On the Fundamental Approximation Theorems of D. Jackson, S. N. Bernstein and Theorems of M. Zamansky and S. B. Stečkin

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To A. M. Ostrowski on the occasion of his 75th birthday, September 25 1968

1. Introduction

The direct theorems of D. JACKSON and the inverse theorems of S. N. BERNSTEIN as well as their generalizations by A. ZYGMUND play a fundamental role in the theory of approximation of periodic functions by trigonometric polynomials. Of further importance is an interesting theoıem by M. ZAMANSKY on derivatives of trigonometric polynomials which converge uniformly towards a periodic function with a given order of approximation. This result has so far played a somewhat isolated role in approximation theory. Then there is a theorem of S. B. STEČKIN on estimations of the convergence of the rth derivative of these trigonometric polynomials towards the rth derivative of the function.

It is the purpose of this paper to establish a converse not only to the Zamansky result for polynomials of best approximation (as well as for a general class of linear approximation processes) but also to the theorem of Stečkin. These results enable one to state connections between the theorems in question that do not seem to have been observed before. Indeed, it may (roughly) be said that the assertions of the theorems of Jackson, Bernstein, Zamansky and Stečkin are equivalent to another for polynomials of best approximation (cf. Theorem 2.2) as well as for a general class of linear approximation processes. The results presented will be established not only for $C_{2\pi}$ or $L^p_{2\pi}$ functions but also in the setting of the theory of intermediate spaces. The essential aim is to examine the inner structure of the classical proofs in order to be assured which elements can be carried over to the abstract setting.

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1) The results of Theorems 2.2 and 2,3 for $L^2{}_{\pi}$-spaces (apart from part (b)) were presented by the first named author in a colloquium lecture held at the Mathematics Research Center, University of Wisconsin, on Nov. 3, 1966. The material of Sec. 2 was presented at the University of Stuttgart on Febr. 15, 1968.

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2. Interconnections Among the Fundamental Theorems

Let us denote a trigonometrical polynomial of degree $\leqslant n$ by $t_n(x)$ and the corresponding linear space by T_n. For the sake of simplicity we restrict the discussion in this section to the uniform norm $\|f\| = \sup_x |f(x)|$ for the elements $f \in \mathsf{C}_{2\pi}$. Setting

$$\mathrm{E}_n[f] = \inf_{t_n \in \mathsf{T}_n} \|f - t_n\| \qquad (f \in \mathsf{C}_{2\pi};\, n \in \mathbb{N}) \tag{2.1}$$

then $\mathrm{E}_0[f] \geqslant \mathrm{E}_1[f] \geqslant \cdots$, and by the theorem of WEIERSTRASS

$$\lim_{n \to \infty} \mathrm{E}_n[f] = 0. \tag{2.2}$$

Here $\mathbb{N}$ denotes the set of all non-negative integers. A result of P. KIRCHBERGER asserts that for every $f \in \mathsf{C}_{2\pi}$ and $n \in \mathbb{N}$ $\mathrm{E}_n(f)$ is attained, i.e. there exists a $t_n^*(x) = t_n^*(f; x) \in \mathsf{T}_n$ such that

$$\mathrm{E}_n[f] = \|f - t_n^*\|. \tag{2.3}$$

Moreover, the polynomial t_n^* of best approximation is unique.

We first state the cited theorems in a form needed below. For $f \in \mathsf{C}_{2\pi}$ we write: $f \in \mathrm{Lip}\,\alpha$ if there is a constant $M > 0$ such that $|f(x+h) - f(x)| \leqslant M|h|^\alpha$ for all x; $f \in \mathrm{Lip}^*\,\alpha$ if $|f(x+h) + f(x-h) - 2f(x)| \leqslant M^*|h|^\alpha$ for all x ; $f \in \mathrm{W}$ if $|f(x+h) - f(x)| \leqslant M'|h \log|1/h||$ for all x.

THEOREM (JACKSON). *If* $f \in \mathsf{C}_{2\pi}$ *and* $f^{(r)} \in \mathrm{Lip}\,\alpha$ *where* $0 < \alpha \leqslant 1$ *and* $r \in \mathbb{N}$, *then* $\mathrm{E}_n[f] = O(n^{-r-\alpha})$.

In the course of the proof (see e.g. CHENEY [6, p. 145], MEINARDUS [11, p. 55]) one may derive the important inequality

$$\mathrm{E}_n[f] \leqslant (c/n)^r \|f^{(r)}\| \qquad (c = 1 + \pi^2/2) \tag{2.4}$$

which is valid if $f^{(r)} \in \mathsf{C}_{2\pi}$. We refer to it as the J a c k s o n inequality.

THEOREM (BERNSTEIN). *If* $f \in \mathsf{C}_{2\pi}$ *and* $\mathrm{E}_n[f] = O(n^{-r-\alpha})$ *where* $r \in \mathbb{N}$ *and* $0 < \alpha \leqslant 1$, *then f possesses continuous derivatives of orders* 1, 2, ... *r and*

$$f^{(r)} \in \begin{cases} \mathrm{Lip}\,\alpha & \text{for} \quad 0 < \alpha < 1 \\ \mathrm{W} & \text{for} \quad \alpha = 1. \end{cases}$$

It is to be noted that the very elegant proof by S. N. BERNSTEIN makes use of the well-known 'inequality of Bernstein' valid for any $t_n \in \mathsf{T}_n$:

$$\|t_n^{(r)}\| \leqslant n^r \|t_n\|. \tag{2.5}$$

The latter theorems are converses of each other only for $0<\alpha<1$. Indeed, if $f\in \mathrm{Lip}\,1$, then $\mathrm{E}_n[f]=O(n^{-1})$, but not conversely as the particular function $f(x)=\sum_{k=1}^{\infty} k^{-2}\sin kx$ shows. However, in the case $\alpha=1$ we have (see [19])

THEOREM (ZYGMUND). *Let* $f\in \mathsf{C}_{2\pi}$ *and* $r\in\mathbb{N}$. *Then* $\mathrm{E}_n[f]=O(n^{-r-1})\Leftrightarrow f^{(r)}\in \mathrm{Lip}^*\,1$. *More generally, for* $0<\alpha<2$, $\mathrm{E}_n[f]=O(n^{-r-\alpha})\Leftrightarrow f^{(r)}\in\mathrm{Lip}^*\,\alpha$.

It is easy to see that

$$\begin{array}{ll}\mathrm{Lip}\,1\subset \mathrm{Lip}^*\,1\subset \mathrm{W}\subset \mathrm{Lip}\,\alpha & (0<\alpha<1),\\ \mathrm{Lip}^*\,\alpha=\mathrm{Lip}\,\alpha & (0<\alpha<1).\end{array}$$

THEOREM (ZAMANSKY). *Let* $f\in\mathsf{C}_{2\pi}$ *and* $t_n\in\mathsf{T}_n$ *such that*

$$\|f-t_n\|<n^{1-l}\varphi(n)\qquad (n,l\in\mathbb{N},\ l\geqslant 1),$$

where $\varphi(x)$ *is a positive strictly increasing or decreasing continuous function of* x. *Then the lth derivative of* t_n *satisfies the inequality*

$$\|t_n^{(l)}\|<A_1+A_2\,n\,\varphi(n)+A_3\int_1^n \varphi(x)\,dx\,,$$

where A_1, A_2 *and* A_3 *are constants.*

For a proof, see M. ZAMANSKY [17, p. 26].

COROLLARY 2.1. *If* $f\in\mathsf{C}_{2\pi}$ *is such that* $\|f-t_n\|=O(n^{-\beta})$ *for* $\beta>0$, *then* $\|t_n^{(l)}\|=O(n^{l-\beta})$ *for* $l\in\mathbb{N}$ *with* $\beta<l$.

THEOREM (STEČKIN). *Let* $f\in\mathsf{C}_{2\pi}$ *and* $t_n\in\mathsf{T}_n$ *such that*

$$\|f-t_n\|\leqslant F_{n+1}\qquad (n\in\mathbb{N}),$$

where F_{n+1} *is non-increasing with respect to n and* $\sum_{n=1}^{\infty} n^{k-1}F_n<\infty$ *for some* $k\in\mathbb{N}$. *Then the kth derivative of f exists as an element of* $\mathsf{C}_{2\pi}$ *and satisfies*

$$\|f^{(k)}-t_n^{(k)}\|\leqslant A_4\left\{n^kF_{n+1}+\sum_{\nu=n+1}^{\infty}\nu^{k-1}F_\nu\right\}\qquad (n\in\mathbb{N}),$$

A_4 *being a constant.*

For a proof see S. B. STEČKIN [14, p. 236].

COROLLARY 2.2. *If for* $f\in\mathsf{C}_{2\pi}$ *holds* $\mathrm{E}_n[f]=O(n^{-\gamma})$, *where* $\gamma>0$, *then* $f^{(k)}\in\mathsf{C}_{2\pi}$ *and* $\|f^{(k)}-t_n^{*(k)}(f)\|=O(n^{k-\gamma})$ *for* $k<\gamma$ *and* $k\in\mathbb{N}$.

This statement may also be derived by a theorem of J. CZIPSZER – G. FREUD [7] and its generalization by A. L. GARKAVI [9].

We next state the converse to the corollary of the Zamansky theorem in case $t_n(x)$ is replaced by the polynomial $t_n^*(f; x)$ of best approximation to $f \in C_{2\pi}$.

THEOREM 2.1. *If $f \in C_{2\pi}$ is such that*

$$\|t_n^{*(l)}(f)\| = O(n^{l-\beta}) \qquad (0 < \beta < l,\ l \in \mathbb{N})$$

then

$$\|f - t_n^*(f)\| = O(n^{-\beta}).$$

This theorem was first established by P. L. BUTZER – S. PAWELKE [4] for functions f belonging to the Hilbert space $L_{2\pi}^2$ as a special instance of a general theorem giving necessary and sufficient conditions upon trigonometric approximation processes such that they possess the property of the corollary to the Zamansky theorem and its converse.[2]) In the case of functions f belonging to $C_{2\pi}$ or $L_{2\pi}^p$, $1 \leqslant p < \infty$, we refer the reader to the interesting direct proof by G. SUNOUCHI [15] which rests upon the Jackson inequality (2.4) applied to $t_{2^n}^*(f; x)$. We remark further that H. BERENS [1] has proved a counterpart of the Zamansky result and its converse for holomorphic semi-groups of bounded linear operators (see Sec. 4).

We now come to the first fundamental theorem which links the results of Zamansky and its converse and of Stečkin with those of Jackson and Bernstein.

THEOREM 2.2. *Let $f \in C_{2\pi}$ and let $t_n^*(f)$ be the polynomial of best approximation to f. The following assertions are equivalent to another for $0 < k < r + \alpha < l$ and $0 < \alpha < 2$ (r, k and $l \in \mathbb{N}$):*

a) $$\|f - t_n^*(f)\| = O(n^{-r-\alpha});$$

b) $$f^{(k)} \in C_{2\pi} \text{ and } \|f^{(k)} - t_n^{*(k)}(f)\| = O(n^{k-r-\alpha});$$

c) $$\|t_n^{*(l)}(f)\| = O(n^{l-r-\alpha});$$

d) $$f^{(r)} \in \mathrm{Lip}^*\alpha.$$

Proof. The implication a) ⇒ b) follows by Corollary 2.2 (with $\gamma = r + \alpha$). The fact that b) ⇒ c) follows immediately by

LEMMA 2.1. *Let $f^{(k)} \in C_{2\pi}$. If $\|f^{(k)} - t_n^{(k)}\| = O(n^{k-r-\alpha})$ for $0 < k < r + \alpha$, then $\|t_n^{(l)}\| = O(n^{l-r-\alpha})$ for $l \in \mathbb{N}$ with $r + \alpha < l$.*

Indeed, apply Corollary 2.1 to the $(l-k)$th derivative of $t_n^{(k)}$, giving $\|[t_n^{(k)}]^{(l-k)}\| = O(n^{l-k+(k-r-\alpha)})$ for $0 < r + \alpha - k < l - k$.

Furthermore, the implication c) ⇒ a) is a consequence of Theorem 2.1 (with $\beta = r + \alpha$). Finally, the equivalence a) ⇔ d) is asserted by the theorems of Jackson, Bernstein and Zygmund. The proof of the theorem is thereby complete.

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2) Note that Theorem 2.1 does not hold for arbitrary $t_n \in T_n$; cf. [4, p. 174f.].

The next problem is to see whether Theorem 2.2 may also be formulated for particular trigonometric approximation processes. For a general answer to this question we refer to Section 4; here we consider the typical means.

Denoting the nth partial sum of the Fourier series of $f\in C_{2\pi}$ by $S_n(f;x)$, thus

$$S_n(f;x)=\sum_{\nu=-n}^{n} f^\wedge(\nu)\,e^{i\nu x},\quad f^\wedge(\nu)=(1/2\pi)\int_{-\pi}^{\pi} f(x)\,e^{-i\nu x}\,dx,$$

their typical means are defined by

$$R_{n,l}(f;x)=\sum_{\nu=-n}^{n}\{1-(|\nu|/n+1)^l\}\,f^\wedge(\nu)\,e^{i\nu x}\qquad (l>0;\ n=1,2,\ldots).$$

In case $l=1$ these are the Fejér means. Denoting the conjugate function of f by $f^\sim$, we also set

$$f^{\{r\}}(x)=\begin{cases} f^{(r)}(x), & r \text{ even}\\ (f^\sim)^{(r)}(x), & r \text{ odd}\end{cases},\qquad f^{\{0\}}(x)=f(x).$$

THEOREM 2.3. *If* $f\in C_{2\pi}$, *the following assertions are equivalent for* $0<k<r+\alpha<l$ *and* $0<\alpha<2$ (r, k *and* $l\in\mathbb{N}$):

a) $$\|f-R_{n,l}(f)\|=O(n^{-r-\alpha});$$

b) $$f^{(k)}\in C_{2\pi}\text{ and }\|f^{(k)}-R_{n,l}^{(k)}(f)\|=O(n^{k-r-\alpha});$$

c) $$\|R_{n,l}^{\{l\}}(f)\|=O(n^{l-r-\alpha});$$

d) $$f^{(r)}\in \mathrm{Lip}^*\alpha.$$

Concerning the proof, the equivalence of the assertions a), c), d) follows by P. L. BUTZER – S. PAWELKE [4] and the implication a) ⇒ b) by the theorem of Stečkin. The fact that b) ⇒ c) may be derived for even l by Lemma 2.1. If l is odd and b) is valid for such l, then it follows readily that b) is also valid for l replaced by $l+1$. Thus again by Lemma 2.1 the assertion c) holds for l replaced by $l+1$. But this is equivalent to d) and therefore to c).

3. The Fundamental Theorems for Approximation and Intermediate Spaces

In this section we treat the problems discussed in Section 2 in the case of best approximation in the setting of the theory of intermediate spaces.

Let X be a Banach space and $P_0=\{0\}\subset P_1\subset\cdots P_n\subset\cdots$ a sequence of subspaces of X. The best approximation of order n to an element $f\in X$ is defined by the quantity

$$E_n[f]=\inf_{p_n\in P_n}\|f-p_n\|\qquad (n\in\mathbb{N}).\tag{3.1}$$

It is useful to introduce the notion of an approximation space Y of X. Such a space is a Banach space with the property[3])

$$\mathsf{P}_n \subset \mathsf{Y} \subset \mathsf{X} \quad (n \in \mathbb{N}). \tag{3.2}$$

With the help of the quantities $\mathrm{E}_n[f]$ we can construct special approximation spaces introduced by J. PEETRE [13].

DEFINITION: *We define by* $\mathsf{X}_{\theta,q}$ *the collection of all elements* $f \in \mathsf{X}$ *for which*

$$\|f\|_{\theta,q} = \|f\| + \left\{ \sum_{n=1}^{\infty} (n^\theta \mathrm{E}_n[f])^q \frac{1}{n} \right\}^{1/q} < \infty \tag{3.3}$$

(θ *real*; $1 \leqslant q < \infty$, *usual modification for* $q = \infty$).

One can show that the $\mathsf{X}_{\theta,q}$ are non-trivial approximation spaces under the norm $\|f\|_{\theta,q}$ for $\theta > 0$. In view of the monotonicity of $\mathrm{E}_n[f]$ and a slight modification of the Cauchy condensation criterium it follows that

$$\|f\| + \left\{ \sum_{n=0}^{\infty} (2^{n\theta} \mathrm{E}_{2^n}[f])^q \right\}^{1/q} \tag{3.4}$$

is an equivalent norm to $\|f\|_{\theta,q}$. In the following we need two assumptions on $\mathrm{E}_n[f]$ (see also (2.2) and (2.3)):

$$\lim_{n \to \infty} \mathrm{E}_n[f] = 0 \quad (f \in \mathsf{X}), \tag{3.5}$$

and to each given $f \in \mathsf{X}$ and $n \in \mathbb{N}$ there exists[4]) an element $p_n^*(f) \in \mathsf{P}_n$ such that

$$\mathrm{E}_n[f] = \|f - p_n^*(f)\|. \tag{3.6}$$

Generalizing the notions of the Jackson and Bernstein inequalities (2.4) and (2.5) of Section 2 we say that a *Jackson-type inequality of order* σ, $\sigma \geq 0$, is satisfied with respect to the approximation space Y of X if

$$\mathrm{E}_n[f] \leqslant C n^{-\sigma} \|f\|_{\mathsf{Y}} \quad (n \in \mathbb{N};\ f \in \mathsf{Y}), \tag{3.7}$$

and an *inequality of Bernstein-type of order* σ, $\sigma \geqslant 0$, if

$$\|p_n\|_{\mathsf{Y}} \leqslant D n^\sigma \|p_n\|^{5)} \quad (n \in \mathbb{N};\ p_n \in \mathsf{P}_n). \tag{3.8}$$

Here, C and D are positive constants depending only upon Y and σ. Finally, we

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3) Here and in the following the symbol '$\subset$' means a continuous embedding.

4) For questions concerning the existence of an element $p_n^*(f) \in P_n$ see e.g. the recent book by I. SINGER: *Cea mai bună approximare in spatii vectoriale normate prin elemente din subspatii vectoriale*, Bucuresti 1967 (386 pp.), and the literature cited there.

5) Norms taken with respect to X are denoted by $\|\cdot\|$, with respect to Y by $\|\cdot\|_{\mathsf{Y}}$.

consider for $0<t<\infty$ and every $f\in X$ the function norm

$$K(t,f;X,Y)=\inf_{f=f_1+f_2,\,f_2\in Y}(\|f_1\|+t\|f_2\|_Y) \tag{3.9}$$

introduced by J. PEETRE [12]. It can be shown that $K(t,f;X,Y)$ is a monotone continuous function in t for every fixed $f\in X$.

DEFINITION: *The collection of all elements* $f\in X$ *with*

$$\|f\|_{\theta,q;K}=\left\{\int_0^\infty [t^{-\theta}K(t,f)]^q\,dt/t\right\}^{1/q}<\infty \tag{3.10}$$

is denoted by $(X,Y)_{\theta,q;K}$ (θ *real*; $1\leqslant q<\infty$, *usual modification for* $q=\infty$).

For $0<\theta<1$, $1\leqslant q<\infty$ and $0\leqslant\theta\leqslant 1$, $q=\infty$ the $(X,Y)_{\theta,q;K}$ are intermediate spaces of X and Y under the norm (3.10), i.e. Banach spaces with the property

$$Y\subset(X,Y)_{\theta,q;K}\subset X.$$

We are now able to prove the following

THEOREM 3.1. *Let* Y *be an approximation space of* X *such that the properties* (3.5) *and* (3.6) *are given with respect to* X *and the properties* (3.7) *and* (3.8) *are given with respect to* Y *with order* σ. *Then the following assertions are equivalent for* $\theta>0$ *and* $1\leqslant q\leqslant\infty$:

i) $$\left\{\sum_{n=0}^\infty (2^{n\theta}E_{2^n}[f])^q\right\}^{1/q}<\infty;$$

ii) $$\left\{\sum_{n=0}^\infty (2^{n(\theta-\sigma)}\|p_{2^n}^*(f)-p_2^{*n-1}(f)\|_Y)^q\right\}^{1/q}<\infty;$$

iii) $$f\in Y,\left\{\sum_{n=0}^\infty (2^{n(\theta-\sigma)}\|p_{2^n}^*(f)-f\|_Y)^q\right\}^{1/q}<\infty\qquad(0\leqslant\sigma<\theta);$$

iv) $$\left\{\sum_{n=0}^\infty (2^{n(\theta-\sigma)}\|p_{2^n}^*(f)\|_Y)^q\right\}^{1/q}<\infty\quad .\qquad(0<\theta<\sigma).$$

Proof: We start with the implication i)$\Leftrightarrow$ii) where no restriction is made upon σ. By the Bernstein-type inequality (3.8) it follows that

$$\|p_{2^{n+1}}^*(f)-p_{2^n}^{*(f)}\|_Y\leqslant D\cdot 2^{(n+1)\sigma}\|p_{2^{n+1}}^*(f)-p_{2^n}^*(f)\|\leqslant 2\cdot 2^\sigma D\,2^{n\sigma}E_{2^n}(f), \tag{3.11}$$

and therefore i)$\Rightarrow$ii). On the other hand, using the Jackson-type inequality (3.7) we have

$$\begin{aligned}E_{2^n}(f)&\leqslant E_{2^n}(f-p_{2^{n+1}}^*(f))+E_{2^n}(p_{2^{n+1}}^*(f)-p_{2^n}^*(f))\\&\leqslant E_{2^{n+1}}(f)+2^{-n\sigma}\|p_{2^{n+1}}^*(f)-p_{2^n}^*(f)\|_Y.\end{aligned}$$

In virtue of (3.5),

$$E_{2^n}(f) \leqslant C\,2^{\sigma} \sum_{k=n}^{\infty} 2^{-k\sigma} \| p^*_{2^k}(f) - p^*_{2^{k-1}}(f) \|_Y , \tag{3.12}$$

whence by an inequality of Minkowski (cf. [10, p. 227]) ii)⇒i).

Now the equivalence of these assertions with iii) and iv) follows rather simply. Indeed, by

$$\| p^*_{2^n}(f) - f \|_Y \leqslant \sum_{k=n}^{\infty} \| p^*_{2^{k+1}}(f) - p^*_{2^k}(f) \|_Y \tag{3.13}$$

and

$$\| p^*_{2^n}(f) \|_Y \leqslant \sum_{k=0}^{n} \| p^*_{2^k}(f) - p^*_{2^{k-1}}(f) \|_Y , \tag{3.14}$$

respectively, where $p^*_{1/2}(f)=0$, and again by the inequality of Minkowski, iii) follows by ii) in case $0 \leqslant \theta < \sigma$ and iv) by ii) in case $0 < \sigma < \theta$ (the inequality (3.13) is valid as the sum converges by ii) if $\sigma < \theta$). The converse directions are obvious.

If we examine the structure of the above proof, we see that the implication i) ⇒ ii) is a direct consequence of the Bernstein-type inequality, while for its converse there is needed the Jackson-type inequality and the classical Bernstein method of representing a function by the limit of its polynomials of best approximation. The remaining non-trivial conclusions ii) ⇒ iii) and ii) ⇒ iv) then follow by the Bernstein-type representation (3.13) and the Zamansky type representation (3.14), respectively.

While this theorem illustrates the function of the classical methods of proof in the abstract setting the next theorem to be formulated presents the above statements in a more convenient form.

THEOREM 3.2. *Let be* Y_1 *and* Y_2 *be two approximation spaces of* X *such that the properties* (3.5) *and* (3.6) *are satisfied with respect to* X *and the properties* (3.7) *and* (3.8) *with respect to* Y_1 *and* Y_2 *with orders* σ_1 *and* σ_2*, respectively. Then the following assertions are equivalent for* $0 \leqslant \sigma_1 < \theta < \sigma_2$ *and* $1 \leqslant q \leqslant \infty$:

a) $$\left\{ \sum_{n=1}^{\infty} (n^{\theta} E_n[f])^q \frac{1}{n} \right\}^{1/q} < \infty ;$$

b) $$f \in Y_1 \text{ and } \left\{ \sum_{n=1}^{\infty} (n^{(\theta-\sigma_1)} \| p^*_n(f) - f \|_{Y_1})^q \frac{1}{n} \right\}^{1/q} < \infty ;$$

c) $$\left\{ \sum_{n=1}^{\infty} (n^{(\theta-\sigma_2)} \| p^*_n(f) \|_{Y_2})^q \frac{1}{n} \right\}^{1/q} < \infty ;$$

d) $$f \in (X, Y_2)_{\theta/\sigma_2, q; K}, \quad \text{i.e.} \quad \|f\|_{\theta/\sigma_2, q; K} < \infty .$$

Proof: Here we need a refinement of the above proof. First of all, we replace *u* by a continuous parameter t by defining

$$p_t^*(f) = \begin{cases} p_n^*(f), & n \leqslant t < n+1 \\ 0, & 0 < t < 1 \end{cases}.$$

Then one can easily verify that a), b) and c), respectively, are equivalent to

a)* $$\left\{ \int_0^\infty (t^\theta \mathrm{E}_t[f])^q \, dt/t \right\}^{1/q} < \infty,$$

b)* $$\left\{ \int_0^\infty (t^{(\theta-\sigma_1)} \| p_t^*(f) - f \|_{Y_1})^q \, dt/t \right\}^{1/q} < \infty,$$

and

c)* $$\left\{ \int_0^\infty (t^{(\theta-\sigma_2)} \| p_t^*(f) \|_{Y_2})^q \, dt/t \right\}^{1/q} < \infty,$$

respectively, so that we need only prove the equivalence of the latter three assertions. Furthermore, as in (3.12) we can derive for $0<t<\infty$

$$\mathrm{E}_t[f] \leqslant 2 C_1 4^{\sigma_1} \sum_{k=0}^\infty (t 2^k)^{-\sigma_1} \| p_{t 2^k}^*(f) - f \|_{Y_1} \tag{3.15}$$

and

$$\mathrm{E}_t[f] \leqslant 2 C_2 4^{\sigma_2} \sum_{k=0}^\infty (t 2^k)^{-\sigma_2} \| p_{t 2^k}^*(f) \|_{Y_2}. \tag{3.16}$$

Using the generalized Minkowski inequality we have by (3.15)

$$\left\{ \int_0^\infty (t^\theta \mathrm{E}_t[f])^q \, dt/t \right\}^{1/q} \leqslant 2 C_1 4^{\sigma_1} \sum_{k=0}^\infty \left\{ \int_0^\infty (t^\theta (t 2^k)^{-\sigma_1} \| p_{t 2^k}^*(f) - f \|_{Y_1})^q \, dt/t \right\}^{1/q}$$

$$= \frac{2 C_1 4^{\sigma_1}}{1 - 2^{-\theta}} \left\{ \int_0^\infty (t^{(\theta-\sigma_1)} \| p_t^*(f) - f \|_{Y_1})^q \, dt/t \right\}^{1/q},$$

and thus b)* implies a)*. In an analogous way we obtain the implication c)* $\Rightarrow$ a)* by (3.16). For the converses, inequalities corresponding to (3.13) and (3.14) combined with the estimate (3.11) give

$$\| p_t^*(f) - f \|_{Y_1} \leqslant 2 D_1 2^{\sigma_1} \sum_{k=0}^\infty (t 2^k)^{\sigma_1} \mathrm{E}_{t 2^k}[f] \tag{3.17}$$

and

$$\| p_t^*(f) \|_{Y_2} \leqslant 2 D_2 2^{\sigma_2} \sum_{k=0}^\infty (t 2^{-k})^{\sigma_2} \mathrm{E}_{t 2^{-k}}[f].$$

The implications a)* $\Rightarrow$ b)* and a)* $\Rightarrow$ c)*, respectively, now follow as in Theorem 3.1. Finally, we remark that the equivalence a) $\Leftrightarrow$ d) has been shown in [13].

The assertion b) in Theorem 3.2 is a characterization of assertion a) of the Stečkin type, while c) is of the Zamansky type. The former one could also be called of reduction type as it reduces the order of approximation.

If we consider the case $\mathsf{X}=\mathsf{C}_{2\pi}$ where P_n is the class of all trigonometric polynomials of order $n-1$ then we obtain by Theorem 3.2

THEOREM 3.3. *Let $f\in\mathsf{C}_{2\pi}$ and k, l and $r\in\mathbb{N}$. The following assertions are equivalent for $1\leqslant q\leqslant\infty$:*

a) $$\left\{\sum_{n=1}^{\infty}(n^{(r+\alpha)}\,\mathrm{E}_n[f])^q\,\frac{1}{n}\right\}^{1/q}<\infty\,;$$

b) $$\left\{\sum_{n=1}^{\infty}[n^{(r+\alpha-k)}\,(\|t_n^*(f)-f\|+\|t_n^{*(k)}(f)-f^{(k)}\|)]^q\,\frac{1}{n}\right\}^{1/q}<\infty\qquad(k<r+\alpha)\,;$$

c) $$\left\{\sum_{n=1}^{\infty}(n^{(r+\alpha-l)}\,\|t_n^{*(l)}(f)\|)^q\,\frac{1}{n}\right\}^{1/q}<\infty\qquad(0<r+\alpha<l)\,;$$

d) $$f^{(r)}\in\mathrm{Lip}(\alpha,2,q;\mathsf{C}_{2\pi})\qquad(0<\alpha<2).$$

Here the equivalences a) $\Leftrightarrow$ b) and a) $\Leftrightarrow$ c) follow by the corresponding equivalences of Theorem 3.2 if we choose $\mathsf{Y}_1=\mathsf{C}_{2\pi}^{(k)}$ and $\mathsf{Y}_2=\mathsf{C}_{2\pi}^{(l)}$. Then all conditions of Theorem 3.2 are satisfied with $\sigma_1=k$, $\sigma_2=l$ and $\theta=r+\alpha$. Concerning assertion b) we remark that the graph norm can be replaced by $\|t_n^{*(k)}(f)-f^{(k)}\|$ (see [5], Kap. 4). This has already been carried out in c) since $r+\alpha-l<0$. The assertion d) is a consequence of

$$(\mathsf{C}_{2\pi},\mathsf{C}_{2\pi}^{(l)})_{\theta/l;\,q;\,\mathrm{K}}=\mathrm{Lip}(\theta,l,q;\mathsf{C}_{2\pi})\qquad(0<\theta<l,\,1\leqslant q\leqslant\infty)\tag{3.18}$$

Here the generalized Lipschitz space Lip $(\theta, l, q;\mathsf{C}_{2\pi})$ is defined as the collection of all elements $f\in\mathsf{C}_{2\pi}$ for which

$$\left\{\int_0^\infty(t^{-\theta}\,\|\Delta_t^l f(\cdot)\|)^q\,dt/t\right\}^{1/q}<\infty\qquad(0<\theta<l,\,1\leqslant q\leqslant\infty),$$

where $\Delta_t^l f(x)=\sum_{m=0}^{l}\binom{l}{m}(-1)^m f(x+mt)$. Furthermore, with $\theta=r+\alpha$ and $0<\alpha<2$, $f\in\mathrm{Lip}\,(\theta,l,q;\mathsf{C}_{2\pi})$ is equivalent to $f^{(r)}\in\mathrm{Lip}\,(\alpha,2,q;\mathsf{C}_{2\pi})$ and to $f^{(r)}\in\mathrm{Lip}(\alpha,1,q;\mathsf{C}_{2\pi})$ if $0<\alpha<1$. In the case $q=\infty$ the spaces Lip $(\alpha,1,\infty;\mathsf{C}_{2\pi})$ and Lip $(\alpha,2,\infty;\mathsf{C}_{2\pi})$ are the classical Lipschitz spaces Lip α and Lip* α, respectively, of Section 2. For all these facts see P. L. BUTZER – H. BERENS [3] and the literature cited there.

As Theorem 3.3 contains Theorem 2.2 for $q=\infty$, Theorem 3.2 is a generalization of the classical results of Jackson, Bernstein, Zamansky and Stečkin in the case of best approximation to approximation and intermediate spaces.

Finally, we wish to stress the fact that further applications of Theorem 3.1 are possible if $X = L^p_{2\pi}$, $1 \leqslant p < \infty$ and $X = L^p(-\infty, \infty)$, $1 \leqslant p < \infty$. In the latter case we use as P_n the class of all integral functions of finite degree $\leqslant n\tau$, where $\tau > 0$ is fixed. Then all assumptions in Theorem 3.1 are fulfilled (cf. A. F. TIMAN [16], Sections 4.8 and 5.1) and we may obtain theorems similar to Theorem 3.2. For applications to the n-dimensional torus see [10a].

4. The Fundamental Theorems for General Approximation Processes

In this section we consider the material of the preceding section for a very general class of approximation processes on Banach spaces. Such a process V is defined by a sequence of bounded and commutative linear operators $\{V_n\}_0^\infty$ on a Banach space X to itself having the approximation property

$$\lim_{n \to \infty} V_n(f) = f \quad (f \in X). \tag{4.1}$$

We also need information which connects the rate of approximation of the process V with certain subspaces of X. Inspired by the methods in Section 3 we here assume that there exists a Banach space $Y \subset X$ such that V satisfies the inequalities

$$\|V_n(f) - f\| \leqslant C_V n^{-\varrho} \|f\|_Y \quad (f \in Y), \tag{4.2}$$

$$V_n(f) \in Y, \quad \|V_n(f)\|_Y \leqslant D_V n^{\varrho} \|f\| \quad (f \in X) \tag{4.3}$$

for a certain $\varrho > 0$, where C_V, D_V only depend on V, Y and ϱ.

We are now able to establish a theorem analogous to Theorem 3.1 for the approximation process V and we shall see that thereby (4.2) and (4.3) play the part of (3.7) and (3.8), respectively, of Section 3.

THEOREM 4.1. *Given an approximation process* V *on* X *and a Banach subspace* Y *of* X *with the properties* (4.2) *and* (4.3). *The following assertions are equivalent for* $0 < \theta < \varrho$, $1 \leqslant q \leqslant \infty$:

i) $$\|f\| + \left\{ \sum_{n=1}^{\infty} (n^\theta \|V_n(f) - f\|)^q \frac{1}{n} \right\}^{1/q} < \infty ;$$

ii) $$\left\{ \sum_{n=0}^{\infty} (2^{n\theta} \|V_{2^n}(f) - V_{2^{n-1}}(f)\|)^q \right\}^{1/q} < \infty ;^{6)}$$

iii) $$\|f\| + \left\{ \sum_{n=0}^{\infty} (2^{n\theta} \|V_{2^n}(f) - f\|)^q \right\}^{1/q} < \infty ;$$

iv) $$\|f\| + \left\{ \sum_{n=0}^{\infty} (2^{n(\theta-\varrho)} \|V_{2^n}(f)\|_Y)^q \right\}^{1/q} < \infty ;$$

6) As before we define $V_{1/2}(f) = 0$ for all $f \in X$.

v) $$\|f\| + \left\{ \sum_{n=1}^{\infty} \left(n^{(\theta-\varrho)} \|V_n(f)\|_Y\right)^q \frac{1}{n} \right\}^{1/q} < \infty ;$$

vi) $$f \in (X, Y)_{\theta/\varrho, q; K}, \quad \text{i.e.} \quad \|f\|_{\theta/\varrho, q; K} < \infty .$$

All quantities i)–v) *define equivalent norms for* $\|f\|_{\theta/\varrho, q; K}$.

Proof: We begin with the proof of the implications ii) ⇒ iii) ⇒ iv) ⇒ ii). For the first one we observe that by (4.1)

$$\|V_{2^n}(f) - f\| \leqslant \sum_{k=n}^{\infty} \| V_{2^{k+1}}(f) - V_{2^k}(f)\| , \tag{4.4}$$

and apply the inequality of Minkowski. Next we use a slight modification of the Zamansky method of proof and have

$$\left.\begin{aligned} \|V_{2^n}(f)\|_Y &\leqslant \sum_{k=0}^{n} \|V_{2^k}(f) - V_{2^{k-1}}(f)\|_Y \\ &\leqslant \sum_{k=0}^{n} \|V_{2^k}(f - V_{2^{k-1}}(f))\|_Y + \|V_{2^{k-1}}(V_{2^k}(f) - f)\|_Y \\ &\leqslant 2 \cdot 2^{\varrho} D_V \left(\|f\| + \sum_{k=0}^{n} 2^{k\varrho} \|V_{2^k}(f) - f\|\right) \end{aligned}\right\} \tag{4.5}$$

using property (4.3). Then we apply the inequality of Minkowski once more. To complete the cyclic argument we use property (4.2) to deduce

$$\left.\begin{aligned} &\|V_{2^n}(f) - V_{2^{n-1}}(f)\| \\ &\quad\leqslant \|V_{2^n}(f) - V_{2^{n-1}}(V_{2^n}(f))\| + \|V_{2^n}(V_{2^{n-1}}(f)) - V_{2^{n-1}}(f)\| \\ &\quad\leqslant 2^{\varrho} C_V \left(2^{-n\varrho} \|V_{2^n}(f)\|_Y + 2^{-(n-1)\varrho} \|V_{2^{n-1}}(f)\|_Y\right). \end{aligned}\right\} \tag{4.6}$$

We now establish the equivalence i) ⇔ v) by proceeding analogously as in Theorem 3.1. We define

$$V_t(f) = \begin{cases} V_n(f), & n \leqslant t < n+1 \\ 0 , & 0 < t < 1 \end{cases} ,$$

and easily verify the equivalence of i) and v) with

i)* $$\left\{ \int_0^{\infty} \left(t^{\theta} \|V_t(f) - f\|\right)^q dt/t \right\}^{1/q} < \infty$$

and

v)* $$\|f\| + \left\{ \int_0^{\infty} \left(t^{(\theta-\varrho)} \|V_t(f)\|_Y\right)^q dt/t \right\}^{1/q} < \infty ,$$

respectively, so that we need only prove i)* ⇔ v)*. Combining the steps (4.4) and (4.6) we now have

$$\|V_t(f)-f\| \leqslant 2\cdot 4^{\varrho} C_V \sum_{k=0}^{\infty} (t2^k)^{-\varrho} \|V_{t2^k}(f)\|_Y,$$

and analogously to (4.5)

$$\|V_t(f)\|_Y \leqslant 2\cdot 2^{\varrho} D_V \sum_{k=0}^{\infty} (t2^{-k})^{\varrho} \|V_{t2^{-k}}(f)-f\|.$$

The rest of the argument is the same as in the corresponding part of Theorem 3.1.

Finally, we show the equivalence with vi). For every representation $f=f_1+f_2$, $f_1\in X$, $f_2\in Y$ we can estimate by (4.1) (using the uniform boundedness principle) and (4.2) the quantity

$$\begin{aligned}\|V_n(f)-f\| &\leqslant \|V_n(f_1)-f_1\| + \|V_n(f_2)-f_2\| \\ &\leqslant M_V\|f_1\| + C_V n^{-\varrho}\|f_2\|_Y\end{aligned}$$

to obtain

$$\|V_n(f)-f\| \leqslant \max(C_V, M_V)\,K(n^{-\varrho}, f; X, Y).$$

On the other hand, we have by $f=(f-V_n(f))+V_n(f)$, $V_n(f)\in Y$

$$K(n^{-\varrho}, f; X, Y) \leqslant \|f-V_n(f)\| + n^{-\varrho}\|V_n(f)\|_Y.$$

These inequalities show that iii) and iv) are together equivalent to

$$\|f\| + \left\{\sum_{n=0}^{\infty} (2^{n\theta} K(2^{-n\varrho}, f; X, Y))^q\right\}^{1/q} < \infty$$

and that i) and v) are equivalent to

$$\|f\| + \left\{\sum_{n=1}^{\infty} (n^{\theta} K(n^{-\varrho}, f; X, Y))^q \frac{1}{n}\right\}^{1/q} < \infty.$$

But by the monotonicity of $K(t,f)$ the latter two assertions are equivalent to $\|f\|_{\theta/\varrho,q;K}$ being finite. This completes the proof.

This theorem contains those results for approximation processes which correspond to the Jackson and Bernstein theorems in the case of best approximation. It also includes the theorem of Zamansky and its converse for approximation processes. Furthermore, we remark that after some small modifications Theorem 4.1 could also be formulated and established for an approximation process generated by a family of operators $\{V_t; 0\leqslant t<\infty\}$ depending upon the continuous parameter t. In particular, if we take a holomorphic semi-group of bounded linear operators $\{T(t); 0\leqslant t<\infty\}$ with infinitesimal generator A, we have (see e. g. P. L. Butzer – H. Berens [3]) $T(t)f-f=\int_0^t T(u)Af\,du$ and $\|A\,T(t)f\|=0(t^{-1}\|f\|)$, $t\to 0+$, so that (4.2) and (4.3) are satisfied with

$$\|T(t)f - f\| \leqslant C_T t \|f\|_{D(A)} \qquad (f \in D(A))$$

and

$$\|T(t)f\|_{D(A)} \leqslant D_T t^{-1} \|f\| \qquad (f \in X),$$

where $\|f\|_{D(A)}$ is the graph norm $\|f\| + \|Af\|$. The theorem corresponding to Theorem 4.1 which can be deduced for such operators has already been shown by H. BERENS [1].

Another large class of approximation processes is that given by the so called 'polynomial' operators, which have the property

$$V_n(f) \in P_n \qquad (n \in \mathbb{N}; f \in X), \tag{4.7}$$

and for which an approximation space Y of X exists such that (3.8) and (4.2) are satisfied with a certain $\varrho > 0$. Since $E_n(f) \leqslant \|V_n(f) - f\|$ for such operators, the inequality (3.7) is satisfied with $\sigma = \varrho$. On account of the inequality

$$\|V_n(f)\|_Y \leqslant D n^\varrho \|V_n(f)\| \leqslant D M_V n^\varrho \|f\| \equiv D_V n^\varrho \|f\|,$$

which follows by (3.8) and (4.1), (4.3) is also satisfied. Thus, if $V_n(f) \in P_n$ and $\sigma = \varrho$, (4.2) may be regarded as a stronger version of the Jackson-type inequality (3.7) while (4.3) is a weaker version of the Bernstein-type inequality (3.8). It follows that Theorem 4.1 is applicable. Under these stronger hypotheses one may also state a reduction theorem of the Stečkin-type.

We have

THEOREM 4.2. *Let there be given an approximation process* V *on* X *with* (4.7) *and two approximation spaces* Y, Y_1 *of* X *such that* (3.8) *and* (4.2) *hold with orders* ϱ *and* ϱ_1, *respectively. The following assertions are equivalent to those of Theorem* 4.1 *for* $0 < \varrho_1 < \theta < \varrho$ *and* $1 \leqslant q \leqslant \infty$:

vii) $$f \in Y_1, \quad \|f\| + \left\{ \sum_{n=0}^{\infty} \left(2^{n(\theta-\varrho_1)} \|V_{2^n}(f) - f\|_{Y_1}\right)^q \right\}^{1/q} < \infty\,;$$

viii) $$f \in Y_1, \quad \|f\| + \left\{ \sum_{n=1}^{\infty} \left(n^{(\theta-\varrho_1)} \|V_n(f) - f\|_{Y_1}\right)^q \frac{1}{n} \right\}^{1/q} < \infty\,.$$

Proof: Proceeding along the lines of the proof of inequality (3.17) we may deduce

$$\|V_t(f) - f\|_{Y_1} \leqslant 2 \cdot 2^{\varrho_1} D_{1,V} \sum_{k=0}^{\infty} (t\,2^k)^{\varrho_1} \|V_{t\,2^k}(f) - f\|\,.$$

If we set $t = 2^n$, this gives the implication iii)⇒vii) by the Minkowski inequality. On the other hand, the usual argument as in Theorem 3.1 yields i)⇒viii). Now to the converses. We have by (4.1) and (3.7) for $0<t<\infty$

$$\begin{aligned} \mathrm{E}_t(f) &\leqslant \sum_{k=0}^{\infty} \mathrm{E}_t\big(V_{t\,2^{k+1}}(f) - V_{t\,2^k}(f)\big) \\ &\leqslant 2\cdot 2^{\varrho_1} C\, t^{-\varrho_1} \sum_{k=0}^{\infty} \| V_{t\,2^k}(f) - f \|_{\mathsf{Y}_1}, \end{aligned}$$

so that $\int_0^\infty (t^\theta \mathrm{E}_t[f])^q\, dt/t$ is finite by viii). Analogously, if we set $t=2^n$, $\sum_{n=0}^{\infty} (2^{n\theta}\mathrm{E}_{2^n}[f])^q$ is finite by vii). But by Theorem 3.2 the latter two assertions are equivalent to assertion vi) of Theorem 4.1, which completes the proof.

Theorem 4.2 is applicable to the approximation process $\{R_{n,l}\}_{n=0}^{\infty}$ of the typical means defined in Section 2 for $\mathsf{C}_{2\pi}$-space. (4.2) is the only condition which still has to be verified. In this respect see A. ZYGMUND [20] and M. ZAMANSKY [18]. Thus, if we take $\mathsf{Y}=\mathsf{C}_{2\pi}^{(l)}$, $\mathsf{Y}_1=\mathsf{C}_{2\pi}^{(k)}$, where k, $l\in\mathbb{N}$, and use (3.18), the generalization of Theorem 2.3 is given by

THEOREM 4.3 *If* $f\in\mathsf{C}_{2\pi}$, *the following statements are equivalent for* $0<k<r+\alpha<l$, $0<\alpha<2$ *and* $1\leqslant q\leqslant\infty$:

a) $$\Big\{\sum_{n=1}^{\infty} (n^{(r+\alpha)} \| R_{n,l}(f) - f \|)^q \frac{1}{n}\Big\}^{1/q} < \infty;$$

b) $$f^{(k)}\in C_{2\pi},\ \Big\{\sum_{n=1}^{\infty} (n^{(r+\alpha-k)} \| R_{n,l}^{(k)}(f) - f^{(k)} \|)^q \frac{1}{n}\Big\}^{1/q} < \infty;$$

c) $$\Big\{\sum_{n=1}^{\infty} (n^{(r+\alpha-l)} \| R_{n,l}^{\{l\}}(f) \|)^q \frac{1}{n}\Big\}^{1/q} < \infty;$$

d) $f^{(r)}\in \mathrm{Lip}(\alpha, 2, q; \mathsf{C}_{2\pi})$.[7]

Theorem 4.3 also holds for $\mathsf{L}_{2\pi}^p$ spaces, $1\leqslant p<\infty$, and for other approximation processes such as the singular integrals of Jackson, Rogosinski-Bernstein etc.

We further remark that the singular integral of de La Vallée Poussin represents an approximation process for which Theorem 4.1 but not Theorem 4.2 is applicable (see [5]). Finally, we mention that there are also approximation processes generated by a family of operators $\{V_t;\, 0<t<\infty\}$ which do not belong to any one of the above classes although Theorem 4.1 is applicable. An example is the process of the typical means for the Fourier-inversion-integral of functions in $\mathsf{L}^p(-\infty,\infty)$, $1\leqslant p\leqslant\infty$, which was recently investigated from our point of view by H. BERENS [2].

7) In the case of the Fejér means ($l=1$) the equivalence of a), b) and d) was first shown by A. FETZER [8], using different methods. These results are employed there to prove the equivalence of the assertions a) and d) of Theorem 3.2 (Theorems of Jackson and Bernstein) without using the K-functional.

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