# On the General Solution of a Functional Equation in the Domain of Distributions

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*To Professor A. M. Ostrowski in honour of the 75th anniversary of his birthday*

**1.** At a conference on functional equations in Zakopane (Poland) in 1967, M. Hosszú posed the following problem: what is the most general solution of the functional equation

$$f(x+y-xy)+f(xy)=f(x)+f(y) \tag{1}$$

in the field of measurable functions? Hosszú solved equation (1) under the assumption of differentiability.

The aim of this paper is to answer the question as originally posed. We will give the most general solution of (1) in the domain of distributions.

Let us introduce the following notations.

$\Delta_k$: the linear space of L. Schwartz test functions of $k$ variables;

$\Delta_k'$: the linear space of distributions on $\Delta_k$;

$D^r = d^r/dt^r$; $D_1^r = \partial^r/\partial x^r$; $D_2^r = \partial^r/\partial y^r$ $(r=1, 2, \ldots)$;

$D^0$: the identity operator;

$\Delta_1(\alpha) = \{h : h \in \Delta_1$ and $D^r h = 0$ for $x=\alpha$, $r=0, 1, 2, \ldots\}$;

$\gamma(\alpha) = \{h : h \in \Delta_1(\alpha)$ and $\alpha \notin \operatorname{supp} h\}$ (i.e. there exists a neighborhood of $\alpha$ in which $h$ vanishes identically);

$\Delta_2^x(\alpha)$: $g \in \Delta_2^x(\alpha)$ iff $g \in \Delta_2$ and $D_1^n D_2^m g = 0$ along the straight line $x=\alpha$ $(n, m=0, 1, 2, \ldots)$;

$\Delta_2^y(\alpha)$: $g \in \Delta_2^y(\alpha)$ iff $g \in \Delta_2$ and $D_1^n D_2^m g = 0$ along the straight line $y=\alpha$ $(n, m=0, 1, 2, \ldots)$;

$C_k^\infty$: the linear space of all infinitely differentiable functions of $k$ variables (with arbitrary support), all derivatives of which are everywhere continuous;

$$I_1, I_2 : (I_1 f)(y) = \int_{-\infty}^{\infty} f(x, y)\,dx, \quad (I_2 f)(x) = \int_{-\infty}^{\infty} f(x, y)\,dy.$$

**2.** We define the following mapping of $\Delta_1$ onto $\Delta_1$ denoted by $T_1$:

$$T_1 = \{\Delta_1 \mapsto \Delta_1 : \psi(t) \mapsto \psi(1-t);\ \psi(t) \in \Delta_1\}.$$

$T_1$ is linear and continuous for the topology of $\Delta_1'$. For distributions we define

$$T_1 = \{\Delta_1' \mapsto \Delta_1' : T_1(V)\cdot\psi = V\cdot T_1(\psi);\ V \in \Delta_1',\ \psi \in \Delta_1\}.$$

*Received Dec. 19, 1967 and in revised form April 22, 1969*

If $V = f(t)$ (i.e. the distribution $V$ is generated by the function $f(t)$), then

$$T_1(f)\cdot\psi = f\cdot T_1(\psi) = \int_{-\infty}^{\infty} f(t)\,\psi(1-t)\,dt = \int_{-\infty}^{+\infty} f(1-t)\,\psi(t)\,dt,$$

which means that $T_1(f)$ is the regular distribution corresponding to the function $t \mapsto f(1-t)$.

Since

$$DT_1(V)\cdot\psi = -\,T_1(V)\cdot D\psi = -\,V\cdot T_1(D\psi) = V\cdot DT_1(\psi) = \\ -\,DV\cdot T_1(\psi) = -\,T_1(DV)\cdot\psi$$

holds for every $\psi \in \Delta_1$, we have

$$DT_1(V) = -\,T_1(DV). \tag{2}$$

A similar transformation denoted by $T_2$ is defined for test functions of two variables in the following way:

$$T_2 = \{\Delta_2 \mapsto \Delta_2 : \varphi(x, y) \mapsto \varphi(1-x, 1-y);\ \varphi(x, y) \in \Delta_2\},$$

and the definition of the transformation $T_2$ for distributions (of two variables) is:

$$T_2 = \{\Delta_2' \mapsto \Delta_2' : T_2(V)\cdot\varphi = V\cdot T_2(\varphi);\ V \in \Delta_2',\ \varphi \in \Delta_2\}.$$

As previously, if $V = f(x, y)$, then

$$T_2(f)(x, y) = f(1-x, 1-y)$$

If $\alpha(x, y) \in C_2^\infty$ and $V \in \Delta_2'$, then

$$T_2(\alpha V) = T_2(\alpha)\,T_2(V), \tag{3}$$

because

$$T_2(\alpha)\,T_2(V)\cdot\varphi = T_2(V)\cdot T_2(\alpha)\,\varphi = V\cdot T_2(T_2(\alpha)\,\varphi) = V\cdot\alpha T_2(\varphi) = \\ = \alpha V T_2(\varphi) = T_2(\alpha V)\cdot\varphi, \quad (\varphi \in \Delta_2).$$

Since $T_2(\alpha) \in C_2^\infty$ the product $T_2(\alpha)\,T_2(V)$ is of course defined.

Furthermore

$$D_i T_2(V) = -\,T_2(D_i V) \quad (i = 1, 2) \tag{4}$$

holds. The proof of (4) is the same as that of (2).

**3.** Let us now consider the following mapping of $\Delta_2^x(0)$ resp. $\Delta_2^y(0)$ into $\Delta_1$ denoted by $p$:

$$p = \left\{\varphi(x, y) \mapsto \int_{-\infty}^{\infty} \varphi\left(u, \frac{t}{u}\right)\frac{du}{|u|} = \psi(t);\ \varphi(x, y) \in \Delta_2^x(0)\right\}$$

resp.

$$p = \left\{\varphi(x, y) \mapsto \int_{-\infty}^{+\infty} \varphi\left(\frac{t}{v}, v\right) \frac{dv}{|v|} = \psi(t);\ \varphi(x, y) \in \Delta_2^y(0)\right\}$$

We shall also use the notation $\psi(t) = p(\varphi)(t)$.

PROPOSITION 1. *$p(\varphi) = \psi \in \Delta_1$ for every $\varphi \in \Delta_2^x(0) \cup \Delta_2^y(0)$.*

*Proof.* The proof is based on the fact that

$$I_k \Phi(x, y) \in \Delta_1 \quad (k = 1, 2) \quad \text{for} \quad \Phi(x, y) \in \Delta_2 .$$

Let us now consider a function e.g. $\varphi(x, y) \in \Delta_2^x(0)$; we show that

$$\frac{1}{|u|} \varphi\left(u, \frac{t}{u}\right) = \Phi(u, t) \in \Delta_2^x(0) \quad (\Phi(0, t) = 0).$$

Since $\varphi(x, y)$ vanishes in infinite order for $x \to 0$,

$$\frac{1}{u} \varphi\left(u, \frac{t}{u}\right)$$

is bounded, and as for positive and negative values of $u$,

$$\frac{1}{u} \varphi\left(u, \frac{t}{u}\right)$$

has a derivative of any order with respect to $u$ which converges to zero for $u \to 0$, $\Phi(u, t)$ has a derivative of any order with respect to $u$ (and also with respect to $t$).

Since the support of $\varphi$ is bounded let

$$\operatorname{supp} \varphi(x, y) \subset (-a \leqq x \leqq a) \times (-a \leqq y \leqq a) \quad (a > 0).$$

If $|u| > a$, $\Phi(u, t) \equiv 0$ for arbitrary $t$; on the other hand if $|t| > a^2$ and at the same time $|u| < a$,

$$\left|\frac{t}{u}\right| > \frac{a^2}{a} = a ,$$

then

$$\Phi(u, t) \equiv \frac{1}{|u|} \varphi\left(u, \frac{t}{u}\right) \equiv 0$$

and hence

$$\operatorname{supp} \Phi(u, t) \subset (-a \leqq u \leqq a) \times (-a^2 \leqq t \leqq a^2).$$

Because of these two facts $\Phi(u, t) \in \Delta_2^x(0)$ and therefore

$$(I_1 \Phi)(u) = \int_{-\infty}^{\infty} \varphi\left(u, \frac{t}{u}\right) \frac{du}{|u|} \in \Delta_1 .$$

The same argument shows that $p(\varphi) \in \Delta_1$ for $\varphi \in \Delta_2^y(0)$.

The mapping $p$ is linear, but it is also continuous with respect to the sequential topology introduced in the space of test functions. As this fact will not be used in our considerations, we will not give the proof here.

**4.** With help of the mapping $p$ we introduce now an operation defined for every distribution $V \in \Delta_1'$ associates with every $V \in \Delta_1'$ a functional $P(V)$ on $\Delta_2^x(0) \cap \Delta_2^y(0)$:

$$P(V)\cdot\varphi = V\cdot p(\varphi), \quad (\varphi \in \Delta_2^x(0) \cap \Delta_2^y(0)).$$

$P(V)$ is a linear (and, as can be proved, also a continuous) functional on the space $\Delta_2^x(0) \cap \Delta_2^y(0)$. For such functionals we define the derivatives similarly as for distributions.

PROPOSITION 2.

$$D_1 P(V)\cdot\varphi = yP(DV)\cdot\varphi \quad \text{and} \quad D_2 P(V)\cdot\varphi = xP(DV)\cdot\varphi \quad (\varphi \in \Delta_2^x(0) \cap \Delta_2^y(0)) \tag{5}$$

*Proof.* It will suffice to prove one of the formulas (5); the other can be proved in the same way.

By the definition of the derivative of a functional over a subspace of $\Delta_2$ we have

$$D_2 P(V)\cdot\varphi = -P(V)\cdot D_2\varphi = -V\cdot p(D_2\varphi), \quad (\varphi \in \Delta_2^x(0) \cap \Delta_2^y(0)).$$

On the other hand we have

$$p(D_2\varphi) = \int_{-\infty}^{\infty} D_2\varphi\left(u, \frac{t}{u}\right)\frac{du}{|u|} = D\int_{-\infty}^{\infty} u\varphi\left(u, \frac{t}{u}\right)\frac{du}{|u|} = Dp(x\varphi(x, y))$$

and thus can write

$$D_2 P(V)\cdot\varphi = -V\cdot p(D_2\varphi) = -V\cdot Dp(x\varphi(x, y)) = DV\cdot p(x\varphi(x, y)) =$$
$$P(DV)\cdot x\varphi(x, y) = xP(DV)\cdot\varphi(x, y). \qquad \text{Q.e.d.}$$

It is important to remark that in the case $V = f(t)$ for every $\varphi(x, y) \in \Delta_2^x(0)$ the following relation holds:

$$P(f)\cdot\varphi = f\cdot p(\varphi) = \int_{-\infty}^{\infty} f(t)\, p(\varphi)\, dt = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} \varphi\left(u, \frac{t}{u}\right)\frac{du}{|u|}\, dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(xy)\,\varphi(x, y)\, dx\, dy.$$

This means that the functional $P(f)$ corresponds to the functional on $\Delta_2^x(0) \cap \Delta_2^y(0)$

generated by $f(xy)$. (Of course the previous argument can be repeated with a test-function of the class $\Delta_2^y(0)$.)

PROPOSITION 3. *Let* $\alpha(t)\in C_1^\infty$, *then*

$$P(\alpha V) = P(\alpha)\, P(V) \qquad (V\in\Delta_1').$$

*Proof.* As

$$\alpha(t)\, p(\varphi) = p(\alpha(xy)\,\varphi(x, y)) \qquad (\varphi\in\Delta_2^x(0)\cup\Delta_2^y(0))$$

it follows that

$$P(\alpha V)\cdot\varphi = \alpha V\cdot p(\varphi) = V\cdot\alpha p(\varphi) = V\cdot p(\alpha(xy)\,\varphi(x, y)) =$$
$$P(V)\cdot\alpha(xy)\,\varphi(x, y) = \alpha(xy)\, P(V)\cdot\varphi = P(\alpha)\, P(V)\,\varphi. \qquad \text{Q.e.d.}$$

The following rule will be useful:

PROPOSITION 4.

$$D_1D_2P(V) = D_2D_1P(V) = P(Dt\, DV). \qquad (V\in\Delta_1') \tag{6}$$

*Proof.* By proposition 3 and (5)

$$D_1D_2P(V) = D_1xP(DV) = P(DV) + xD_1P(DV) = P(DV) + xyP(D^2V)$$
$$= P(DV) + P(t)\, P(D^2V) = P(DV) + P(tD^2V) = P(DV + tD^2V) = P(Dt\, DV).$$

Q.e.d.

**5.** We shall need a further transformation $M$, on $\Delta_1'$ defined as follows:

$$M = T_2PT_1,$$

more detailed:

$$M(V) = T_2\{P[T_1(V)]\}. \qquad (V\in\Delta_1')$$

The following rules hold:

$$D_1M(V) = (1-y)\, M(DV) \quad \text{and} \quad D_2M(V) = (1-x)\, M(DV). \qquad (V\in\Delta_1') \tag{7}$$

In fact, we have by (2), (3), (4) and (5):

$$D_1M(V) = -\, T_2(D_1P(T_1(V))) = -\, T_2(yP(DT_1(V))) = T_2(yP(T_1(DV)))$$
$$= (1-y)\, T_2(P(T_1(DV))) = (1-y)\, M(DV). \qquad \text{Q.e.d.}$$

Also the following simple relation will be useful:

$$D_1D_2M(V) = D_2D_1M(V) = T_2P(Dt\, DT_1(V)). \qquad (V\in\Delta_1') \tag{8}$$

Using (6) and (4), we get (8) by direct verification.

In view of the remarks made in **2.** and **4.** we see that for $V=f(t)$ the mapping $M$ is given by

$$f(t) \mapsto f(1-(1-x)(1-y)) \equiv f(x+y-xy).$$

**6.** The mapping $S$ is a mapping from $\Delta_1' \times \Delta_1'$ into $\Delta_2'$ defined as follows ([1], p. 385)

$$S(U, V) \cdot \varphi = U \cdot I_2 \varphi + V \cdot I_1 \varphi. \quad (U, V \in \Delta_1'; \varphi \in \Delta_2)$$

It is easy to prove

$$D_1 D_2 S(U, V) = D_2 D_1 S(U, V) = 0. \quad (U, V \in \Delta_1') \tag{9}$$

In the special case $U=f(t)$, $V=g(t)$ ($f, g$ are functions which generate distributions) we see directly

$$S(f(t), g(t)) = f(x) + g(y).$$

**7.** Let us now consider instead of (1) the following distributional functional equation:

$$M(F) + P(F) = S(F, F) \tag{10}$$

in which $F$ is an unknown distribution of one variable. Because of our previous remarks on the meaning of the operators $M, P, S$ we recognize that in the case $F=f(t)$, (10) goes over into (1).

The aim of our investigations is to give the most general solution of (10) in the domain of distributions.

The right-hand side of (10) is a distribution (of two variables), i.e. a linear and continuous functional on the space $\Delta_2$, while the left-hand side is only defined on the subspace

$$[\Delta_2^x(0) \cap \Delta_2^y(0)] \cap [\Delta_2^x(1) \cap \Delta_2^y(1)].$$

From that we conclude: if there exists a distribution $F \in \Delta_1'$ which satisfies (10), then for this distribution the functional $M(F)+P(F)$ can be extended uniquely from the space

$$[\Delta_2^x(0) \cap \Delta_2^y(0)] \cap [\Delta_2^x(1) \cap \Delta_2^y(1)]$$

to $\Delta_2$, i.e. to a distribution.

**8.** Let us now assume that (10) has a solution $F$. If we operate with $D_1 D_2$ on both sides of (10), we get after considering (6), (8) and (9):

$$P(Dt\, DF) + T_2(P(Dt\, DT_1(F))) = 0. \tag{11}$$

Introducing the following notations

$$Dt\, DF = U \quad \text{and} \quad Dt\, DT_1(F) = V$$

we can write for (11)

$$P(U) = - T_2 P(V). \tag{12}$$

Because of (4) and (5), partial differentiation of (12) yields

$$D_1 P(U) = yP(DU) = - D_1 T_2(P(V)) = T_2(yP(DV))$$
$$D_2 P(U) = xP(DU) = - D_2 T_2(P(V)) = V_2(xP(DV)).$$

By combining these two equations

$$0 = xT_2(yP(DV)) - yT_2(xP(DV)) = T_2[(1-x)\, yP(DV) - x(1-y)\, P(DV)]$$
$$= T_2[(y-x)\, P(DV)].$$

is obtained, and from this

$$(y-x)\, P(DV) = 0 \tag{13}$$

follows.

**9.** In order to solve the equation (13) we use the following approximation theorem.

PROPOSITION 5. *If $\chi(t) \in \Gamma(0) \cap \Delta_1(1)$ is a given testfunction, then there exists a family of functions $\varphi_\varepsilon(x, y) \in \Delta_2^x(0) \cup \Delta_2^y(0)$ for which*

$$p((x-y)\, \varphi_\varepsilon(x, y)) \overset{\Delta_1}{\to} \chi(t). \quad (\varepsilon \to 0)$$

*Proof.* It is no restriction of generality if we suppose that $\chi(t) \equiv 0$ for $t \leqq 0$, because every function of $\gamma(0)$ can be written in the form $\chi(t) + \psi(t)$ where $\chi(t) \equiv 0$ for $t \leqq 0$ and $\psi(t) \equiv 0$ for $t \geqq 0$ and $\chi(t)$, $\psi(t) \in \gamma(0)$.

Consider e.g. a function $\varphi(x, y) \in \Delta_2^x(0)$ for which $\varphi(x, y) \equiv 0$, if $x \leqq 0$. Then

$$p((x-y)\, \varphi(x, y)) = \int_{-\infty}^{\infty} \left(u - \frac{t}{u}\right) \varphi\left(u, \frac{t}{u}\right) \frac{du}{|u|} = \int_{0}^{\infty} \left(u - \frac{t}{u}\right) \varphi\left(u, \frac{t}{u}\right) \frac{du}{u} =$$
$$= \int_{-\infty}^{+\infty} (e^v - e^{\tau - v})\, \varphi(e^v, e^{\tau - v})\, dv = \int_{-\infty}^{+\infty} e^v \tilde{\varphi}(v, \tau - v)\, dv - \int_{-\infty}^{+\infty} \tilde{\varphi}(v, \tau - v)\, e^{\tau - v}\, dv \tag{14}$$

$(t = e^\tau,\ \varphi(e^\xi, e^\eta) = \tilde{\varphi}(\xi, \eta))$.

Instead of $\chi(t)$ we introduce

$$\omega(t) = \frac{\chi(t)}{t-1}$$

which is, of course, also a $\gamma(0)$ function. Let the interval $0 < \alpha \leqq t \leqq \beta$ contain the support of $\tilde{\omega}(t)$. Then $\tilde{\omega}(e^\tau) = \tilde{\omega}(\tau)$ is a $\Delta_1$-test function whose support lies in the interval $\log\alpha \leqq \tau \leqq \log\beta$. Another test function $\kappa_\varepsilon(t) \in \Delta_1$ may now be defined in the following way:

$$\kappa_\varepsilon(t) = \begin{cases} e^{-(\varepsilon^2/(\varepsilon^2 - \log^2 t))} & \text{for} \quad e^{-\varepsilon} \leqq t \leqq e^{\varepsilon} \\ 0 & \text{elsewhere} \end{cases} .$$

With the help of $\omega(t)$ and $\kappa_\varepsilon(t)$ we construct the function

$$\varphi_\varepsilon(x, y) = \omega(x)\, \kappa_\varepsilon(y).$$

We see at once that $\varphi_\varepsilon(x, y) \in \Delta_2^x(0)$ and $\varphi_\varepsilon(x, y) \equiv 0$ for $x \leqq 0$. Therefore we can write according to (14)

$$p((x-y)\, \varphi_\varepsilon(x, y)) = \int_{-\infty}^{\infty} \left(u - \frac{t}{u}\right) \varphi_\varepsilon\left(u, \frac{t}{u}\right) \frac{du}{u} =$$

$$= \int_{-\infty}^{\infty} e^{v} \tilde{\omega}(v)\, \tilde{\kappa}_\varepsilon(\tau - v)\, dv - e^{\tau} \int_{-\infty}^{\infty} e^{-v} \tilde{\omega}(v)\, \tilde{\kappa}_\varepsilon(\tau - v)\, dv .$$

Let now $\varepsilon \to 0$, then after a well-known theorem ([2] p. 142)

$$\int_{-\infty}^{\infty} e^{v} \tilde{\omega}(v)\, \tilde{\kappa}_\varepsilon(\tau - v)\, dv \overset{\Delta_1}{\to} e^{\tau} \tilde{\omega}(\tau) = t\omega(t)$$

and

$$\int_{-\infty}^{\infty} e^{-v} \tilde{\omega}(v)\, \tilde{\kappa}_\varepsilon(\tau - v)\, dv \overset{\Delta_1}{\to} e^{-\tau} \tilde{\omega}(\tau) = \frac{1}{t}\, \omega(t),$$

and therefore

$$p((x-y)\, \varphi_\varepsilon(x, y)) \overset{\Delta_1}{\to} t\omega(t) - \omega(t) = (t-1)\, \omega(t) = \chi(t).$$

The same argument applies to $\psi(t) \in \gamma(0)$, $\psi(t) \equiv 0$ for $t \geqq 0$. Hence our assertion is valid for every $\gamma(0)$-test function of the type under consideration.

**10.** Returning to equation (13), we assert

PROPOSITION 6. *The support of the distribution $W = DV$ cannot be larger than the set $\{x=0, x=1\}$.*

*Proof.* For an arbitrary $\varphi(x, y) \in \Delta_2^x(0) \cup \Delta_2^y(0)$, we have after (13)

$$(x-y)\, P(W) \cdot \varphi = P(W) \cdot (x-y)\, \varphi(x, y) = W \cdot p[(x-y)\, \varphi(x, y)] = 0 .$$

If $\chi(t) \in \Gamma(0)$ and vanishes in infinite order in $t=1$, then from proposition 5 it follows that

$$W \cdot \chi = 0 .$$

This shows that the support of $W$ is concentrated in $(-\alpha, \alpha) \cup \{1\}$ for any $\alpha$ $(0 \leqq \alpha < 1)$. Therefore supp $W \subset \{0\} \cup \{1\}$.

The proposition just proved can be interpreted as follows: $W$ is a distribution of the form

$$W = R_0 + R_1 \tag{15}$$

where $R_i$ is a distribution concentrated at the point $i$ $(i=0, 1)$.

Now we show that $W=0$. By (15) $W$ has the following form:

$$W = \lambda_0 \delta + \lambda_1 \delta' + \cdots + \mu_0 \delta_1 + \mu_1 \delta_1' + \cdots$$

when $\lambda_0, \lambda_1, \ldots; \mu_0, \mu_1, \ldots$ are constant coefficients.

In order to prove that all these coefficients vanish, we consider a polynomial $B(y)$ of sufficiently high degree that $D^r B(y)=0$ for $y=0$ and $y=1$, $r=0, 1, 2, \ldots, k-1, k+1, \ldots$ and $D^k B(y)=1$ for $y=0$, $D^k B(y)=0$ for $y=1$. Extend $B(y)$ outside the interval $(0, 1)$ in such a way that we obtain a test function $b(y) \in \Delta_1$. Let $a=a(x) \in \gamma(0)$ be another test function such that

$$1^\circ \quad (\operatorname{supp} a \times \operatorname{supp} b) \cap \{y = x\} = \emptyset$$

$$2^\circ \int_{-\infty}^{+\infty} \frac{a(u)}{|u|\, u^{k-1}}\, du \neq 0.$$

For

$$\varphi(x, y) = a(x)\, b(y),$$

$\varphi(x, y) \in \Delta_2^x(0)$ and

$$P(W) \cdot \varphi = \mu_k \int_{-\infty}^{+\infty} \frac{a(u)}{|u|\, u^{k-1}}\, du. \tag{16}$$

But on the other hand, (13) implies $P(W) \cdot \varphi = 0$ for every test function vanishing at the points of the line $y=x$. Therefore from (16) $\mu_k=0$ follows. A similar construction can be made also for every $\lambda_k$ and so the assertion is proved.

**11.** Since $W=DV=0$, $V=K$ (a constant), and by the definition of $V$

$$V = Dt\, DT_1(F) = K.$$

Hence

$$T_1(F) = Kt + L \log|t| + MY(t) + N \; (L, M, N = \text{constants})$$

and therefore

$$F = K(1-t) + L \log|1-t| + MY(1-t) + N \tag{17}$$

$(Y(t)$ is the Heaviside function).

Thus we have proved, if the equation (10) has at all a solution (in the domain of distributions) it can only be of the form (17), which is a locally integrable function. But for distributions generated by locally integrable functions, (10) goes over into (1). Substituting (17) into (1) we see that (17) is a solution of (1) (resp. of (10)) for every value of $x$ and $y$ if and only if $L=0$ and $M=0$. Thus we have proved the following

THEOREM 1. *The most general solution of the distributional functional equation* (10) *is the distribution generated by a function of the form* $F(t)=kt+m$, *where* $k$ *and* $m$ *are arbitrary constants. The same class of functions represents the most general locally integrable solution of* (1).

**12.** The functional equation (1) can be generalized to

$$f(x+y-xy)+g(xy)=h(x)+k(y), \tag{18}$$

where $f$, $g$, $h$ and $k$ are unknown functions. (18) corresponds to the distributional functional equation

$$M(F)+P(G)=S(H, K) \tag{19}$$

in which $F$, $G$, $H$, $K$ are unknown distributions of one variable.

If we assume that there exist distributions satisfying (19), we obtain by the same method as in 7. the following relation:

$$P(Dt\, DF)+T_2(P(Dt\, DT_1(G)))=0,$$

and introducing the notations

$$Dt\, DF=U \quad \text{and} \quad Dt\, DT_1(G)=V$$

we obtain equation (12) which leads to (13). Therefore $V=C$ (a constant), hence $G$ is a distribution generated by a function of the form

$$G=\beta_1(1-t)+\beta_2 \log|1-t|+\beta_3 Y(1-t)+\beta_4 \tag{20}$$

($\beta_i$ are constants).

Since $V$ is a constant, the relation (12) implies

$$P(U)=L \quad (L=\text{constant}).$$

By (5) we deduce

$$yP(DU)=0 \quad \text{and} \quad xP(DU)=0$$

whence

$$(y-x)\, P(DU)=0$$

which yields as in 10. $DU=0$, i.e. $U=M$ (=constant), and from the definition of $U$

we obtain

$$F = \alpha_1 t + \alpha_2 \log|t| + \alpha_3 Y(t) + \alpha_4 \tag{21}$$

($\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constants).

Let us substitute (20) and (21) in the left-hand side of (18) (as for distributions generated by locally integrable functions the equation (19) goes over into (18)), we get[1])

$$\alpha_1(x + y - xy) + \alpha_2 \log|x + y - xy| + \alpha_4 + \beta_1(1 - xy) + \beta_2 \log|xy| + \beta_3, \tag{22}$$

The partial derivative of (22) with respect to $y$ is

$$\alpha_1(1 - x) + \alpha_2 \frac{1 - x}{x + y - xy} - \beta_1 x + \beta_2 \frac{1}{y}.$$

We see that the above expression depends only on $x$ if and only if $\alpha_2 = \beta_2 = 0$ and therefore (22) reduces to

$$\alpha_1(x + y) - (\alpha_1 + \beta_1) xy + \varrho,$$

which can be written in the form $h(x) + k(y)$ if and only if $\alpha_1 + \beta_1 = 0$. So we proved the

THEOREM 2. *The general solution of* (18) *resp.* (19) *in the domain of distributions consists of distributions generated by the following type of functions*

$$f(t) = at + m, \quad g(t) = at + l, \quad h(t) = at + r, \quad k(t) = at + s$$

*if* $m + l = r + s$ *hold.*

The idea used for the solution of (1) and (18) is also suitable to treat the functional equation

$$f(r_1 + r_2 x + r_3 y + r_4 xy) + g(s_1 + s_2 x + s_3 y + s_4 xy) = h(x) + k(y).$$

By operations similar to the $T_2$, $P$ and $M$-transformations the last functional equation can be rewritten into a distributional functional equation which can be solved with a similar method as above. These considerations will appear in a later paper.

## REFERENCES

[1] Fenyő, I., *Über eine Lösungsmethode gewisser Funktionalgleichungen*, Acta Math. Hung. *7*, 383–396 (1957).

[2] Gelfand, I. M. and Schilow, E. G., *Verallgemeinerte Funktionen*, Bd. I (Berlin 1960).

[1]) The term containing $Y(t)$ is omitted, because equation (18) is homogeneous, and $Y$ is not a solution of (18).