibc} = -\frac{1}{2}B$, whence (from the angle-shape formula) $\lambda := \triangle_{\mathbf{icb}} = (e^{2i(-1/2C)})''(e^{2i(-1/2B)})' = (e^{-iC})''(e^{-iB})'$. Then

$$\mathbf{i}_\triangle = \frac{1-e^{iC}}{1-e^{-iB}}\,\triangle = \frac{1+e^{-iB}}{1+e^{iC}}.$$

Similarly, the excentre of $\triangle\mathbf{abc}$ opposite vertex $\mathbf{a}$ has coordinate

$$\frac{1+e^{iC}}{1+e^{-iB}}\,\triangle = \frac{1-e^{-iB}}{1-e^{iC}}. \qquad \square$$

Example 2.2: Orthocentres. The altitudes of $\triangle\mathbf{abc}$ meet at its orthocentre $\mathbf{h}$, so the quantity $(\mathbf{a}-\mathbf{c})/(\mathbf{b}-\mathbf{h}) = [\infty, \mathbf{b}; \mathbf{h}, \mathbf{c}]/[\infty, \mathbf{c}; \mathbf{a}, \mathbf{b}]$ is imaginary. Taking arguments, we get $\measuredangle\,\mathbf{hbc} = C \pm \frac{1}{2}\pi$; similarly, $\measuredangle\,\mathbf{bch} = B \pm \frac{1}{2}\pi$. From the angle-shape formula, then, we have $\lambda := \triangle_{\mathbf{hcb}} = (e^{2i(B \pm 1/2\pi)})''(e^{2i(C \pm 1/2\pi)})' = (-e^{2iB})''(-e^{2iC})'$. Then

$$\mathbf{h}_\triangle = \frac{1+e^{-2iB}}{1+e^{2iC}}\,\triangle = \frac{1-e^{-4iB}}{1-e^{4iC}}. \qquad \square$$

The triangle $\triangle\mathbf{zcb}$ may be degenerate, i.e. $\lambda := \triangle_{\mathbf{zcb}}$ may be real.

THEOREM 2.1. *Point* $\mathbf{z}$ *is on side* **bc**, **ca** *or* **ab** *of* $\triangle\mathbf{abc}$ *if and only if* $\mathbf{z}_\triangle/\triangle$, $\mathbf{z}_{\triangle'}/\triangle'$ *or* $\mathbf{z}_{\triangle''}/\triangle''$ *respectively is real. In this case,* $-[\infty, \mathbf{z}; \mathbf{c}, \mathbf{b}]$ *gives the signed ratio into which* $\mathbf{z}$ *divides* **bc**. *The mid-points of sides* **bc**, **ca** *and* **ab** *have coordinates* $-\triangle$, $2-\triangle$ *and* $\triangle(2\triangle - 1)^{-1}$ *respectively.*

Proof. We have $\mathbf{z}_\triangle/\triangle = \lambda = [\infty, \mathbf{z}; \mathbf{c}, \mathbf{b}]$, so $\mathbf{z}_\triangle/\triangle$ is real if and only if $\mathbf{z}$ is on the side **bc**. Similar arguments apply to the other two sides. If $\lambda := [\infty, \mathbf{z}; \mathbf{c}, \mathbf{b}]$ is real, then $-\lambda = (\mathbf{b}-\mathbf{z})/(\mathbf{z}-\mathbf{c})$ gives the signed ratio into which $\mathbf{z}$ divides **bc**. The mid-point of **bc** is the point $\mathbf{p}$ satisfying $[\infty, \mathbf{p}; \mathbf{c}, \mathbf{b}] = -1$, so $\mathbf{p}_\triangle = (-1)\triangle = -\triangle$. The mid-points of the other sides can be found by "cycling"; for example, the mid-point of **ca** is the point $\mathbf{q}$ satisfying $[\infty, \mathbf{q}; \mathbf{a}, \mathbf{c}] = -1$, so $\mathbf{q}_{\triangle'} = -\triangle'$, from which $\mathbf{q}_\triangle = (\mathbf{q}_{\triangle'})'' = (-\triangle')'' = 2 - \triangle$. Similarly, the mid-point of **ab** has coordinate $\triangle(2\triangle - 1)^{-1}$. □

Other conditions on $\lambda := [\infty, \mathbf{z}; \mathbf{c}, \mathbf{b}]$ give other loci relative to $\triangle\mathbf{abc}$; for example $\mathbf{z}$ is on the perpendicular bisector of **bc** whenever $|\lambda| = 1$, etc.

The next theorem gives triangle coordinate of the foot of a perpendicular to side **bc**.

THEOREM 2.2. *For any point* $\mathbf{z}$, *the foot of the perpendicular from* $\mathbf{z}$ *to side* **bc** *of* $\triangle\mathbf{abc}$ *has coordinate*

$$\mathbf{f}_\triangle = -|\lambda|^2 \frac{\mathrm{Re}\{\lambda''\}}{\mathrm{Re}\{1/\lambda'\}} \triangle = \frac{\mathrm{Re}\{\bar{\lambda}(1-\lambda)\}}{\mathrm{Re}\{1-\lambda\}} \triangle.$$

where $\lambda := \triangle_{\mathbf{zcb}} = \mathbf{z}_\triangle/\triangle$.

Proof. For $\mathbf{z}$ on **bc**, λ is real and the formula gives $\mathbf{f}_\triangle = \lambda\triangle = \mathbf{z}_\triangle$ as required. For $\mathbf{f} = \mathbf{c}$, $\lambda' = \triangle_{\mathbf{cbz}}$ is imaginary, and the formula gives $\mathbf{f}_\triangle = \infty = \mathbf{c}_\triangle$. For $\mathbf{z}$ not on **bc** and $\mathbf{f} \neq \mathbf{c}$, λ is not real, and the quanity $(\mathbf{f}-\mathbf{c})/(\mathbf{f}-\mathbf{z})$ is imaginary. Solve the equation $\mathbf{z}_\triangle = \lambda\triangle$ for $\mathbf{z}$: $\mathbf{z} = (\lambda - 1)^{-1}\{\lambda\mathbf{c} - \mathbf{b}\}$. For some real μ, $\mathbf{f}_\triangle = \mu\triangle$: solve this equation for $\mathbf{f}$: $\mathbf{f} = (\mu - 1)^{-1}\{\mu\mathbf{c} - \mathbf{b}\}$. Then $(\mathbf{f}-\mathbf{c})/(\mathbf{f}-\mathbf{z}) = (1-\lambda)/(\mu-\lambda)$, so

$$\frac{1-\lambda}{\mu-\lambda} = -\frac{1-\bar{\lambda}}{\mu-\bar{\lambda}}.$$

Solve for μ and set $\mathbf{f}_\triangle = \mu\triangle$. □

(The feet of the perpendiculars to the other sides may be found by cycling, as in the previous theorem.)

EXAMPLE 2.3. The foot of the altitude from **a** to side **bc** has coordinate

$$\mathbf{f}_\triangle = \frac{1-e^{2iC}}{1+e^{2iC}} \cdot \frac{1+e^{-2iB}}{1-e^{-2iB}} \triangle = \frac{1+e^{-2iB}}{1+e^{2iC}}.$$

(Put $\lambda = \mathbf{a}_\triangle / \triangle = \triangle^{-1}$ in Theorem 2.3; use the angle-shape formula for $\triangle$.) □

EXAMPLE 2.4. The point of tangency of the incircle and side **bc** has coordinate

$$\mathbf{t}_\triangle = \frac{1-e^{iC}}{1+e^{iC}} \cdot \frac{1+e^{-iB}}{1-e^{-iB}} \triangle = \left(\frac{1+e^{-iB}}{1+e^{iC}}\right)^2.$$

(The proof is direct calculation from Example 2.1 and Theorem 2.2; **t** is the foot of the perpendicular from the incentre **i** to side **bc**.) Similarly, the point of tangency of the excircle opposite vertex **a** and side **bc** has coordinate

$$\frac{1+e^{iC}}{1-e^{iC}} \cdot \frac{1-e^{-iB}}{1+e^{-iB}} \triangle = \left(\frac{1-e^{-iB}}{1-e^{iC}}\right)^2.$$ □

Triangle coordinates give a simple criterion for points inverse in the circumcircle of the base triangle.

THEOREM 2.3. *Two points are inverses in the circumcircle of the base triangle* $\triangle$**abc** *whenever their triangle coordinates are conjugate. A point lies on the circumcircle whenever its coordinate is real. The circumcentre* **o** *of* $\triangle$**abc** *has coordinate* $\mathbf{o}_\triangle = \overline{\triangle}$.

Proof. Since inversion in the circumcircle fixes the vertices and conjugates cross ratios, points **z** and **w** are inverses whenever $[\mathbf{w}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = \overline{[\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}]}$, i.e. whenever $\mathbf{w}_\triangle = \overline{\mathbf{z}_\triangle}$. A point lies on the circumcircle whenever it is self-inverse, i.e. whenever its coordinate is self-conjugate, or real. The circumcentre is the inverse of ∞, so $\mathbf{o}_\triangle = \overline{\infty_\triangle} = \overline{\triangle}$. □

3. Applications of the coordinate map

The *coordinate map* is the transformation $\mathbf{z} \to \mathbf{z}_\triangle$. As well as providing a coordinatization of $\mathbb{C}_\infty$, this transformation serves another very useful function: since it is linear fractional, the coordinate map preserves cross ratios, i.e. for all **a**, **b**, **c** and **d** in $\mathbb{C}_\infty$, with at most two equal,

$$[\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}] = [\mathbf{a}_\triangle, \mathbf{b}_\triangle; \mathbf{c}_\triangle, \mathbf{d}_\triangle].$$

This means that any geometric property expressed in terms of cross ratios of points can be expressed in terms of cross ratios of their triangle coordinates. We illustrate with a simple example.

EXAMPLE 3.1. THE NINE POINT CENTRE. The nine point circle (or Feuerbach circle) of a triangle contains the mid-points of its sides, the feet of its altitudes and the mid-points of the segments between the vertices and the orthocentre. We show that the centre $\mathbf{n}$ of the nine-point circle of $\triangle\mathbf{abc}$ has coordinate

$$\mathbf{n}_\triangle = \frac{1 + e^{-2iB} - e^{2iC}}{1 - e^{-2iB} + e^{2iC}} \triangle.$$

Let, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ be the mid-points of sides $\mathbf{bc}$, $\mathbf{ca}$ and $\mathbf{ab}$ respectively. Since $\mathbf{n}$ is the circumcentre of $\triangle\mathbf{rpq}$, $[\mathbf{n}, \mathbf{r}; \mathbf{p}, \mathbf{q}] = \overline{[\infty, \mathbf{r}; \mathbf{p}, \mathbf{q}]}$. Since $\triangle\mathbf{rpq}$ is similar to $\triangle\mathbf{cba}$, $[\infty, \mathbf{r}; \mathbf{p}, \mathbf{q}] = \triangle_{\mathbf{rpq}} = \triangle''$, so

$$[n, \infty; \mathbf{p}, \mathbf{q}] = \frac{[\mathbf{n}, \mathbf{r}; \mathbf{p}, \mathbf{q}]}{[\infty, \mathbf{r}; \mathbf{p}, \mathbf{q}]} = \frac{\overline{[\infty, \mathbf{r}; \mathbf{p}, \mathbf{q}]}}{[\infty, \mathbf{r}; \mathbf{p}, \mathbf{q}]} = \frac{\bar{\triangle}''}{\triangle''} = e^{-2iC}.$$

Now apply the coordinate map: $[\mathbf{n}_\triangle, \infty_\triangle; \mathbf{p}_\triangle, \mathbf{q}_\triangle] = e^{-2iC}$. We have that $\infty_\triangle = \triangle$ and (from Theorem 2.1) $\mathbf{p}_\triangle = -\triangle$, $\mathbf{q}_\triangle = 2 - \triangle$. Set $\mathbf{n}_\triangle = \lambda\triangle$ for some $\lambda \in \mathbb{C}$; then we get $[\lambda\triangle, \triangle; -\triangle, 2 - \triangle] = e^{-2iC}$. Solve for λ:

$$\lambda = \frac{2 - e^{2iC} - \triangle(1 - e^{2iC})}{\triangle(1 - e^{2iC}) + e^{2iC}} = \frac{1 + e^{-2iB} - e^{2iC}}{1 - e^{-2iB} + e^{2iC}}$$

(use the angle-shape formula $\triangle = (e^{2Bi})''(e^{2iC})'$); then set $\mathbf{n}_\triangle = \lambda\triangle$. □

To do Euclidean geometry in triangle coordinates, it is in principle only necessary that we be able to calculate the angle between any two vectors and the ratio of their lengths. This is done in part (a) of the following theorem; the remaining parts translate other geometric relations into triangle coordinate form.

THEOREM 3.1. *(a) For points* $\mathbf{p} \neq \mathbf{q}$ *and* $\mathbf{r} \neq \mathbf{s}$ *in* $\mathbb{C}$, *set*

$$\mathscr{R} := \frac{[\triangle, \mathbf{s}_\triangle; \mathbf{p}_\triangle, \mathbf{r}_\triangle]}{[\triangle, \mathbf{p}_\triangle; \mathbf{s}_\triangle, \mathbf{q}_\triangle]};$$

then $|\mathscr{R}|$ *gives the ratio of the length of* $\overrightarrow{\mathbf{rs}}$ *to that of* $\overrightarrow{\mathbf{pq}}$, *and* arg $\mathscr{R}$ *gives the angle from* $\overrightarrow{\mathbf{pq}}$ *to* $\overrightarrow{\mathbf{rs}}$. *In particular, the vectors are parallel when* $\mathscr{R}$ *is real and positive, antiparallel when* $\mathscr{R}$ *is real and negative, and perpendicular when* $\mathscr{R}$ *is imaginary.*

(*b*) *For distinct* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *in* $\mathbb{C}$,

$$\measuredangle\mathbf{qpr} \equiv \arg[\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle]$$

(*c*) *Points* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *in* $\mathbb{C}$ *are collinear whenever* $[\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle]$ *is real. In this case,* $\mathbf{p}$ *divides segment* $\mathbf{qr}$ *the signed ratio being* $-[\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle]$, *so* $\mathbf{p}$ *is between* $\mathbf{q}$ *and* $\mathbf{r}$ *whenever* $[\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle]$ *is negative, and is the mid-point of* $\mathbf{qr}$ *whenever* $[\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle] = -1$.

(*d*) *Points* $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ *and* $\mathbf{s}$ *are concyclic or collinear whenever* $[\mathbf{p}_\triangle, \mathbf{q}_\triangle; \mathbf{r}_\triangle, \mathbf{s}_\triangle]$ *is real. In this case, the pairs* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$, $\mathbf{s}$ *separate each other whenever* $[\mathbf{p}_\triangle, \mathbf{q}_\triangle; \mathbf{r}_\triangle, \mathbf{s}_\triangle]$ *is negative, and* $\mathbf{p}$, $\mathbf{q}$ *are harmonic conjugates with respect to* $\mathbf{r}$, $\mathbf{s}$ *whenever* $[\mathbf{p}_\triangle, \mathbf{q}_\triangle; \mathbf{r}_\triangle, \mathbf{s}_\triangle] = -1$.

(*e*) *For distinct points* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$, *the mapping* $\mathbf{z} \to \mathbf{w}$ *given by*

$$[\mathbf{w}_\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle] = \overline{[\mathbf{z}_\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle]}$$

is the reflection in the line containing the points if they are collinear, or the inversion in the circle through them otherwise.

Proof. With the exception of (a), these statements are just applications of the coordinate map to the corresponding statements in §1. Statement (a) is the application of the coordinate map to the quantity

$$\mathscr{R} = \frac{[\infty, \mathbf{s}; \mathbf{p}, \mathbf{r}]}{[\infty, \mathbf{p}; \mathbf{s}, \mathbf{q}]} = \frac{\mathbf{s} - \mathbf{r}}{\mathbf{q} - \mathbf{q}},$$

so the conclusions of (a) follow directly from the geometric properties of complex numbers. □

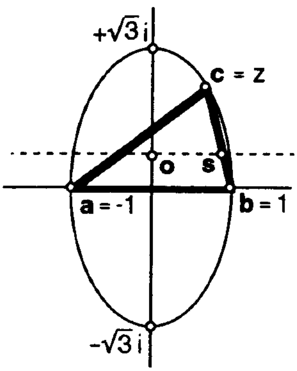
We look at some applications of this theorem.

EXAMPLE 3.2. The Euler line of a non-equilateral triangle is the line through its circumcentre and its centroid, and contains several other interesting special points of the triangle (e.g. the orthocentre and the nine point centre). We identify those triangles for which the Euler line is parallel to a side. In $\triangle\mathbf{abc}$, suppose that this line is parallel to side $\mathbf{ab}$. Since the centroid trisects the median to side $\mathbf{ab}$, the Euler line trisects side **bc** at a point **s** with $\mathbf{s}_\triangle = -\frac{1}{2}\triangle$ (Theorem 2.1). From Theorem 3.1, (a) since $\mathbf{o}_\triangle = \triangle$, $\mathbf{a}_\triangle = 1$ and $\mathbf{b}_\triangle = 0$, the quotient

$$\frac{[\triangle, \mathbf{o}_\triangle; \mathbf{b}_\triangle, \mathbf{s}_\triangle]}{[\triangle, \mathbf{b}_\triangle; \mathbf{o}_\triangle, \mathbf{a}_\triangle]} = \frac{(2\bar{\triangle} + \triangle)(\triangle - 1)}{3(\triangle - \bar{\triangle})}$$

must be real. Its denominator is imaginary, so its numerator must be as well; set the real part of the numerator equal to 0 and simplify:

$$(\triangle + \bar{\triangle})^2 - 3(\triangle + \bar{\triangle}) + 2\triangle\bar{\triangle} = 0. \tag{1}$$



To see which triangles this determines, coordinatize with $\mathbf{a} = -1$, $\mathbf{b} = 1$ and $\mathbf{c} = z = x + iy$. Then we have $\triangle = \frac{1}{2}[(x+1)^2 + iy]$, and (1) becomes

$$x^2 + \frac{y^2}{(\sqrt{3})^2} = 1,$$

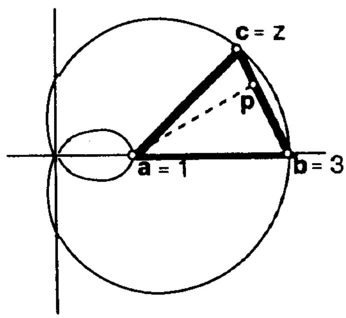
i.e. we have an ellipse. The side **ab** lies along its minor axis, and vertex **c** can be any other point on the ellipse with the exception of $\pm\sqrt{3}i$ (for which no Euler line exists). Note that $\triangle\mathbf{abc}$ must be acute angled. □

EXAMPLE 3.3: TRISECTING ANGLES. Beginning geometry students often believe they can trisect the angle at the vertex of a triangle by trisecting the opposite side and joining the trisection point to the vertex. This belief is of course a fallacy in general, but does this construction *ever* work? We identify those triangles for which it does.

Let **p** be a point on side **bc** of $\triangle$**abc** such that $\measuredangle$**pac** $= \frac{1}{3}\measuredangle$**bac**, and such that **p** trisects side **bc**; more specifically, the point with $\mathbf{p}_\triangle = -2\triangle$ (Theorem 2.1). Now $\measuredangle$**bac** $= \arg\triangle$ and from Theorem 3.1, (b) $\measuredangle$**pac** $= \arg[\triangle, 1; -2\triangle, \infty] = \arg\{3\triangle/(1+2\triangle)\}$, so the condition $3\measuredangle$**pac** $= \measuredangle$**bac** implies that $\arg\triangle = 3\arg\{3\triangle/(1+2\triangle)\} = \arg\{\triangle/(1+2\triangle)\}^3$. Two numbers have the same argument whenever their quotient is real and positive, so we must have $\mathscr{P} := \triangle^2/(1+2\triangle)^3$ real and positive.

Hindsight shows us the most appropriate coordinatization: $\mathbf{a} = 1$. $\mathbf{b} = 3$ and $\mathbf{c} = z$ for some non real z. Then $\triangle = [\infty, 1; 3, z] = \frac{1}{2}(z-1)$, so $\mathscr{P} = (z-1)^2/4z^3$, and we must have $\mathscr{A} := 4|z|^6\mathscr{P} = \bar{z}^3(z-1)^2$ real and positive. See $\mathscr{A} = \bar{\mathscr{A}}$, expand, divide off a factor $z - \bar{z} \neq 0$ from each pair of like powered terms, and rearrange to get

$$(z\bar{z})^2 - 2(z\bar{z})(z+\bar{z}) - (z+\bar{z})^2 - (z\bar{z}) = 0.$$



Set $z = r\,e^{i\theta}$ (so $z\bar{z} = r^2$ and $z + \bar{z} = 2r\cos\theta$), simplify, divide by r^2 and rearrange to get $(r - 2\cos\theta)^2 = 1$. The two equations $r = \pm 1 + 2\cos\theta$ represent the same curve, since (r, θ) satisfies one whenever the equivalent point $(-r, \theta + \pi)$ satisfies the other, so we have the limaçon $r = 1 + 2\cos\theta$.

We have thus far identified the points $\mathbf{c} = z$ with $\mathscr{A}$ *real*; we must still determine which of them make $\mathscr{A}$ *positive*. We have $z = r\,e^{i\theta}$ for $r = 1 - e^{i\theta} - e^{-i\theta}$, so $z - 1 = e^{2i\theta} + e^{i\theta} = 2\,e^{3i\theta/2}\cos(\frac{1}{2}\theta)$. Then $\mathscr{A} = 4\cos^2(\frac{1}{2}\theta)/r^3$, so $\mathscr{A} > 0$ if and only if $r > 0$. The allowable **c**'*s* thus lie on the outer loop of the limaçon. □

For our last example, we apply the cross ratio preserving property of the coordinate map to the proof of a theorem.

EXAMPLE 3.4: AN ANALYTIC PROOF OF FEUERBACH'S THEOREM. Feuerbach's theorem states that the nine-point circle of a triangle is tangent to its incircle and its three excircles. We prove the incircle case only; the excircle case is similar.

In $\triangle \mathbf{abc}$, let $\mathbf{p}$ be the midpoint of side $\mathbf{bc}$ and $\mathbf{t}$ the point of tangency of the incircle and side $\mathbf{bc}$. If $\mathbf{i}$ and $\mathbf{n}$ are the incentre and nine point centre, then the incircle and nine-point circle have radii $r := |\mathbf{t} - \mathbf{i}|$ and $R := |\mathbf{p} - \mathbf{n}|$, and the distance between their centres is $d := |\mathbf{n} - \mathbf{i}|$. It now suffices to prove that

$$|[\infty, \mathbf{p}; \mathbf{c}, \mathbf{n}]| \pm |[\infty, \mathbf{p}; \mathbf{c}, \mathbf{i}] - [\infty, \mathbf{p}; \mathbf{c}, \mathbf{t}]| = |[\infty, \mathbf{p}; \mathbf{c}, \mathbf{i}] - [\infty, \mathbf{p}; \mathbf{c}, \mathbf{n}]| \tag{2}$$

since this reduces to $R \pm r = d$.

We calculate the three cross ratios in terms of $\beta := e^{iB}$ and $\gamma := e^{iC}$. Note that each has the form $[\infty, \mathbf{p}; \mathbf{c}, \mathbf{z}]$ for some $\mathbf{z}$. Apply the coordinate map (recall from Theorem 2.1 that $\mathbf{p}_\triangle = -\triangle$); then the cross ratios have the form $[\triangle, -\triangle; \infty, \mathbf{z}_\triangle] = [1, -1; \infty, \mathbf{z}_\triangle/\triangle]$ for $\mathbf{z} = \mathbf{i}, \mathbf{t}$ or $\mathbf{n}$. From Examples 2.1, 2.4 and 3.1,

$$\frac{\mathbf{i}_\triangle}{\triangle} = \frac{1-\gamma}{1-\bar{\beta}}, \qquad \frac{\mathbf{t}_\triangle}{\triangle} = \frac{1-\gamma}{1+\gamma} \cdot \frac{1+\bar{\beta}}{1-\bar{\beta}}, \qquad \frac{\mathbf{n}_\triangle}{\triangle} = \frac{1-\gamma^2+\bar{\beta}^2}{1+\gamma^2-\bar{\beta}^2},$$

from which

$$[\infty, \mathbf{p}; \mathbf{c}, \mathbf{i}] = \frac{2-\gamma-\bar{\beta}}{\bar{\beta}-\gamma}, \qquad [\infty, \mathbf{p}; \mathbf{c}, \mathbf{t}] = \frac{1-\bar{\beta}\gamma}{\bar{\beta}-\gamma}, \qquad [\infty, \mathbf{p}; \mathbf{c}, \mathbf{n}] = \frac{1}{\bar{\beta}^2-\gamma^2}.$$

Now we calculate: For

$$\varrho := |[\infty, \mathbf{p}; \mathbf{c}, \mathbf{n}]| = |\bar{\beta}^2 - \gamma^2|^{-1},$$

we get

$$|[\infty, \mathbf{p}; \mathbf{c}, \mathbf{i}] - [\infty, \mathbf{p}; \mathbf{c}, \mathbf{t}]| = \varrho |(1-\bar{\beta})(1-\gamma)(\bar{\beta}+\gamma)|$$

and

$$|[\infty, \mathbf{p}; \mathbf{c}, \mathbf{i}] - [\infty, \mathbf{p}; \mathbf{c}, \mathbf{n}]| = \varrho |\bar{\beta} + \gamma - 1|^2.$$

Since the right-hand side of the identity

$$\bar{\gamma}\beta\{(1-\bar{\beta})(1-\gamma)(\bar{\beta}+\gamma)\} = 1 - |\bar{\beta}+\gamma-1|^2$$

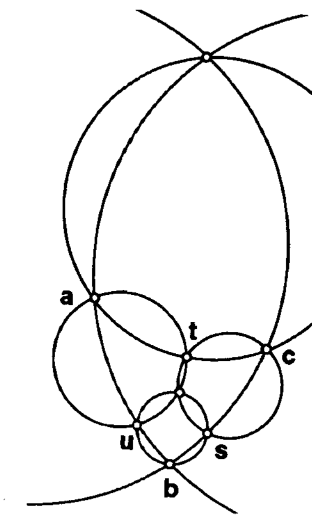
is real, the modulus of the left hand side is either itself or its negative. Thus, since $|\bar{\gamma}\beta| = |e^{i(B-C)}| = 1$, we have

$$\mp|(1-\bar{\beta})(1-\gamma)(\bar{\beta}+\gamma)| = 1 - |\bar{\beta}+\gamma-1|^2.$$

Multiply by ϱ and rearrange to get (2). □

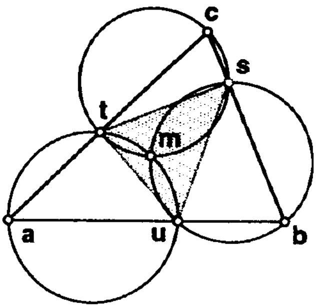
4. Miquel triangles and isogonal conjugates

Miquel's theorem may be stated loosely as follows: for points **a**, **b**, **c**, **s**, **t** and **u** in $\mathbb{C}_\infty$, if the circles **bsc**, **cta** and **aub** meet in a point, then so do circles **aut**, **bsu** and **cts**. (The circles may degenerate into lines and some pairs of points may coincide; see [7] or [8] for details.)



If the first three circles meet at ∞, they become lines and we have a theorem about $\triangle$**abc**: for distinct points **s**, **t** and **u** on sides **bc**, **ca** and **ab** of $\triangle$**abc**, circles **aut**, **bsu** and **cts** meet in a point **m**. The triangle $\triangle$**stu** is called a *Miquel triangle* of point **m** with respect to $\triangle$**abc**. Any nonvertex point has infinitely many Miquel triangles; its pedal triangle is an example.

Miquel triangles are related to triangle coordinates through the following theorem.

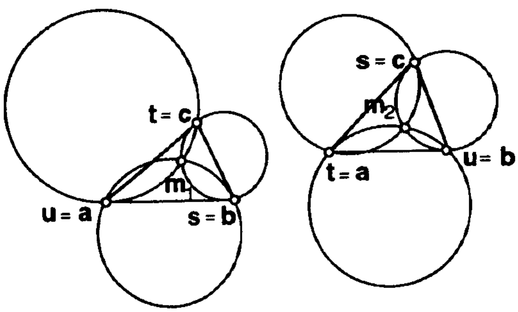


MIQUEL TRIANGLE THEOREM (MTST) [4, section 4]. *Let* $\triangle\mathbf{stu}$ *be a Miquel triangle of the nonvertex point* **m** *with respect to the nondegenerate triangle* $\triangle\mathbf{abc}$. *Then triangle* $\triangle\mathbf{stu}$ *has shape*

$$\triangle_{\mathbf{stu}} = \overline{[\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}]}.$$

Thus from MTST, it follows that $\mathbf{m}_\triangle = [\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = \overline{\triangle_{\mathbf{stu}}}$, i.e. the triangle coordinate of any nonvertex point is the conjugate of the shape of its Miquel triangles. (See also [9], where invariant properties of certain geometric configurations were investigated using pedal triangle shapes as coordinates.)

EXAMPLE 4.1. The **Brocard points** occur when each of **s**, **t** and **u** coincide with a vertex. This can happen in two ways. The Miquel triangles of each point are then $\triangle\mathbf{abc}$ with the vertices cycled, so from MTST, they have coordinates $\bar{\triangle}'$ and $\bar{\triangle}''$.



□

EXAMPLE 4.2. The **isodynamic points** of a triangle occur as the points common to its three Apollonian circles, and have antisimilar equilateral Miquel triangles (see [3], points 15 and 16 for further information). They thus have coordinates $\omega := e^{\pi i/3}$ and $\bar{\omega}$. □

We now look at isogonal conjugates. With respect to a pair of intersecting lines, any two lines through the intersection point are *isogonal* whenever they are reflections in the angle bisectors of the given lines. (The angle bisectors are perpendicular, so both reflections give the same result.) When the given lines form the sides of a triangle, we have the following property: if lines through the three vertices of a triangle meet at a nonvertex point, then so do the lines isogonal to them (with respect to the pair of sides through the same vertex). This second point is the *isogonal conjugate* of the first. The relation between a point and its isogonal conjugate is symmetric and bijective, with a few exceptions:

- the isogonal conjugate of all nonvertex points on a side is the opposite vertex
- the isogonal conjugate of all nonvertex points on the circumcircle is ∞
- no nonvertex point on a side or on the circumcircle is the isogonal conjugate of any point in $\mathbb{C}$.

We denote the isogonal conjugate of the point $\mathbf{z}$ by $\tilde{\mathbf{z}}$, and determine its coordinate.

Isogonal Conjugate Formula. *For any nonvertex point* $\mathbf{z} \in \mathbb{C}$,

$$\tilde{\mathbf{z}}_\triangle = \frac{\operatorname{Im}[\triangle, \infty; 1, \mathbf{z}_\triangle]}{\operatorname{Im}[\triangle, 0; 1, \mathbf{z}_\triangle]} \triangle \overline{\mathbf{z}_\triangle}.$$

Proof. Set $\alpha := \mathbf{z}_\triangle$, $\beta := \tilde{\mathbf{z}}_\triangle$, and $M := (\triangle - 1)/(\triangle - \alpha)$. Since $\mathbf{z}$ and $\tilde{\mathbf{z}}$ are on isogonal lines through $\mathbf{b}$, $\measuredangle \mathbf{cb\tilde{z}} \equiv \measuredangle \mathbf{zba} \pmod \pi$ so, from Theorem 3.1, (b), the quotient

$$\mathscr{B} := \frac{[\infty_\triangle, \mathbf{b}_\triangle; \mathbf{c}_\triangle, \tilde{\mathbf{z}}_\triangle]}{[\infty_\triangle, \mathbf{b}_\triangle; \mathbf{z}_\triangle, \mathbf{a}_\triangle]} = \frac{[\triangle, 0; \infty, \beta]}{[\triangle, 0; \alpha, 1]} = \frac{\beta - \triangle}{M\alpha\beta}$$

is real. Similarly, since $\mathbf{z}$ and $\tilde{\mathbf{z}}$ are on isogonal lines through $\mathbf{c}$, the quotient

$$\mathscr{C} := \frac{[\infty_\triangle, \mathbf{c}_\triangle; \mathbf{a}_\triangle, \tilde{\mathbf{z}}_\triangle]}{[\infty_\triangle, \mathbf{c}_\triangle; \mathbf{z}_\triangle, \mathbf{b}_\triangle]} = \frac{[\triangle, \infty; 1, \beta]}{[\triangle, \infty; \alpha, 0]} = \frac{M\triangle}{\triangle - \beta}$$

is real.

From $\mathscr{C}$ real follows that $\triangle\bar{\triangle}(M - \bar{M}) = M\triangle\bar{\beta} - \bar{M}\bar{\triangle}\beta$, and from $\mathscr{B}\mathscr{C}$ real follows that $\triangle\bar{\alpha}\bar{\beta} = \bar{\triangle}\alpha\beta$. Eliminate $\bar{\beta}$ to get

$$\beta = \frac{\operatorname{Im}\{M\}}{\operatorname{Im}\{\alpha M\}} \triangle\bar{\alpha}.$$

But $M = [\triangle, \infty; 1, \alpha]$ and $\alpha M = [\triangle, 0; 1, \alpha]$, so the formula follows immediately upon replacing $\alpha := \mathbf{z}_\triangle$ and $\beta := \tilde{\mathbf{z}}_\triangle$. □

EXAMPLE 4.3: THE CENTROID AND THE SYMMEDIAN POINT. The centroid of $\triangle\mathbf{abc}$ (its centre of gravity) is the point $\mathbf{g} := \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Now

$$\frac{\mathbf{g}_\triangle}{\triangle} = \triangle_{\mathbf{gcb}} = \frac{\mathbf{g} - \mathbf{b}}{\mathbf{g} - \mathbf{c}} = \frac{2(\mathbf{a} - \mathbf{b}) - (\mathbf{a} - \mathbf{c})}{2(\mathbf{a} - \mathbf{c}) - (\mathbf{a} - \mathbf{b})} = \frac{2 - \triangle}{2\triangle - 1},$$

so

$$\mathbf{g}_\triangle = -\frac{\triangle - 2}{2\triangle - 1}\triangle.$$

The *symmedian point* (or *Lemoine point*) is the isogonal conjugate of the centroid. Direct calculation from the isogonal conjugate formula then gives its coordinate:

$$\tilde{\mathbf{g}}_\triangle = \frac{\bar{\triangle} - 2}{2\bar{\triangle} - 1}.$$ □

(Since from MTST, the Miquel triangles of the symmedian point then have shape $(\triangle - 2)/(2\triangle - 1)$, this justifies the claim of Example 6.3 of [4].)

The following corollary relates isogonal conjugates to Miquel triangles.

COROLLARY 4.1. *If a non-vertex point* $\mathbf{z}$ *has Miquel triangle* $\triangle\mathbf{stu}$ *with respect to the base triangle, then* $\tilde{\mathbf{z}}_\triangle = [\mathbf{z}, \mathbf{s}; \mathbf{t}, \mathbf{u}]$.

Proof. In the notation of the proof of the isogonal conjugate formula

$$\frac{[\tilde{\mathbf{z}}, \mathbf{a}; \mathbf{b}, \mathbf{c}]}{\triangle\bar{\alpha}} = \frac{\operatorname{Im}\{M\}}{\operatorname{Im}\{\alpha M\}} \in \mathbb{R}. \tag{3}$$

Because of the Miquel configuration of collinear and concyclic points, the right hand side of the identity

$$\frac{[\mathbf{z}, \mathbf{s}; \mathbf{t}, \mathbf{u}]}{[\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}][\infty, \mathbf{s}; \mathbf{t}, \mathbf{u}]} = \frac{[\mathbf{z}, \mathbf{a}; \mathbf{t}, \mathbf{u}]}{[\infty, \mathbf{a}; \mathbf{t}, \mathbf{c}][\infty, \mathbf{a}; \mathbf{b}, \mathbf{u}]}$$

is real, so since $[\infty, \mathbf{s}; \mathbf{t}, \mathbf{u}] = \overline{[\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}]} = \bar{\alpha}$ (from MTST), we have

$$\frac{[\mathbf{z}, \mathbf{s}; \mathbf{t}, \mathbf{u}]}{\triangle \bar{\alpha}} \in \mathbb{R}. \tag{4}$$

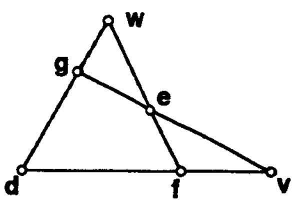
Divide (3) by (4) to show that $[\tilde{\mathbf{z}}, \mathbf{a}; \mathbf{b}, \mathbf{c}]/[\mathbf{z}, \mathbf{s}; \mathbf{t}, \mathbf{u}] \in \mathbb{R}$. Similarly (cycle the vertices) $[\tilde{\mathbf{z}}, \mathbf{b}; \mathbf{c}, \mathbf{a}]/[\mathbf{z}, \mathbf{t}; \mathbf{u}, \mathbf{s}] \in \mathbb{R}$, so since $[\tilde{\mathbf{z}}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$ is not real (no isogonal conjugates lie on the circumcircle), then ECRT implies that $\tilde{\mathbf{z}}_\triangle = [\tilde{\mathbf{z}}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\mathbf{z}, \mathbf{s}; \mathbf{t}, \mathbf{u}]$. □

Many basic properties of isogonal conjugates can be proven with triangle coordinates: Corollary 4.1, for example, shows that any point is located in the same position relative to all its Miquel triangles. We could also discuss the *anti-Miquel* triangles of a point (any triangle for which the base triangle is a Miquel triangle of the point), and prove basic theorems about them (e.g. the anti-Miquel triangles of a point are similar to the Miquel triangles of its isogonal conjugate, etc.).

5. The theorems of Menelaus and Ceva

The triangle coordinate versions of both Menelaus's theorem and Ceva's theorem are consequences of the following lemma.

LEMMA 5.1. *Let* **d**, **e**, **f**, **g** *be finite points with no three collinear. Suppose that lines* **df** *and* **eg** *intersect at* **v** *and lines* **dg** *and* **ef** *intersect at* **w** *(***v** *and* **w** *may be infinite). Then* $[\mathbf{e}, \mathbf{w}; \mathbf{g}, \mathbf{f}] = [\mathbf{v}, \mathbf{d}; \mathbf{g}, \mathbf{f}]$.



Proof. The following identities hold:

$$\frac{[\mathbf{e},\mathbf{g};\mathbf{f},\mathbf{w}]}{[\mathbf{v},\mathbf{g};\mathbf{f},\mathbf{d}]}[\infty,\mathbf{v};\mathbf{d},\mathbf{f}\,] = \begin{cases} \dfrac{[\infty,\mathbf{g};\mathbf{d},\mathbf{w}]}{[\infty,\mathbf{e};\mathbf{f},\mathbf{w}]} & \text{if } \mathbf{w}\neq\infty \\ -\dfrac{[\infty,\mathbf{e};\mathbf{g},\mathbf{f}\,]}{[\infty,\mathbf{g};\mathbf{e},\mathbf{d}]} & \text{if } \mathbf{w}=\infty. \end{cases}$$

If $\mathbf{w}\neq\infty$, then (collinear points) $[\infty,\mathbf{g};\mathbf{d},\mathbf{w}]$ and $[\infty,\mathbf{e};\mathbf{f},\mathbf{w}]$ are real. If $\mathbf{w}=\infty$, then **ef** is parallel to **gd**, so $-[\infty,\mathbf{e};\mathbf{f},\mathbf{g}]/[\infty,\mathbf{g};\mathbf{e},\mathbf{d}] = (\mathbf{e}-\mathbf{f}\,)/(\mathbf{g}-\mathbf{d})$ is real. In both cases, then, the right hand side is real, so since $[\infty,\mathbf{v};\mathbf{d},\mathbf{f}\,]$ is real (collinear points), $[\mathbf{e},\mathbf{w};\mathbf{f},\mathbf{w}]/[\mathbf{v},\mathbf{g};\mathbf{f},\mathbf{d}]$ is real.

Similarly (interchange **d** and **e**, and hence interchange **v** and **w**),

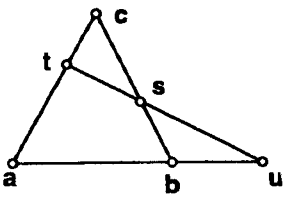
$$\frac{[\mathbf{d},\mathbf{g};\mathbf{f},\mathbf{v}]}{[\mathbf{w},\mathbf{g};\mathbf{f},\mathbf{e}]} = \frac{[\mathbf{e},\mathbf{f};\mathbf{w},\mathbf{g}]}{[\mathbf{v},\mathbf{f};\mathbf{d},\mathbf{g}]}$$

is real. Since **e**, **g**, **f** and **w** are neither collinear nor concyclic, $[\mathbf{e};\mathbf{g};\mathbf{f},\mathbf{w}]$ is not real, so from ECRT, $[\mathbf{e},\mathbf{w};\mathbf{g},\mathbf{f}\,] = [\mathbf{v},\mathbf{d};\mathbf{g},\mathbf{f}\,]$. □

COMPLEX MENELAUS'S THEOREM. *Let* **s**, **t** *and* **u** *be finite, nonvertex points on the sides* **bc**, **ca** *and* **ab** *of* $\triangle$**abc** *respectively. Then* **s**, **t** *and* **u** *are collinear if and only if* $\mathbf{s}_\triangle\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''} = -1$.

Proof. Let $\hat{\mathbf{s}}$ be the point of intersection of lines **tu** and **cb**. Now

$$\hat{\mathbf{s}}_\triangle\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''} = -(\hat{\mathbf{s}}_\triangle/\mathbf{u}_\triangle)(\mathbf{t}_{\triangle'}/\mathbf{u}_{\triangle''}) = -[\hat{\mathbf{s}},\mathbf{u};\mathbf{b},\mathbf{c}][\mathbf{t},\mathbf{u};\mathbf{c},\mathbf{a}].$$



But from Lemma 5.1, $[\hat{\mathbf{s}},\mathbf{b};\mathbf{u},\mathbf{c}] = [\mathbf{t},\mathbf{a};\mathbf{u},\mathbf{c}]$, so $[\hat{\mathbf{s}},\mathbf{u};\mathbf{b},\mathbf{c}] = [\mathbf{t},\mathbf{u};\mathbf{a},\mathbf{c}] = [\mathbf{t},\mathbf{u};\mathbf{c},\mathbf{a}]^{-1}$, from which $\hat{\mathbf{s}}_\triangle\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''} = -1$.

If **s**, **t** and **u** are collinear, then $\hat{\mathbf{s}} = \mathbf{s}$, from which $\mathbf{s}_\triangle\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''} = -1$. Conversely, if $\mathbf{s}_\triangle\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''} = -1$, then $\hat{\mathbf{s}}_\triangle = -(\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''})^{-1} = \mathbf{s}_\triangle$, so $\hat{\mathbf{s}} = \mathbf{s}$, whence **s**, **t** and **u** are collinear. □

(The classical form of Menelaus's theorem may be found by taking moduli in the relation $\mathbf{s}_{\triangle}\mathbf{t}_{\triangle'}\mathbf{u}_{\triangle''} = -1$, expanding, and adjusting signs as necessary.)

The example which follows uses Menelaus's theorem, Miquel's theorem and Lemma 4.1 from [4], which states:

Let **v**, **w**, **p**, **q** and **v**, **w**, **d**, **e** be quadruples of concyclic or collinear points. Assume that all points are distinct, except that

(a) one circle/line may be tangent to the other at $\mathbf{v} = \mathbf{w}$
(b) if the two are not tangent, then possibly $\mathbf{v} = \mathbf{q}$ or $\mathbf{w} = \mathbf{e}$.

Then $[\mathbf{v}, \mathbf{q}; \mathbf{e}, \mathbf{p}]/[\mathbf{w}, \mathbf{e}; \mathbf{q}, \mathbf{d}]$ is real and non-zero.

EXAMPLE 5.1. *Let* $\triangle\mathbf{jkm}$ *be an arbitrary triangle and* **e** *any non-vortex point. Construct on the sides of* $\triangle\mathbf{abc}$ *triangles* $\triangle\mathbf{pcb}$, $\triangle\mathbf{cqa}$ *and* $\triangle\mathbf{bar}$ *all similar to* $\triangle\mathbf{jkm}$, *and points* **s**, **t** *and* **u** *in the same position relative to each as* **e** *is to* $\triangle\mathbf{jkm}$. *Assume that* **s**, **t** *and* **u** *are distinct and not on the circumcircle of* $\triangle\mathbf{abc}$. *Then the circles/lines* **atu**, **bus** *and* **cst** *meet at a point on the circumcircle of* $\triangle\mathbf{abc}$.

Proof. By construction, we have that $\triangle_{\mathbf{scb}}\triangle_{\mathbf{tac}}\triangle_{\mathbf{uba}} = \triangle_{\mathbf{ekm}}\triangle_{\mathbf{emj}}\triangle_{\mathbf{ejk}} = 1$. Suppose that circles **bus** and **cst** meet at **s** and **m** then, from the lemma, $[\mathbf{m}, \mathbf{b}; \mathbf{t}, \mathbf{u}]/[\mathbf{s}, \mathbf{t}; \mathbf{b}, \mathbf{c}]$ is real. It then follows from the identity

$$[\mathbf{m}, \mathbf{a}; \mathbf{t}, \mathbf{u}] = \triangle_{\mathbf{scb}}\triangle_{\mathbf{tac}}\triangle_{\mathbf{uba}}\frac{[\mathbf{m}, \mathbf{b}; \mathbf{t}, \mathbf{u}]}{[\mathbf{s}, \mathbf{t}; \mathbf{b}, \mathbf{c}]},$$

that $[\mathbf{m}, \mathbf{a}; \mathbf{t}, \mathbf{u}]$ is real, so **m** lies on all three circles.

Miquel's theorem now implies that the circles/lines **aub**, **bsc** and **cta** meet at some non-vertex point **w**. (If **w** were a vertex, say $\mathbf{w} = \mathbf{a}$, then **s** would lie on the circumcircle of $\triangle\mathbf{abc}$, contrary to assumption.) Apply a linear fractional transformation $z \to \hat{z}$ with $\mathbf{w} \to \infty$; then $\hat{\mathbf{s}}$, $\hat{\mathbf{t}}$ and $\hat{\mathbf{u}}$ are non-vertex points on the sides of $\hat{\triangle} := \triangle\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ and

$$[\hat{\mathbf{s}}, \hat{\mathbf{a}}; \hat{\mathbf{b}}, \hat{\mathbf{c}}][\hat{\mathbf{t}}, \hat{\mathbf{b}}; \hat{\mathbf{c}}, \hat{\mathbf{a}}][\hat{\mathbf{u}}, \hat{\mathbf{c}}; \hat{\mathbf{a}}, \hat{\mathbf{b}}] = [\mathbf{s}, \mathbf{a}; \mathbf{b}, \mathbf{c}][\mathbf{t}, \mathbf{b}; \mathbf{c}, \mathbf{a}][\mathbf{u}, \mathbf{c}; \mathbf{a}, \mathbf{b}],$$

i.e.

$$\hat{\mathbf{s}}_{\hat{\triangle}}\hat{\mathbf{t}}_{\hat{\triangle}'}\hat{\mathbf{u}}_{\hat{\triangle}''} = (\triangle_{\mathbf{scb}}\triangle)(\triangle_{\mathbf{tac}}\triangle')(\triangle_{\mathbf{uba}}\triangle'') = -1.$$

From Menelaus' theorem, then, $\hat{\mathbf{s}}$, $\hat{\mathbf{t}}$ and $\hat{\mathbf{u}}$ are collinear, so since $\triangle\hat{\mathbf{s}}\hat{\mathbf{t}}\hat{\mathbf{u}}$ is a Miquel triangle of $\hat{\mathbf{m}}$ with respect to $\triangle\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$, we have (MTST),

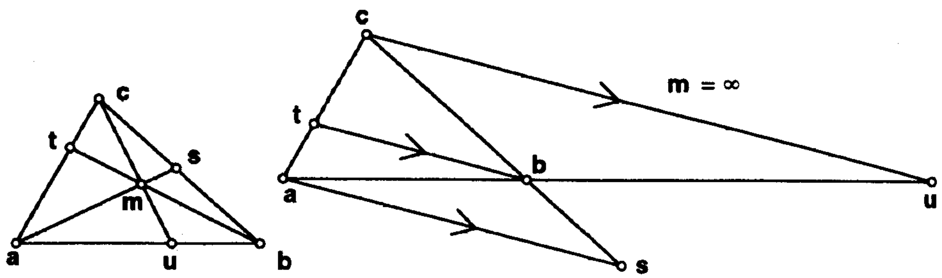
$$\mathbf{m}_\triangle := [\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\hat{\mathbf{m}}, \hat{\mathbf{a}}; \hat{\mathbf{b}}, \hat{\mathbf{c}}] = \overline{[\infty, \hat{\mathbf{s}}; \hat{\mathbf{t}}, \hat{\mathbf{u}}]} \in \mathbb{R}.$$

Then (Theorem 2.3) **m** is on the circumcircle of $\triangle$**abc**. □

This result appears to hold even when **s**, **t** or **u** are on the circumcircle; however a different proof is necessary to show that **m** lies on the circumcircle.

For Ceva's theorem, some terminology: a *cevian* through a vertex of a triangle is any line which is not a side, and its *side point* is its point of intersection with the opposite side. The triangle coordinate version of Ceva's theorem not only tells us when cevians meet at a point, but also gives the coordinate of that point.

COMPLEX CEVA'S THEOREM (CCEV). *Suppose that cevians through* **a**, **b** *and* **c** *have side-points* **s**, **t** *and* **u** *respectively. Then these cevians meet at some point* $\mathbf{m} \in \mathbb{C}_\infty$ *if and only if* $\mathbf{s}_\triangle \mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''} = 1$, *in which case* $\mathbf{m}_\triangle = \mathbf{t}_\triangle \mathbf{u}_\triangle$. (*We include the case* $\mathbf{m} = \infty$, *when the cevians are parallel.*)



Proof. Suppose that the cevians **cu** and **bt** meet at a point **m**. (If **m** is finite, then (Lemma 5.1) $[\mathbf{m}, \mathbf{u}; \mathbf{b}, \mathbf{c}] = [\mathbf{t}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$. If **m** is infinite (i.e. if **cu** and **bt** are parallel), a similar triangle argument shows that the same relation holds: alternately, since cross ratios are continuous functions of their arguments, we may take the limit as $\mathbf{m} \to \infty$ (e.g. keep **a**, **b**, **c** and **t** fixed and let **m** approach ∞ along **bt**). For either case, then, $[\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\mathbf{m}, \mathbf{u}; \mathbf{b}, \mathbf{c}][\mathbf{u}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\mathbf{t}, \mathbf{a}; \mathbf{b}, \mathbf{c}][\mathbf{u}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$, i.e. $\mathbf{m}_\triangle = \mathbf{t}_\triangle \mathbf{u}_\triangle$. Now suppose that lines **am** and **bc** meet at some point $\hat{\mathbf{s}}$. (If $\mathbf{m} = \infty$, take **am** to be parallel to **cu** and **bt**.) As above, $\mathbf{m}_{\triangle'} = \mathbf{u}_{\triangle'} \hat{\mathbf{s}}_{\triangle''}$ and $\mathbf{m}_{\triangle''} = \hat{\mathbf{s}}_{\triangle''} \mathbf{t}_{\triangle''}$, so

$$-1 = \mathbf{m}_\triangle \mathbf{m}_{\triangle'} \mathbf{m}_{\triangle''} = (\hat{\mathbf{s}}_{\triangle'} \hat{\mathbf{s}}_{\triangle''})(\mathbf{t}_\triangle \mathbf{t}_{\triangle''})(\mathbf{u}_\triangle \mathbf{u}_{\triangle''}) = -(\hat{\mathbf{s}}_\triangle \mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''})^{-1},$$

whence $\hat{\mathbf{s}}_\triangle \mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''} = 1$.

If the cevians **as**, **bt** and **cu** meet at **m**, then $\hat{\mathbf{s}} = \mathbf{s}$, so $\mathbf{s}_\triangle \mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''} = 1$. Conversely, if $\mathbf{s}_\triangle \mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''} = 1$, then $\hat{\mathbf{s}}_\triangle = (\mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''})^{-1} = \mathbf{s}_\triangle$, so $\hat{\mathbf{s}} = \mathbf{s}$, whence the cevians meet at **m**. □

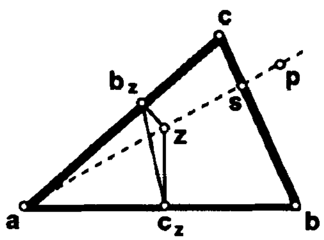
(As with Menelaus's theorem, the classical form of Ceva's theorem may be found by taking moduli in the complex form.)

When the cevians are given by their side-points, CCEV may be used to prove the usual theorems about intersecting medians, angle bisectors, etc. More generally, the cevians are given by other points on them, in which case we may use the following formulas.

SIDE POINT FORMULAS. *If the cevian* **ap** *intersects side* **bc** *of* $\triangle$**abc** *at* **s**, *then*

$$\mathbf{s}_\triangle = \frac{\mathbf{p}_\triangle}{\overline{\tilde{\mathbf{p}}_\triangle}} = \frac{\operatorname{Im}[\triangle, 0; 1, \mathbf{p}_\triangle]}{\operatorname{Im}[\triangle, \infty; 1, \mathbf{p}_\triangle]} \frac{1}{\overline{\triangle}}.$$

Proof. For any arbitrary non-vertex **z** on **ap**, let $\mathbf{a}_\mathbf{z}$, $\mathbf{b}_\mathbf{z}$, $\mathbf{c}_\mathbf{z}$ be the feet of the perpendiculars from **z** to sides **bc**, **ca** and **ab** of $\triangle$**abc**.



Then (from MTST) $\overline{\mathbf{z}_\triangle} = [\infty, \mathbf{a}_\mathbf{z}; \mathbf{b}_\mathbf{z}, \mathbf{c}_\mathbf{z}]$ and (Corollary 4.1) $\tilde{\mathbf{z}}_\triangle = [\mathbf{z}, \mathbf{a}_\mathbf{z}; \mathbf{b}_\mathbf{z}, \mathbf{c}_\mathbf{z}]$, so $\overline{\mathbf{z}_\triangle}/\tilde{\mathbf{z}}_\triangle = [\infty, \mathbf{z}; \mathbf{b}_\mathbf{z}, \mathbf{c}_\mathbf{z}]$. But all triangles $\triangle \mathbf{z}\mathbf{b}_\mathbf{z}\mathbf{c}_\mathbf{z}$ for **z** on **ap** have the same shape, so for $\mathbf{z} = \mathbf{p}$ and $\mathbf{z} = \mathbf{s}$ in particular,

$$\overline{\mathbf{p}_\triangle}/\tilde{\mathbf{p}}_\triangle = [\infty, \mathbf{p}; \mathbf{b}_\mathbf{p}, \mathbf{c}_\mathbf{p}] = [\infty, \mathbf{s}; \mathbf{b}_\mathbf{s}, \mathbf{c}_\mathbf{s}] = \overline{\mathbf{s}_\triangle}/\tilde{\mathbf{s}}_\triangle = \overline{\mathbf{s}_\triangle},$$

since $\tilde{\mathbf{s}}_\triangle = \mathbf{a}_\triangle = 1$. Thus $\mathbf{s}_\triangle = \mathbf{p}_\triangle/\overline{\tilde{\mathbf{p}}_\triangle}$. The second form for $\mathbf{s}_\triangle$ follows immediately from the isogonal conjugate formula. □

EXAMPLE 5.2. Suppose that two isogonal cevians through **a** have side points **e** and **f**. Any point **p** on the first but not on a side or the circumcircle has its isogonal conjugate on the second, so from the side point formulae, $\mathbf{e}_\triangle = \mathbf{p}_\triangle / \overline{\tilde{\mathbf{p}}_\triangle}$ and $\mathbf{f}_\triangle = \tilde{\mathbf{p}}_\triangle / \overline{\mathbf{p}_\triangle}$, whence $\mathbf{e}_\triangle \overline{\mathbf{f}_\triangle} = 1$. Now $\mathbf{e}_\triangle = -\sigma \triangle$ and $\mathbf{f}_\triangle = -\varrho \triangle$ where $\sigma := -[\infty, \mathbf{e}, \mathbf{c}, \mathbf{b}]$ and $\varrho := -[\infty, \mathbf{f}; \mathbf{c}, \mathbf{b}]$ are the signed ratios into which $\mathbf{e}$ and $\mathbf{f}$ divide $\mathbf{bc}$. Since ϱ is real and $\triangle \bar{\triangle} = |\triangle|^2 = |\mathbf{a} - \mathbf{c}|^2 / |\mathbf{a} - \mathbf{b}|^2$, we have

$$\left\{\sigma \frac{|\mathbf{a} - \mathbf{c}|}{|\mathbf{a} - \mathbf{b}|}\right\}\left\{\varrho \frac{|\mathbf{a} - \mathbf{c}|}{|\mathbf{a} - \mathbf{b}|}\right\} = 1. \tag{5}$$

In the particular case that the cevians coincide (i.e. with either of the angle bisectors through $\mathbf{a}$), we have $\mathbf{e} = \mathbf{f}$, $\varrho = \sigma$, the brackets coincide, and (5) becomes $\sigma = \pm |\mathbf{a} - \mathbf{b}| / |\mathbf{a} - \mathbf{c}|$, i.e. we get the classical theorem that the two angle bisectors divide the opposite side into the signed ratio of the remaining two sides. Relation (5) is thus a generalization of this theorem. □

EXAMPLE 5.3. Let **ap**, **bq** and **cr** be chords of a circle with distinct endpoints. Then the lines **ap**, **bq** and **cr** are concurrent if and only if

$$(\mathbf{p} - \mathbf{c})(\mathbf{q} - \mathbf{a})(\mathbf{r} - \mathbf{b}) = (\mathbf{p} - \mathbf{b})(\mathbf{q} - \mathbf{c})(\mathbf{r} - \mathbf{a}).$$

Proof. Take $\triangle\mathbf{abc}$ as base triangle then, since $\mathbf{p}$ is on its circumcircle, $\tilde{\mathbf{p}} = \infty$ and $\tilde{\mathbf{p}}_\triangle = \triangle$ so, from the side point formula, the side point $\mathbf{s}$ of cevian $\mathbf{ap}$ is given by $\mathbf{s}_\triangle = \mathbf{p}_\triangle / \bar{\triangle}$. Similarly, the side points $\mathbf{t}$ and $\mathbf{u}$ of cevians $\mathbf{bq}$ and $\mathbf{cr}$ satisfy $\mathbf{t}_{\triangle'} = \mathbf{q}_{\triangle'} / \bar{\triangle}'$ and $\mathbf{u}_{\triangle''} = \mathbf{r}_{\triangle''} / \bar{\triangle}''$ so, from CCEV, the cevians meet if and only if $\mathbf{p}_\triangle \mathbf{q}_{\triangle'} \mathbf{r}_{\triangle''} = \overline{\triangle \triangle' \triangle''} = -1$. Expand, simplify and rearrange to get the required relation. □

Since the length of a chord of a circle is proportional to the sine of the angle it subtends at the circumference, this last example may be reworked to give a proof of the classical sine version of Ceva's theorem.

Note that any point not on a side of the triangle is the intersection of concurrent cevians.

COROLLARY 5.1. *For any* $\mathbf{m}$ *not on a side of* $\triangle\mathbf{abc}$, *the side points* $\mathbf{s}$, $\mathbf{t}$ *and* $\mathbf{u}$ *of the cevians through* $\mathbf{m}$ *are given by*

$$\mathbf{s}_\triangle = \frac{\mathbf{m}_\triangle}{\overline{\tilde{\mathbf{m}}_\triangle}}, \qquad \mathbf{t}_{\triangle'} = \frac{\mathbf{m}_{\triangle'}}{\overline{\tilde{\mathbf{m}}_{\triangle'}}}, \qquad \mathbf{u}_{\triangle''} = \frac{\mathbf{m}_{\triangle''}}{\overline{\tilde{\mathbf{m}}_{\triangle''}}}.$$

Proof. The formulae are a direct consequence of the side point formulae. □

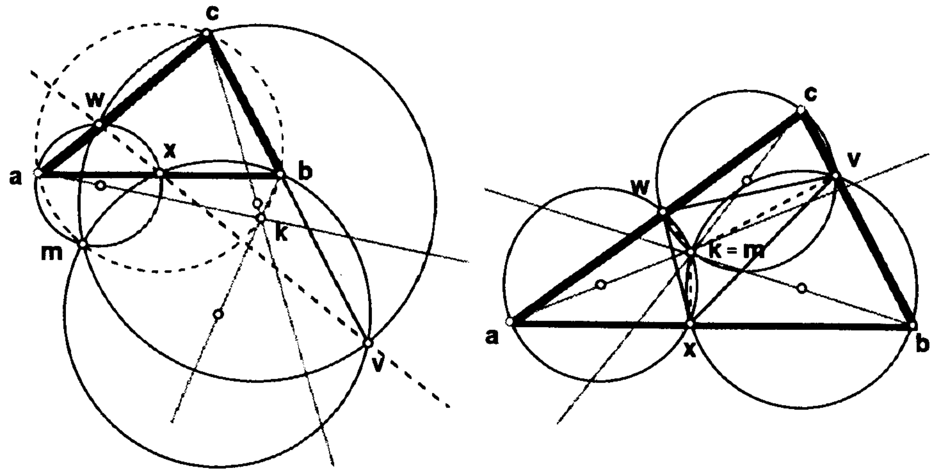
6. Further applications of Ceva's Theorem

In this section, we look at a few less elementary consequences of CCEV.

EXAMPLE 6.1. *Let* $\triangle\mathbf{vwx}$ *be a Miquel triangle of the point* $\mathbf{m}$ *with respect to the triangle* $\triangle\mathbf{abc}$; *then the lines from* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *through the centres of the circles* **axmw**, **bvmx** *and* **cwvm** *respectively meet at a point* $\mathbf{k}$ *in exactly two cases*:

(i) $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{x}$ *are collinear, in which case* $\mathbf{k}$ *is on the circumcircle*

(ii) $\triangle\mathbf{vwx}$ *is the pedal triangle of* $\mathbf{m}$, *in which case* $\mathbf{k}=\mathbf{m}$.



Proof. If one of the lines coincides with a side, the statement is easily checked, so assume otherwise; we may assume also that $\mathbf{w}\neq\mathbf{c}$. Set

$$\varrho:=[\infty,\mathbf{v};\mathbf{w},\mathbf{x}],\qquad \alpha:=\frac{\mathbf{x}-\mathbf{w}}{\mathbf{b}-\mathbf{c}}\qquad \beta:=\frac{\mathbf{v}-\mathbf{x}}{\mathbf{c}-\mathbf{a}}\qquad \gamma:=\frac{\mathbf{w}-\mathbf{v}}{\mathbf{a}-\mathbf{b}};$$

then

$$\frac{\gamma}{\beta}=\frac{\Delta}{\varrho}\qquad \frac{\alpha}{\gamma}=\frac{\Delta'}{\varrho'}\qquad \frac{\beta}{\alpha}=\frac{\Delta''}{\varrho''}. \tag{6}$$

Let $\mathbf{p}$ be the centre of circle **axmw**. If $\mathbf{w}\neq\mathbf{a}$, then triangles $\triangle\mathbf{awp}$ and $\triangle\mathbf{wap}$ are anti-similar, so $[\infty, \mathbf{a}; \mathbf{w}, \mathbf{p}] = \overline{[\infty, \mathbf{w}; \mathbf{a}, \mathbf{p}]}$. Multiply by $[\infty, \mathbf{a}; \mathbf{c}, \mathbf{w}] = \overline{[\infty, \mathbf{a}; \mathbf{c}, \mathbf{w}]}$ to get

$$[\infty, \mathbf{a}; \mathbf{c}, \mathbf{p}] = -\overline{\left(\frac{\mathbf{w}-\mathbf{p}}{\mathbf{a}-\mathbf{c}}\right)}. \tag{7}$$

This relation still holds if the circle is tangent to side **ac** at $\mathbf{w} = \mathbf{a}$; it then states that $(\mathbf{a}-\mathbf{p})/(\mathbf{a}-\mathbf{c})$ is imaginary, i.e. that radius **ap** is perpendicular to side **av** of $\triangle\mathbf{abc}$. Similarly

$$[\infty, \mathbf{a}; \mathbf{b}, \mathbf{p}] = -\overline{\left(\frac{\mathbf{x}-\mathbf{p}}{\mathbf{a}-\mathbf{b}}\right)}. \tag{8}$$

Divide (7) and (8) and rearrange to get

$$\frac{\mathbf{p}-\mathbf{w}}{\overline{\mathbf{p}-\mathbf{w}}} = \frac{\triangle}{\overline{\triangle}}, \qquad \mathbf{p} = \frac{\triangle\mathbf{x} - \overline{\triangle}\mathbf{w}}{\triangle - \overline{\triangle}},$$

from which (7) and (8) become

$$(2i \operatorname{Im} \triangle)[\infty, \mathbf{a}; \mathbf{c}, \mathbf{p}] = \overline{\left(\frac{\mathbf{w}-\mathbf{x}}{\mathbf{a}-\mathbf{b}}\right)} = \overline{\left(\frac{\gamma}{\varrho'}\right)}$$

and

$$(2i \operatorname{Im} \triangle)[\infty, \mathbf{a}; \mathbf{b}, \mathbf{p}] = \triangle\overline{\left(\frac{\mathbf{w}-\mathbf{x}}{\mathbf{a}-\mathbf{b}}\right)} = \overline{\left(\frac{\gamma}{\varrho'\triangle}\right)}|\triangle|^2.$$

The side-point **s** of the cevian **ap** is given by

$$\mathbf{s}_\triangle = \frac{\operatorname{Im}[\triangle, 0; 1, \mathbf{p}_\triangle]}{\operatorname{Im}[\triangle, \infty; 1, \mathbf{p}_\triangle]}\frac{1}{\overline{\triangle}} = \frac{\operatorname{Im}[\infty, \mathbf{b}; \mathbf{a}, \mathbf{p}]}{\operatorname{Im}[\infty, \mathbf{c}; \mathbf{a}, \mathbf{p}]}\frac{1}{\overline{\triangle}} = \frac{\operatorname{Im}[\infty, \mathbf{a}; \mathbf{b}, \mathbf{p}]}{\operatorname{Im}[\infty, \mathbf{a}; \mathbf{c}, \mathbf{p}]}\frac{1}{\overline{\triangle}},$$

so we calculate that

$$\mathbf{s}_\triangle = \frac{|\triangle|^2 \operatorname{Re}(\gamma/\varrho'(\triangle)}{\operatorname{Re}(\gamma/\varrho')}\frac{1}{\overline{\triangle}} = \frac{\operatorname{Re}(\gamma/\varrho'\triangle)}{\operatorname{Re}(\gamma/\varrho')}\triangle.$$

Similarly (cycle and use (6)), the side-points **t** and **u** of the other cevians are given by

$$\mathbf{t}_{\triangle'} = \frac{\operatorname{Re}(\alpha\varrho''/\triangle')}{\operatorname{Re}(\alpha/\varrho'')}\triangle' = \frac{\operatorname{Re}(\gamma\varrho)}{\operatorname{Re}(\gamma\triangle'\varrho)}\triangle',$$

$$\mathbf{u}_{\triangle''} = \frac{\operatorname{Re}(\beta/\varrho\triangle'')}{\operatorname{Re}(\beta/\varrho)}\triangle'' = -\frac{\operatorname{Re}(\gamma\triangle')}{\operatorname{Re}(\gamma/\triangle)}\triangle''.$$

From CCEV, we have that the cevians intersect whenever $\mathbf{s}_\triangle \mathbf{t}_{\triangle''} \mathbf{u}_{\triangle''} = \mathcal{N}/\mathcal{D} = 1$ for $\mathcal{N} := \operatorname{Re}(\gamma/\varrho'\triangle)\operatorname{Re}(\gamma\varrho)\operatorname{Re}(\gamma\triangle')$ and $\mathcal{D} := \operatorname{Re}(\gamma/\varrho')\operatorname{Re}(\gamma\triangle'\varrho)\operatorname{Re}(\gamma/\triangle)$. Direct calculation gives

$$\mathcal{N} - \mathcal{D} = -\left\{\frac{|\gamma|^2 \operatorname{Im}\triangle}{|\triangle|^2|1-\triangle|^2}\right\}\cdot \operatorname{Im}\varrho \cdot \operatorname{Re}\{\gamma(\varrho - \bar{\triangle})\},$$

so the cevians intersect at some point $\mathbf{k}$ if and only if either ϱ is real or $\gamma(\varrho - \bar{\triangle})$ is imaginary.

If $\varrho = [\infty, \mathbf{v}; \mathbf{w}, \mathbf{x}]$ is real, then $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ are collinear. Furthermore, (using (6)),

$$\mathbf{t}_{\triangle'} = \frac{\operatorname{Re}(\alpha/\triangle')}{\operatorname{Re}(\alpha)}\triangle' = \frac{\operatorname{Re}(\gamma)}{\operatorname{Re}(\alpha)}\frac{\triangle'}{\varrho'} = \frac{\operatorname{Re}(\gamma)}{\operatorname{Re}(\alpha)}\frac{\alpha}{\gamma}$$

and

$$\mathbf{u}_{\triangle''} = \frac{\operatorname{Re}(\beta/\triangle'')}{\operatorname{Re}(\beta)}\triangle'' = \frac{\operatorname{Re}(\alpha)}{\operatorname{Re}(\beta)}\frac{\triangle''}{\varrho''} = \frac{\operatorname{Re}(\alpha)}{\operatorname{Re}(\beta)}\frac{\beta}{\alpha}$$

from which

$$\mathbf{t}_\triangle = (\mathbf{t}_{\triangle'})'' = \frac{\operatorname{Im}(\gamma\bar{\alpha})}{\operatorname{Re}(\gamma)}\frac{i}{\alpha} \quad \text{and} \quad \mathbf{u}_\triangle = (\mathbf{u}_{\triangle''})' = \frac{\operatorname{Re}(\beta)}{\operatorname{Im}(\alpha\bar{\beta})}\frac{\alpha}{i}.$$

Since $\mathbf{k}_\triangle = \mathbf{t}_\triangle \mathbf{u}_\triangle$ is then real, $\mathbf{k}$ is on the circumcircle.

If $\gamma(\varrho - \bar{\triangle})$ is imaginary, then since $[\mathbf{w}, \mathbf{m}; \mathbf{c}, \mathbf{v}]$ and $[\infty, \mathbf{c}; \mathbf{w}, \mathbf{a}]$ are real and since $\bar{\varrho} = [\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$ (from MTST), the quantity

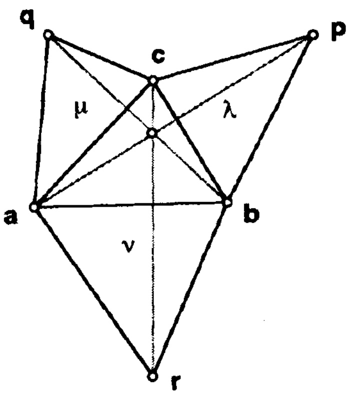
$$\frac{\gamma}{\bar{\varrho} - \triangle}[\mathbf{w}, \mathbf{m}; \mathbf{c}, \mathbf{v}][\infty, \mathbf{c}; \mathbf{w}, \mathbf{a}] = \frac{\mathbf{m} - \mathbf{v}}{\mathbf{b} - \mathbf{c}}$$

is also imaginary. Thus line **mv** is perpendicular to side **bc**, and similarly, lines **mw** and **mx** perpendicular to sides **ca** and **ab**. Then $\triangle$**vwx** is the pedal triangle of **m**, and the segments **am**, **bm** and **cm** are diameters of the circles. These diameters lie along the cevians, which then intersect at $\mathbf{k} = \mathbf{m}$. □

Several special points of a triangle (e.g. its Fermat points) are given by the intersection of cevians from each vertex to the apex of some particular triangle constructed on the opposite side. We consider the general case: which combinations of the four triangle shapes will produce concurrent cevians?

COROLLARY CCEV-1. *If triangles* $\triangle\mathbf{pcb}$, $\triangle\mathbf{qac}$ *and* $\triangle\mathbf{rba}$ *with shapes* λ, μ *and* ν *are erected on the sides of* $\triangle\mathbf{abc}$, *and* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *do not lie on a side through* $\mathbf{a}$, $\mathbf{b}$ *or* $\mathbf{c}$ *respectively, then the cevians* $\mathbf{ap}$, $\mathbf{bq}$ *and* $\mathbf{cr}$ *are concurrent if and only if*

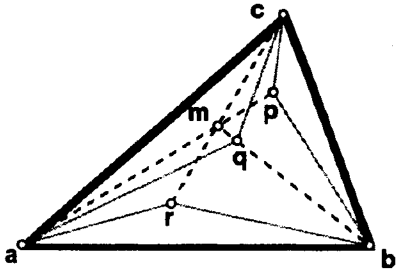
$$\frac{\mathrm{Im}\{(\lambda''\triangle')^{-1}\}}{\mathrm{Im}\{\lambda'\triangle''\}}\cdot\frac{\mathrm{Im}\{(\mu''\triangle'')^{-1}\}}{\mathrm{Im}\{\mu'\triangle\}}\cdot\frac{\mathrm{Im}\{(\nu''\triangle)^{-1}\}}{\mathrm{Im}\{\nu'\triangle'\}}=-1.$$



Proof. In the side point formula, put $\mathbf{p}_\triangle=\lambda\triangle$; then $[\triangle,0;1,\mathbf{p}_\triangle]$ and $[\triangle,\infty;1,\mathbf{p}_\triangle]$ simplify to $(\lambda''\triangle')^{-1}$ and $\lambda'\triangle''$ respectively, so $\mathbf{s}_\triangle=\bar{\triangle}^{-1}\,\mathrm{Im}\{(\lambda''\triangle')^{-1}\}/\mathrm{Im}\{\lambda'\triangle''\}$. Similar expressions result for $\mathbf{t}_{\triangle'}$ and $\mathbf{u}_{\triangle''}$, and the required relations follow immediately from CCEV and the relation $\triangle\triangle'\triangle''=-1$. □

An application:

EXAMPLE 6.2. *Suppose that isogonal pairs of cevians through the vertices of* $\triangle\mathbf{abc}$ *intersect at points* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *as illustrated. Then the cevians* $\mathbf{ap}$, $\mathbf{bq}$ *and* $\mathbf{cr}$ *are concurrent at some point* $\mathbf{m}$.



Proof. We have that $\measuredangle\,\mathbf{bar} = \measuredangle\,\mathbf{qac} =: \theta$, $\measuredangle\,\mathbf{cbp} = \measuredangle\,\mathbf{rba} =: \varphi$ and $\measuredangle\,\mathbf{acq} = \measuredangle\,\mathbf{pcb} =: \psi$. Set $\alpha := e^{2i\theta}$, $\beta := e^{2i\varphi}$ and $\gamma := e^{2i\psi}$. From the angle-shape formula,

$$\nu = \triangle_{\mathbf{rba}} = (e^{2i\measuredangle\,\mathbf{abr}})''(e^{2i\measuredangle\,\mathbf{rab}})' = (\beta^{-1})''(\gamma^{-1})' = \frac{\alpha(1-\beta)}{\alpha - 1},$$

so (using $\alpha\bar{\alpha} = \beta\bar{\beta} = 1$)

$$\nu'' = \frac{1-\alpha\beta}{\alpha(1-\beta)} = \frac{\bar{\alpha}\bar{\beta} - 1}{\bar{\beta} - 1} = \alpha^{-1}\bar{\nu}''$$

and thus

$$2i\,\mathrm{Im}\{(\nu''\triangle)^{-1}\} = -|\nu''\triangle|^{-2}\{\nu''\triangle - \bar{\nu}''\bar{\triangle}\} = -|\triangle|^{-2}(\bar{\nu}'')^{-1}\{\triangle - \alpha\bar{\triangle}\}.$$

Similarly

$$\mu' = \frac{1-\gamma}{1-\gamma\alpha} = \frac{(\bar{\gamma}-1)\bar{\alpha}}{\bar{\gamma}\bar{\alpha} - 1} = \alpha^{-1}\bar{\mu}' \quad \text{and} \quad 2i\,\mathrm{Im}\{\mu\triangle\} = \mu'\{\triangle - \alpha\bar{\triangle}\}$$

so

$$\frac{\mathrm{Im}\{(\nu''\triangle)^{-1}\}}{\mathrm{Im}\{\mu'\triangle\}} = -\frac{1}{|\triangle|^2\mu'\bar{\nu}''} = -\frac{1}{|\triangle|^2}\cdot\frac{1-\gamma\alpha}{1-\gamma}\cdot\frac{1-\beta}{1-\alpha\beta}.$$

Cycle this last relation to get

$$\frac{\mathrm{Im}\{(\lambda''\triangle')^{-1}\}}{\mathrm{Im}\{\nu'\triangle'\}} = -\frac{1}{|\triangle'|^2}\cdot\frac{1-\alpha\beta}{1-\alpha}\cdot\frac{1-\gamma}{1-\beta\gamma}$$

and

$$\frac{\operatorname{Im}\{(\mu''\triangle'')^{-1}\}}{\operatorname{Im}\{\lambda'\triangle''\}}=-\frac{1}{|\triangle''|^2}\cdot\frac{1-\beta\gamma}{1-\beta}\cdot\frac{1-\alpha}{1-\gamma\alpha}.$$

The product of the three right hand sides of these relations is -1, so from CCEV-1, the cevian **ap**, **bq** and **cr** are concurrent. □

In the special case that $\triangle$**pcb**, $\triangle$**acq** and $\triangle$**arb** are similar to each other, Example 6.2 gives Theorem 4.2, (b) of [6]. Example 6.2 has a "dual" nature: if the line pairs **qc-rb**, **ra-pc** and **pb-qa** intersect at $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$ and $\hat{\mathbf{r}}$ respectively, then the cevians **a**$\hat{\mathbf{p}}$, **b**$\hat{\mathbf{q}}$ and **c**$\hat{\mathbf{r}}$ meet at the isogonal conjugate of **m**.

If $\theta=\varphi=\psi=-\pi/6$ in Example 6.2, we get the first Napoleon point of $\triangle$**abc**: the intersection point of the cevians from each vertex through the centre of an equilateral triangle constructed "outward" on the opposite side. Suppose that instead of the centre of an equilateral triangle, we chose a more general point of a more general triangle, and construct similar copies of *that* triangle and point on the sides of $\triangle$**abc**. It is easy to check that, provided we rotate the copies of this new triangle so that a different side of each copy lies along each side of $\triangle$**abc**, the resulting λ, μ and ν must satisfy $\lambda\mu\nu=1$. The following corollary then applies.

COROLLARY CCEV-2. *On the sides of a variable nondegenerate triangle* $\triangle$**abc**, construct triangles $\triangle$**pcb**, $\triangle$**qac** *and* $\triangle$**rba** *of shapes* λ, μ *and* ν *respectively. Assume that* $\lambda\mu\nu=1$. *Then the cevians* **ap**, **bq** *and* **cr** *meet for all triangles* $\triangle$**abc** *if and only if* λ'/ν, μ'/λ *and* ν'/μ *are all imaginary.*

Proof. Note that the assumption $\lambda\mu\nu=1$ implies that

$$\lambda'\mu'\nu'=-(\lambda''\nu''\mu'')^{-1}. \tag{9}$$

From Corollary CCEV-1, the condition to be satisfied is

$$\operatorname{Im}(\lambda'\triangle'')\operatorname{Im}(\mu'\triangle)\operatorname{Im}(\nu'\triangle')=-\operatorname{Im}\left(\frac{1}{\lambda''\triangle'}\right)\operatorname{Im}\left(\frac{1}{\mu''\triangle''}\right)\operatorname{Im}\left(\frac{1}{\lambda''\triangle}\right)$$

for all shapes $\triangle$. Expand the left hand side and simplify using $\triangle\triangle'\triangle''=-1$ to get

$$\frac{1}{4}\operatorname{Im}\left\{\lambda'\mu'\nu'+\frac{\triangle}{\bar{\triangle}}\bar{\lambda}_,'\mu'\bar{\nu}+\frac{\triangle'}{\bar{\triangle}'}\bar{\lambda}'\bar{\mu}'\nu'+\frac{\triangle''}{\bar{\triangle}''}\lambda'\bar{\mu}'\bar{\nu}'\right\}$$

$$=\frac{1}{4}\operatorname{Im}\{\lambda'\mu'\nu'+e^{2iA}\bar{\lambda}'\mu'\bar{\nu}'+e^{2iB}\bar{\lambda}'\bar{\mu}'\nu'+e^{2iC}\lambda'\bar{\mu}'\bar{\nu}'\}.$$

Similarly, the right hand side becomes

$$\frac{1}{4}\,\mathrm{Im}\left\{\frac{-1}{\lambda''\mu''\nu''}+e^{2iA}\frac{1}{\lambda''\mu''\bar{\nu}''}+e^{2iB}\frac{1}{\bar{\lambda}''\mu''\nu''}+e^{2iC}\frac{1}{\lambda''\bar{\mu}''\nu''}\right\},$$

so (using (9)) we must satisfy

$$\mathrm{Im}\left\{e^{2iA}\left[\bar{\lambda}'\mu'\bar{\nu}'-\frac{1}{\lambda''\mu''\bar{\nu}''}\right]+e^{2iB}\left[\bar{\lambda}'\bar{\mu}'\nu'-\frac{1}{\bar{\lambda}''\mu''\nu''}\right]\right.$$
$$\left.+\,e^{2iC}\left[\lambda'\bar{\mu}'\bar{\nu}'-\frac{1}{\lambda''\bar{\mu}''\nu''}\right]\right\}=0$$

for all possible triangles with angles A, B and C. It is not difficult to check that this condition holds if and only if all three square brackets vanish. Using (9), these brackets simplify to

$$-\frac{\mu'}{\bar{\nu}''}\,\mathrm{Re}(\lambda'/\nu),\qquad -\frac{\nu'}{\bar{\lambda}''}\,\mathrm{Re}(\mu'/\lambda)\quad\text{and}\quad -\frac{\lambda'}{\bar{\mu}''}\,\mathrm{Re}(\nu'/\mu),$$

so they vanish if and only if λ'/ν, μ'/λ and ν'/μ are all imaginary. □

There are two possible ways to construct rotated copies of the triangle and point on the sides of $\triangle$**abc**; each gives a different result.

EXAMPLE 6.3. *Let* $\triangle$**jkm** *be any nondegenerate triangle and* **e** *any nonvertex point. On the sides of a variable triangle* $\triangle$**abc**, *construct triangles* $\triangle\widehat{\mathbf{ab}}\mathbf{c}$, $\triangle\widehat{\mathbf{cb}}\mathbf{a}$, *and* $\triangle\mathbf{b}\widehat{\mathbf{ac}}$ *all similar to* $\triangle$**jkm**, *and points* **p**, **q** *and* **r** *in the same position relative to these triangles that* **e** *is to* $\triangle$**jkm**. *Then the cevians* **ap**, **bq** *and* **cr** *meet for all triangles* $\triangle$**abc** *if and only if* **e** *is the orthocentre of* $\triangle$**jkm**.

Proof. We have that $\lambda=[\infty,\mathrm{e};\mathbf{k},\mathbf{m}]$, $\mu=[\infty,\mathrm{e};\mathbf{m},\mathbf{j}]$ and $\nu=[\infty,\mathbf{e};\mathbf{j},\mathbf{k}]$, so $\lambda\mu\nu=1$. From Corollary CCEV-2, then, the cevians meet if and only if the quantities

$$\frac{\lambda'}{\nu}=\frac{\mathbf{e}-\mathbf{j}}{\mathbf{m}-\mathbf{k}},\qquad \frac{\mu'}{\lambda}=\frac{\mathbf{e}-\mathbf{k}}{\mathbf{j}-\mathbf{m}},\quad\text{and}\quad \frac{\nu'}{\mu}=\frac{\mathbf{e}-\mathbf{m}}{\mathbf{k}-\mathbf{j}}$$

are imaginary. This holds if and only if the lines **ej**, **ek** and **em** are perpendicular to sides **mk**, **jm** and **kj** respectively of $\triangle$**jkm**, i.e. if and only if **e** is the orthocentre of $\triangle$**jkm**. □

EXAMPLE 6.4. *Let* $\triangle$**jkm** *be any nondegenerate triangle and* **e** *any nonvertex point. On the sides of a variable triangle* $\triangle$**abc**, *construct triangles* $\triangle\hat{\mathbf{a}}\mathbf{cb}$, $\triangle\mathbf{ac}\hat{\mathbf{b}}$ *and* $\triangle\mathbf{a}\hat{\mathbf{c}}\mathbf{b}$ *all similar to* $\triangle$**jkm**, *and points* **p**, **q** *and* **r** *in the same position relative to these triangles that* **e** *is to* $\triangle$**jkm**. *Then the cevians* **ap**, **bq** *and* **cr** *meet for all triangles* $\triangle$**abc** *if and only if* **e** *is either the incentre of* $\triangle$**jkm** *or one of its excentres.*

Proof. We have that $\lambda = [\infty, \mathrm{e}; \mathbf{k}, \mathbf{m}]$, $\mu = [\infty, \mathrm{e}; \mathbf{j}, \mathbf{k}]$ and $\nu = [\infty, \mathrm{e}; \mathbf{m}, \mathbf{j}]$, so $\lambda\mu\nu = 1$ and $\lambda'\mu'\nu' = -(\lambda''\nu''\mu'')^{-1}$.

We first show that λ'/ν, μ'/λ and ν'/μ are all imaginary if and only if ν''/μ', λ''/ν' and μ''/λ' are all real with a positive product. Note that

$$\frac{\lambda'}{\nu} = -(\lambda'\mu'\nu')\frac{\nu''}{\mu'}, \qquad \frac{\mu'}{\lambda} = -(\lambda'\mu'\nu')\frac{\lambda''}{\nu'}, \qquad \frac{\nu'}{\mu} = -(\lambda'\mu'\nu')\frac{\mu''}{\lambda'}.$$

If λ'/ν, μ'/λ and ν'/μ are all imaginary, then their product $\lambda'\mu'\nu'$ is as well, so λ''/ν' and μ''/λ' are all real, and their product $(\lambda''\nu''\mu'')/(\lambda'\mu'\nu') = -(\lambda'\mu'\nu')^{-2}$ is positive. Conversely, if ν''/μ', λ''/ν' and μ''/λ' are all real with a positive product $-(\lambda'\mu'\nu')^{-2}$, then $\lambda'\mu'\nu'$ is imaginary, so λ'/ν, μ'/λ and ν'/μ are as well.

From Corollary CCEV-2, then, the cevians meet if and only if ν''/μ', λ''/ν' and μ''/λ' are all real with a positive product. The first quantity $\nu''/\mu' = [\infty, \mathbf{j}; \mathbf{e}, \mathbf{m}]/[\infty, \mathbf{j}; \mathbf{k}, \mathbf{e}]$ is real and positive if and only if $\measuredangle$**ejm** = $\measuredangle$**kje**, i.e. if and only if **e** is on the internal bisector through **j**. Similarly, ν''/μ' is real and negative if and only if **e** is on the external bisector through j. Analogous results hold for the angle bisectors through the other vertices, so ν''/μ', λ''/ν' and μ''/λ' are all real with a positive product if and only if **e** is either on all three internal bisectors or on one internal bisector and two external bisectors, i.e. if and only if **e** is either the incentre of $\triangle$**jkm** or one of its excentres. □

If $\triangle$**jkm** is equilateral with centre **e**, then **e** is both incentre and orthocentre, so both of the last two examples specialize to give the Napoleon point.

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