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Survey Papers

On Wigner's theorem: Remarks, complements, comments, and corollaries

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Summary. In this paper we present a unified treatment of Wigner's unitarity-antiunitarity theorem simultaneously in the real and the complex case. Its elementary nature, emphasized by V. Bargmann in 1964, is underlined here by removing unnecessary hypotheses, the most important being the completeness of the inner product spaces involved. At the end, we shall obtain connections to some recent results in geometry.

1. Introduction

Wigner's theorem was first published in 1931 ([13], p. 251). In the early 1960s, several authors began working on Wigner's original idea of proof in order to make the argument rigorous (cf., e.g., [12], [7], [5]). Most recently, there was new movement around Wigner's theorem (cf., e.g., [11], [1], [8]). It seems to the author that Bargmann's paper [5] optimally succeeded in giving a rigorous proof and in revealing the elementary nature of the theorem. We shall take essentially Bargmann's proof as the basis of our procedure till Theorem 7. The results were announced in [9].

2. General hypotheses

Throughout the paper we let $(X, \langle \cdot , \cdot \rangle), (Y, \langle \cdot , \cdot \rangle)$ denote inner product spaces over K $(= \mathbb{R}$ or $\mathbb{C})$ with $\dim_K X \geq 1$, $\dim_K Y \geq 1$ and norm $\| \cdot \|$.

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3. Notations and preliminaries

For the zero vector we write o and, for a subset A of a vector space, $\operatorname{lin}_K A$ denotes the linear span of A. Furthermore, $\mathbb{R}_+^* := \{\lambda \in \mathbb{R};\ \lambda > 0\}$, $\mathbb{R}_-^* := \{\lambda \in \mathbb{R};\ \lambda < 0\}$, $\mathbb{R}_+ := \{\lambda \in \mathbb{R};\ \lambda \geq 0\}$, and $\bar{\cdot}: \mathbb{C} \to \mathbb{C}$ is the ordinary conjugation of $\mathbb{C}$.

For $x, x' \in X$, we define $x' \sim x :\Leftrightarrow \exists \tau \in K$ with $|\tau| = 1$ and $x' = \tau x$,

(1) *the ray* $\dot{x} := \{x' \in X;\ x' \sim x\}$ *of* x $(x \in X)$, and

(2) $\mathscr{S}(X) := X/\!\sim\ = \{\dot{x};\ x \in X\}$, *the set of all rays of* X ([5], p. 862).

If $x, x', z, z' \in X$, $x' \sim x$, $z' \sim z$, then $|\langle x', z' \rangle| = |\langle x, z \rangle|$ and $\|x'\| = \|x\|$, so that

(3) $\dot{x} \cdot \dot{z} := |\langle x, z \rangle|$, $|\dot{x}| := \|x\|$ $(x \in \dot{x},\ z \in \dot{z})$ are well-defined.

(4) $\mathscr{S}_1(X) := \{\dot{x} \in \mathscr{S}(X);\ |\dot{x}| = 1\}$ *is the set of all unit rays of* X, and the elements of the set

(5) $\mathscr{R}(X) := \{Kx;\ x \in X \backslash \{o\}\}$ are called *lines of* X.

Some authors describe states of a physical system by unit rays (e.g., [5]), some by lines (e.g., [12], [7]). The equivalence of these two kinds of description is established by the bijective mapping

(6) $\Phi_X: \mathscr{S}_1(X) \to \mathscr{R}(X)$, $\Phi_X(\dot{x}) := \operatorname{lin}_K \dot{x}\,(\dot{x} \in \mathscr{S}_1(X))$ and by Lemma 1 below.

For the transition probability $p_X(\dot{x}, \dot{z})$ $(\dot{x}, \dot{z} \in \mathscr{S}_1(X))$ or $\tilde{p}_X(Kx, Kz)$ $(Kx, Kz \in \mathscr{R}(X))$ we obtain

(7) $p_X(\dot{x}, \dot{z}) = (\dot{x} \cdot \dot{z})^2 =_{(3)} = (\langle x, z \rangle \cdot \langle z, x \rangle)/(\langle x, x \rangle \cdot \langle z, z \rangle) = \tilde{p}_X(Kx, Kz)$,

so that the preservation of the transition probabilities is expressed by the two functional equations $(\dot{*}_0)$ and $(\tilde{*})$ below.

LEMMA 1. (*a*) *The solutions* T, T_0, $\tilde{T}$ *of the functional equations*

$$(\dot{*})\quad T: \mathscr{S}(X) \to \mathscr{S}(Y), \qquad T\dot{x} \cdot T\dot{z} = \dot{x} \cdot \dot{z} \quad (\forall \dot{x}, \dot{z} \in \mathscr{S}(X)),$$

$$(\dot{*}_0)\quad T_0: \mathscr{S}_1(X) \to \mathscr{S}_1(Y), \qquad T_0\dot{x} \cdot T_0\dot{z} = \dot{x} \cdot \dot{z} \quad (\forall \dot{x}, \dot{z} \in \mathscr{S}_1(X)),$$

$$(\tilde{*})\quad \tilde{T}: \mathscr{R}(X) \to \mathscr{R}(Y), \qquad \tilde{p}_Y(\tilde{T}(Kx), \tilde{T}(Kz)) = \tilde{p}_X(Kx, Kz) \quad (\forall Kx, Kz \in \mathscr{R}(X))$$

are in bijective correspondence by virtue of the diagram

```tikzcd
\mathscr{S}(X) \arrow[r, "T"] & \mathscr{S}(Y) \\
\mathscr{S}_1(X) \arrow[u, hook] \arrow[r, "T_0"] \arrow[d, two heads, tail, "\Phi_X"'] & \mathscr{S}_1(Y) \arrow[u, hook] \arrow[d, two heads, tail, "\Phi_Y"] \\
\mathscr{R}(X) \arrow[r, "\tilde{T}"] & \mathscr{R}(Y).
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(*b*) *T surjective $\Leftrightarrow T_0$ surjective $\Leftrightarrow \tilde{T}$ surjective.*
(*c*) *All solutions of* $(\dot{*})$, $(\dot{*}_0)$, $(\tilde{*})$ *are injective.*

Proof. (a) If T satisfies $(\dot{*})$, then $T[\mathscr{S}_1(X)] \subset \mathscr{S}_1(Y)$, and we take T_0 as the (bilateral) restriction of T. For extending uniquely a solution T_0 of $(\dot{*}_0)$ to a solution T of $(\dot{*})$ see [5], p. 864, section 2. The rest is clear from (6) and (7).

(b) is easily established; it separates the surjectivity question from the main body of the theory.

(c) follows from the equality condition in the Cauchy–Schwarz inequality ([5], p. 863, 1.2). q.e.d.

By the way, $\tilde{p}_X$ is the Cayley measure on metric vector spaces in the sense of [2], p. 4.

Next we are going to connect ray mappings T (and therefore, by Lemma 1, also line mappings $\tilde{T}$) with vector mappings $S: X \to Y$.

DEFINITION 2. The mappings $R: X \to Y$ and $S: X \to Y$ are called *phase-equivalent* if $\exists \tau: X \to K$ such that $|\tau(x)| = 1$, $Rx = \tau(x) \cdot Sx$ $(\forall x \in X)$, in other words, if $Rx \sim Sx$ $(\forall x \in X)$.

LEMMA 3. (*a*) *If $S: X \to Y$ is a solution of the functional equation*

$$|\langle Sx, Sz \rangle| = |\langle x, z \rangle| \qquad (\forall x, z \in X), \qquad (*)$$

then there exists a ray mapping $T: \mathscr{S}(X) \to \mathscr{S}(Y)$ satisfying $(\dot{*})$ and $(Sx)^{\cdot} = T\dot{x}$, i.e., $Sx \in T\dot{x}$ $(\forall x \in X)$. We say then that T is induced by S and that S is compatible with T.

(*b*) *If $T: \mathscr{S}(X) \to \mathscr{S}(Y)$ is a solution of $(\dot{*})$, if $R: X \to Y$, $S: X \to Y$, and if S is compatible with T, then S is a solution of $(*)$, and R is compatible with T if and only if R and S are phase-equivalent.*

(*c*) *If $T: \mathscr{S}(X) \to \mathscr{S}(Y)$ is a solution of $(\dot{*})$, then there exists a solution $S: X \to Y$ of $(*)$ such that S is compatible with T.*

Proof. (a) If $S: X \to Y$ is a solution of $(*)$, the following properties of S are quite immediate:

(8) $x \in X \Rightarrow \|Sx\| = \|x\|$,
(9) $x, z \in X \Rightarrow [x \perp z \Leftrightarrow Sx \perp Sz]$,
(10) $x, z \in X \Rightarrow [x, z$ linearly independent $\Leftrightarrow Sx, Sz$ linearly independent$]$,
(11) $x \in X, \lambda \in K \Rightarrow S(\lambda x) \sim \lambda Sx$,
(12) $x, x' \in X, x' \sim x \Rightarrow Sx' \sim Sx$,
(13) S additive $\Rightarrow S$ injective.

(10) follows from the equality criterion in the Cauchy–Schwarz inequality ([5], p. 866, 6.(a)). By (12), $T\colon \mathscr{S}(X) \to \mathscr{S}(Y)$ is well-defined by $T\dot{x} := (Sx)^{\cdot}$ $(\forall x \in X)$, and by (3) and $(*)$, T satisfies $(\dot{*})$.

(b) For $x, z \in X$ we have $|\langle Sx, Sz \rangle| =_{(3)} = (Sx)^{\cdot} \cdot (Sz)^{\cdot} = T\dot{x} \cdot T\dot{z} = \dot{x} \cdot \dot{z} = |\langle x, z \rangle|$, which proves the first part. $-R$ compatible with $T \Leftrightarrow (Rx)^{\cdot} = T\dot{x}$ $(\forall x \in X)$ $\Leftrightarrow (Rx)^{\cdot} = (Sx)^{\cdot}$ $(\forall x \in X)$ $\Leftrightarrow Rx \sim Sx$ $(\forall x \in X) \Leftrightarrow R$ and S are phase-equivalent.

(c) For every $x \in X$, choose y from the subset $T\dot{x}$ of Y and call it Sx. Then S is compatible with T and by (b) a solution of $(*)$. q.e.d.

REMARK 4. Lemmas 1(a) and 3 require a complement: If $T_0\colon \mathscr{S}_1(X) \to \mathscr{S}_1(Y)$ is a solution of $(\dot{*}_0)$ and if $S\colon X \to Y$ satisfies $S(\varrho x) = \varrho Sx$ $(\forall x \in X, \forall \varrho \in \mathbb{R}_+)$ as well as $(Sx)^{\cdot} = T_0\dot{x}$ $(\forall x \in X, \|x\| = 1)$, then we have $(Sx)^{\cdot} = T\dot{x}$ $(\forall x \in X)$ for the unique solution $T\colon \mathscr{S}(X) \to \mathscr{S}(Y)$ of $(\dot{*})$ which extends T_0. The proof is easy.

REMARK 5. (a) Lemma 3 shows that the solutions T of $(\dot{*})$ and the phase-equivalence classes of solutions of $(*)$ are in bijective correspondence.

(b) The very poor method of proof of Lemma 3(c) does not guarantee at all a good quality of the mapping S because the elements Sx $(x \in X)$ are chosen completely unrelatedly.

(c) If $\chi\colon K \to K$ denotes the identity of K or, if $K = \mathbb{C}$, the ordinary conjugation mapping, then the linear or conjugate-linear isometries $U\colon X \to Y$ are characterized by

$$\langle Ux, Uz \rangle = \chi(\langle x, z \rangle) \quad (\forall x, z \in X), \tag{I_χ}$$

and of course every solution of (I_χ) is a solution of $(*)$.

(d) The question arises whether every solution T of $(\dot{*})$ can be lifted into a "nice" solution of $(*)$, e.g., in the optimal case, into an isometry. Wigner's theorem gives a positive answer to this question.

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d] & Y \arrow[d] \\ \mathscr{S}(X) \arrow[r, "T"'] & \mathscr{S}(Y) \end{tikzcd}$$

There are some auxiliary statements which need explicit mention for later use:

LEMMA 6. *Let* $T\colon \mathscr{S}(X) \to \mathscr{S}(Y)$ *be a solution of* $(\dot{*})$. *Then we have*:

(*a*) *If* $\dim_K X \geq 2$ *and* $R, S\colon X \to Y$ *are additive and compatible with* T, *then* $\exists \theta \in K$ *such that* $|\theta| = 1$ *and* $R = \theta \cdot S$.

(*b*) *If* $x \in X \backslash \{o\}$, $S\colon \mathrm{lin}_K\{x\} \to Y$, $(Sz)^{\cdot} = T\dot{z}$ $(\forall z \in \mathrm{lin}_K\{x\})$, *then there exists a unique function* $\varphi_x\colon K \to K$ *with the properties* $\varphi_x(1) = 1$ *and* $|\varphi_x(\lambda)| = |\lambda|$, $S(\lambda x) = \varphi_x(\lambda) \cdot Sx$ $(\forall \lambda \in K)$.

(*c*) *If* $K = \mathbb{R}$, $\dim_{\mathbb{R}} X = 1$ *and* $R, S\colon X \to Y$ *are additive and compatible with* T, *then* $R = S$ *or* $R = -S$.

(*d*) *If* $K = \mathbb{C}$, $\dim_{\mathbb{C}} X = 1$ *and* $R, S\colon X \to Y$ *are additive and compatible with* T, *then there is in general no* $\theta \in \mathbb{C}$ *such that* $R = \theta \cdot S$.

(*e*) *Let* $S\colon X \to Y$ *be compatible with* T. *Then*:

(*ea*) *If* S *is surjective, so is* T.

(*eb*) *If* T *is surjective, and if* $w \in S(X)$, $\lambda \in K$ *implies* $\lambda w \in S(X)$, *then* S *is surjective.*

(*ec*) *If* T *is bijective, then* S *need not be injective or surjective.*

Proof. (a), (b): The proofs given for $K = \mathbb{C}$ ([5], p. 866, Theorem 2; p. 864, (8), (8a)) work for $K = \mathbb{R}$ as well.

(c) Let $X = \mathrm{lin}_{\mathbb{R}}\{x\}$. By (b) $\exists \varphi_x\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi_x(1) = 1$ and $|\varphi_x(\lambda)| = |\lambda|$, $S(\lambda x) = \varphi_x(\lambda) \cdot Sx$ $(\forall \lambda \in \mathbb{R})$. For any $\lambda, \mu \in \mathbb{R}$ we get $\varphi_x(\lambda + \mu)Sx = S((\lambda + \mu)x) = S(\lambda x + \mu x) = S(\lambda x) + S(\mu x) = \varphi_x(\lambda)Sx + \varphi_x(\mu)Sx = [\varphi_x(\lambda) + \varphi_x(\mu)]Sx$, and Lemma 3(b) and (8) ensure $Sx \neq o$, so $\varphi_x(\lambda + \mu) = \varphi_x(\lambda) + \varphi_x(\mu)$ $(\forall \lambda, \mu \in \mathbb{R})$. Since $|\varphi_x(\lambda)| = |\lambda| \leq 1$ $(0 \leq \lambda \leq 1)$, φ_x is of the form $\varphi_x(\lambda) = \varphi_x(1) \cdot \lambda = \lambda$ $(\forall \lambda \in \mathbb{R})$ by a famous theorem of Darboux ([6]) originating from projective geometry, i.e., $\varphi_x = id_{\mathbb{R}}$. Therefore $S(\lambda x) = \lambda Sx$ $(\forall \lambda \in \mathbb{R})$ and analogously $R(\lambda x) = \lambda Rx$ $(\forall \lambda \in \mathbb{R})$. By Lemma 3(b) $Rx \sim Sx$, which means here $Rx = \pm Sx$, hence $R = \pm S$.

(d) $X = Y = \mathbb{C}^1$ with $\langle x, z \rangle := x\bar{z}$ $(\forall x, z \in X)$, $S = id_{\mathbb{C}}$, R the ordinary conjugation of $\mathbb{C}$ ([5], p. 863, 1.4).

(ea) $\dot{y} \in \mathcal{S}(Y)$, $y \in \dot{y}$. There exists $x \in X$ such that $Sx = y$. Thus $T\dot{x} = (Sx)^{\cdot} = \dot{y}$. — (eb) (For S a (conjugate-)linear isometry cf. [5], p. 863, Corollary). $y \in Y$ arbitrary. So $\dot{y} \in \mathcal{S}(Y)$, and there exists $\dot{x} \in \mathcal{S}(X)$ such that $T\dot{x} = \dot{y}$. For $x \in \dot{x}$ we have $(Sx)^{\cdot} = T\dot{x} = \dot{y}$, i.e., $Sx \sim y$, say $y = \tau Sx$, therefore by the hypothesis on $S(X)$ finally $y \in S(X)$. — (ec) $X = Y = \mathbb{R}^1$ with $\langle x, z \rangle := xz$ $(\forall x, z \in X)$, $Sx := |x|$ $(\forall x \in X)$. Then $Sx \sim x$ $(\forall x \in X)$, so $T\dot{x} = (Sx)^{\cdot} = \dot{x}$ $(\forall x \in X)$, i.e., S induces the bijective mapping $T = id_{\mathcal{S}(X)}$. But S is neither injective nor surjective nor additive. This shows that the special hypotheses in (13) and in (eb) are essential. q.e.d.

4. The main results

THEOREM 7 (E. Wigner). *Under our general hypotheses, let* $T\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ *be a solution of* $(*)$ $T\dot{x} \cdot T\dot{z} = \dot{x} \cdot \dot{z}$ $(\forall \dot{x}, \dot{z} \in \mathcal{S}(X))$. *Then*:

(*a*) *There exists* $U\colon X \to Y$ *with the following properties*:
(*aa*) $(Ux)^{\cdot} = T\dot{x}$ *for all* $x \in X$.
(*ab*) $\langle Ux, Uz \rangle = \chi(\langle x, z \rangle)$ $(\forall x, z \in X)$ *where* $\chi = id_{\mathbb{R}}$ $(K = \mathbb{R})$, $\chi = id_{\mathbb{C}}$ *or* $\chi = \bar{\ }$ $(K = \mathbb{C})$, *i.e.*, U *is a linear or a conjugate-linear isometry.*
(*b*) *If* $K = \mathbb{C}$, $\dim_{\mathbb{C}} X \geq 2$, *then* T *uniquely indicates whether* $\chi = id_{\mathbb{C}}$ *or* $\chi = \bar{\ }$.
(*c*) U *is surjective if and only if* T *is surjective.*
(*d*) *If both* $U_1\colon X \to Y$ *and* $U_2\colon X \to Y$ *satisfy* (*aa*) *and* (*ab*) *above, then they are phase-equivalent, and, except for* $K = \mathbb{C}$, $\dim_{\mathbb{C}} X = 1$, *there exists a* $\theta \in K$ *such that* $U_2 = \theta \cdot U_1$, $|\theta| = 1$.

Proof. [5], pp. 863–866. Specifically: (b) section 1.5; (c), (d): our Lemma 6 (ea), (eb); (a), (c), (d).

COROLLARY 8. *If, under our general hypotheses,* $S\colon X \to Y$ *satisfies* $(*)$ $|\langle Sx, Sz \rangle| = |\langle x, z \rangle|$ $(\forall x, z \in X)$, *then*
(*a*) S *is phase-equivalent to a linear isometry* $U\colon X \to Y$ *if* $K = \mathbb{R}$.
(*b*) S *is a phase-equivalent to a linear or a conjugate-linear isometry* $U\colon X \to Y$ *if* $K = \mathbb{C}$.
(*c*) *If* S *is surjective, then* S *is phase-equivalent to an inner product space isomorphism or anti-isomorphism* $U\colon X \to Y$, *i.e., to a unitary or an anti-unitary mapping if* $X = Y$.
(*d*) *If* $R\colon X \to Y$ *is phase-equivalent to a linear or to a conjugate-linear isometry* $U\colon X \to Y$, *then* R *is a solution of* $(*)$.

Proof. (*a*), (*b*): Let $T\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ be the ray mapping induced by S (Lemma 3(a)). By Theorem 7(a) $\exists U\colon X \to Y$, a linear or conjugate-linear isometry (for $K = \mathbb{R}$, these two variants coincide) such that $(Ux)^{\cdot} = T\dot{x}$ $(\forall x \in X)$. By Lemma 3(b), S is phase-equivalent to U. (c): Directly from Lemma 6(ea) and Theorem 7(c). (d): $|\langle Rx, Rz \rangle| = |\langle Ux, Uz \rangle| = |\chi(\langle x, z \rangle)| = |\langle x, z \rangle|$ $(\forall x, z \in X)$. q.e.d.

REMARK 9. Corollary 8(a) generalizes Theorem 1 of [4], p. 49, and, for $K = \mathbb{R}$, Corollary 8(c) is identical to the first fundamental theorem of projective-metric geometry ([10], p. 145, (10.4)).

REMARK 10. Some proofs either of Theorem 7 or of Corollary 8 in the papers quoted earlier are restricted to the exclusive case $K = \mathbb{C}$, to the special case $X = Y$, to bijective mappings $T\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ or $S\colon X \to Y$, and some use characteristic properties of Hilbert spaces such as

- the Riesz–Fischer theorem,
- the fact that every maximal orthonormal subset is fundamental,
- the weak sequential compactness of every bounded subset, or
- the closedness of the range of an isometry.

These properties fail to hold in non-complete inner product spaces. On the other hand, Bargmann's procedure is easily adapted for the general inner product space case. The main source of this advantage lies in working with appropriate finite orthonormal sets and the corresponding Bessel's identity, and in using $X = M \oplus M^{\perp}$ only for finite-dimensional linear subspaces M of X. This insight should underline the elementary nature of Wigner's theorem.

REMARK 11. In connection with isometries $U: X \to Y$, the following functional equations are important (cf. Remark 5(c)):

$$\|Ux - Uz\| = \|x - z\| \ (\forall x, z \in X) \text{ and } Uo = o, \tag{III}$$

$$Re\langle Ux, Uz\rangle = Re\langle x, z\rangle \ (\forall x, z \in X), \tag{II}$$

$$\langle Ux, Uz\rangle = \chi(\langle x, z\rangle) \ (\forall x, z \in X), \tag{I_χ}$$

$$|\langle Ux, Uz\rangle| = |\langle x, z\rangle| \ (\forall x, z \in X). \tag{$*$}$$

Clearly, $(*) \Leftarrow (\mathrm{I}_\chi) \Rightarrow (\mathrm{II}) \Leftrightarrow (\mathrm{III})$. Wigner's theorem just says that $(*) \Rightarrow (\mathrm{I}_\chi)$ holds up to phase-equivalence. For $K = \mathbb{R}$, (II) $\Rightarrow (\mathrm{I}_{id_{\mathbb{R}}})$ trivially holds if we agree in the convention Re: $\mathbb{R} \to \mathbb{R}$, Re$:= id_{\mathbb{R}}$. For $K = \mathbb{C}$, (II) $\Rightarrow [(\mathrm{I}_{id_{\mathbb{C}}})$ or $(\mathrm{I}_{\bar{\cdot}})]$ does not hold as the following example shows: $X = Y = \mathbb{C}^2$ with $\langle(\xi_1, \xi_2), (\zeta_1, \zeta_2)\rangle := \xi_1\bar{\zeta}_1 + \xi_2\bar{\zeta}_2 \ (\forall(\xi_1, \xi_2), (\zeta_1, \zeta_2) \in \mathbb{C}^2)$, $U(\xi_1, \xi_2) := (\xi_1, \bar{\xi}_2) \ (\forall(\xi_1, \xi_2) \in \mathbb{C}^2)$. This U also violates $(*)$. This phenomenon may explain why in the proof of Theorem 7(b) some effort is needed for establishing the pure linear or the pure conjugate-linear feature of U in the complex case.

5. Continuous solutions of $(*)$

LEMMA 12. *If $(Y, \|\cdot\|)$ is a normed K-vector space and $y_k, y \in Y$, $y \neq o$, $\lambda_k, \lambda \in K$ $(\forall k \in \mathbb{N})$, then $y_k \to y$ $(k \to \infty)$ and $\lambda_k y_k \to \lambda y$ $(k \to \infty)$ imply $\lambda_k \to \lambda$ $(k \to \infty)$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary. $|\lambda| \cdot \|y_k - y\| + \|\lambda_k y_k - \lambda y\| = \|\lambda(y_k - y)\| + \|(\lambda_k - \lambda)y_k + \lambda(y_k - y)\| \geq \|(\lambda_k - \lambda)y_k\| = |\lambda_k - \lambda| \cdot \|y_k\| \geq 0$. As the left-hand side tends to zero, we obtain $|\lambda_k - \lambda| \cdot \|y_k\| \to 0$ $(k \to \infty)$. Since $y_k \to y \neq o$ $(k \to \infty)$, there is no loss of generality in assuming $y_k \neq o$ $(\forall k \in \mathbb{N})$. So $\|y_k\| \to \|y\| > 0$, $1/\|y_k\| \to 1/\|y\|$ $(k \to \infty)$, and finally

$$|\lambda_k - \lambda| = (1/\|y_k\|) \cdot |\lambda_k - \lambda| \cdot \|y_k\| \to (1/\|y\|) \cdot 0 = 0 \ (k \to \infty).$$

THEOREM 13. *If, under our general hypotheses, $S: X \to Y$ is a continuous solution of $(*)$ $|\langle Sx, Sz\rangle| = |\langle x, z\rangle|$ $(\forall x, z \in X)$, then we have:*

(a) *If $K = \mathbb{R}$, $\dim_{\mathbb{R}} X = 1$, say $X = \mathrm{lin}_{\mathbb{R}}\{x_0\}$ with $\|x_0\| = 1$, then there exists $y_0 \in Y$ with $\|y_0\| = 1$ such that $S(\lambda x_0) = \lambda y_0$ $(\forall \lambda \in \mathbb{R})$ or $S(\lambda x_0) = |\lambda| y_0$ $(\forall \lambda \in \mathbb{R})$.*

(*b*) *If* $K = \mathbb{R}$, $\dim_{\mathbb{R}} X \geq 2$, *then* S *is a linear isometry.*

(*c*) *If* $K = \mathbb{C}$, *then there exists a linear or conjugate-linear isometry* $U: X \to Y$ *and* $f: X \to \mathbb{R}$ *such that* $\cos \circ f$ *and* $\sin \circ f$ *are continuous on* $X \backslash \{o\}$ *and* $Sx = e^{if(x)} Ux$ $(\forall x \in X)$.

Conversely, all these mappings are continuous solutions of $(*)$.

Proof. Let S be a continuous solution of $(*)$. By Corollary 8(a), (b) there exists a linear or a conjugate-linear isometry $U: X \to Y$ and a mapping $\tau: X \to K$ with

(14) $|\tau(x)| = 1$ and $Sx = \tau(x) \cdot Ux$ $(\forall x \in X)$.

Let $x \in X \backslash \{o\}$ and (x_k) a sequence of elements of X with $x_k \to x$ $(k \to \infty)$. By (14), $Sx_k = \tau(x_k) \cdot Ux_k$ $(\forall k \in \mathbb{N})$ and $Sx = \tau(x) \cdot Ux$. Continuity of S and U lead to $Sx_k \to Sx$, $Ux_k \to Ux$ $(k \to \infty)$, and we have $\|Ux\| = \|x\| > 0$, i.e., $Ux \neq 0$. By Lemma 12 we get $\tau(x_k) \to \tau(x)$ $(k \to \infty)$. Therefore τ is continuous at x, and since $x \in X \backslash \{o\}$ was arbitrary,

(15) τ is continuous on $X \backslash \{o\}$.

(a) $y_0 := Sx_0$, so $\|y_0\| =_{(8)} = \|x_0\| = 1$. If $x \in X$ is arbitrary, x and x_0 are linearly dependent, and by (10) so are Sx and Sx_0, i.e., $S(X) \subset \mathrm{lin}_{\mathbb{R}} \{y_0\}$. $Ux =_{(14)} = (1/\tau(x)) \cdot Sx \in \mathrm{lin}_{\mathbb{R}} \{y_0\}$ $(\forall x \in X)$. By Corollary 8(a), U is $\mathbb{R}$-linear, thus there exists a unique $\alpha \in \mathbb{R}$ such that $Ux_0 = \alpha y_0$. $|\alpha| = |\alpha| \cdot \|y_0\| = \|\alpha y_0\| = \|Ux_0\| = \|x_0\| = 1$, so $\alpha = \pm 1$. $U(\lambda x_0) = \lambda U x_0 = \lambda \alpha y_0 = \alpha(\lambda y_0)$ $(\forall \lambda \in \mathbb{R})$, i.e.,

(16) $U(\lambda x_0) = \alpha \lambda y_0$ $(\forall \lambda \in \mathbb{R})$ with $\alpha \in \{-1, 1\}$ fixed,

(17) $S(\lambda x_0) =_{(14),(16)} = \tau(\lambda x_0) \alpha \lambda y_0$ $(\forall \lambda \in \mathbb{R})$.

$\lambda = 1$ in (17) yields $Sx_0 = \tau(x_0) \alpha y_0$, and since $Sx_0 = y_0$, we get

(18) $\tau(x_0) \alpha = 1$.

$\lambda = -1$ in (17) yields $S(-x_0) = \tau(-x_0)\alpha(-1)y_0$. On the other hand, $\|S(-x_0)\| =_{(8)} = \|-x_0\| = 1$, so

(19) $S(-x_0) = \beta y_0$ with $\beta = -\tau(-x_0)\alpha \in \{-1, 1\}$.

Now, $\tau(x) \in \{-1, 1\}$ $(\forall x \in X)$, connectedness of $\mathbb{R}_+^* \cdot x_0$ and $\mathbb{R}_-^* \cdot x_0$, and (15) imply $\tau(\lambda x_0) = \tau(x_0)$ $(\forall \lambda \in \mathbb{R}_+^*)$, $\tau(\lambda x_0) = \tau(-x_0)$ $(\forall \lambda \in \mathbb{R}_-^*)$. So $S(\lambda x_0) =_{(17)} = \tau(x_0)\alpha\lambda y_0 =_{(18)} = \lambda y_0$ $(\forall \lambda \in \mathbb{R}_+^*)$ and $S(\lambda x_0) =_{(17)} = \tau(-x_0)\alpha\lambda y_0 =_{(19)} = -\beta\lambda y_0$ $(\forall \lambda \in \mathbb{R}_-^*)$. Since $So =_{(8)} = o$, we have in the total $S(\lambda x_0) = \omega(\lambda) y_0$ $(\forall \lambda \in \mathbb{R})$ with $\omega = id_{\mathbb{R}}$ or $\omega = |\cdot|$. — Conversely, if $S(\lambda x_0) = \omega(\lambda) y_0$ $(\forall \lambda \in \mathbb{R})$ with $\|x_0\| = \|y_0\| = 1$, $\omega = id_{\mathbb{R}}$ or $\omega = |\cdot|$, then S is continuous, and $|\langle S(\lambda x_0), S(\mu x_0) \rangle| = |\langle \omega(\lambda) y_0, \omega(\mu) y_0 \rangle| = |\omega(\lambda) \cdot \omega(\mu)| = |\lambda \cdot \mu| = |\langle \lambda x_0, \mu x_0 \rangle|$ $(\forall \lambda, \mu \in \mathbb{R})$ says that S satisfies $(*)$.

(b) Since $\dim_{\mathbb{R}} X \geq 2$, any two points of $X \backslash \{o\}$ can be joined by a polygon in $X \backslash \{o\}$. Thus $X \backslash \{o\}$ is connected, and $\tau(X \backslash \{o\}) \subset \{-1, 1\}$ and (15) imply that τ is constant on $X \backslash \{o\}$, say $\tau(x) = \tau_0$ $(\forall x \in X \backslash \{o\})$. So $Sx =_{(14)} = \tau_0 Ux$ $(\forall x \in X \backslash \{o\})$. But $So = o = \tau_0 \cdot o = \tau_0 Uo$, i.e., $S = \tau_0 \cdot U$ with $\tau_0 \in \{-1, 1\}$. By Corollary 8(a), U is a linear isometry, and so is S. — The converse statement is valid by Remark 5(c).

(c) For every $x \in X$, choose $f(x) \in \mathbb{R}$ such that $\tau(x) = e^{if(x)}$; $f(x)$ is unique mod 2π. By (15), $\cos \circ f = \operatorname{Re} \tau(\cdot)$ and $\sin \circ f = \operatorname{Im} \tau(\cdot)$ are continuous on $X \backslash \{o\}$, and $Sx =_{(14)} = e^{if(x)} Ux$ ($\forall x \in X$). — Conversely, let $U: X \to Y$, $f: X \to \mathbb{R}$ have the properties required and define $S: X \to Y$ by $Sx = e^{if(x)} Ux$ ($\forall x \in X$). U satisfies $(*)$ by Remark 5(c), and then so does S. $\operatorname{Re} e^{if(\cdot)} = \cos \circ f$ and $\operatorname{Im} e^{if(\cdot)} = \sin \circ f$ are continuous on $X \backslash \{o\}$, and so is $e^{if(\cdot)}$. Since U is continuous, S is continuous on $X \backslash \{o\}$. Let $\varepsilon \in \mathbb{R}_+^*$ be arbitrary. $x \in X$ and $\|x\| < \varepsilon$ imply $\|Ux\| < \varepsilon$, so $\|Sx\| < \varepsilon$. Since $So = o$, S is continuous at o. In the total, S is continuous on the whole of X.

REMARK 14. C. Alsina and J. L. Garcia-Roig ([3], p. 214, Theorem 1) proved the special case $X = Y = \mathbb{R}^n$ ($n \in \mathbb{N}$, $n \geq 2$) of Theorem 13(b) by a different method.

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