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The Lie Bialgebroid of a Poisson–Nijenhuis Manifold

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Abstract. We describe a new class of Lie bialgebroids associated with Poisson–Nijenhuis structures.

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Introduction

Nijenhuis operators have been introduced into the theory of integrable systems in the work of Magri, Gelfand and Dorfman (see the book [4]) and, under the name of hereditary operators, in that of Fuchssteiner and Fokas. Poisson–Nijenhuis structures were defined by Magri and Morosi in 1984 [15] in their study of completely integrable systems. There is a compatibility condition between the Poisson structure and the Nijenhuis structure that is expressed by the vanishing of a rather complicated tensorial expression. In this Letter, we shall prove that this condition can be expressed in a very simple way, using the notion of a Lie bialgebroid [14, 7, 12]. A Lie bialgebroid is a pair of vector bundles in duality, each of which is a Lie algebroid, such that the differential defined by one of them on the exterior algebra of its dual is a derivation of the Schouten bracket. Here we show that a Poisson structure and a Nijenhuis structure constitute a Poisson–Nijenhuis structure if and only if the following condition is satisfied: the cotangent and tangent bundles are a Lie bialgebroid when equipped, respectively, with the bracket of 1-forms defined by the Poisson structure, and with the deformed bracket of vector fields defined by the Nijenhuis structure.

Let me add three 'historical' remarks. This result was first conjectured by Magri during a conversation that we held at the time of the Semestre Symplectique at the Centre Emile Borel. Secondly, the Lie bracket of differential 1-forms on a Poisson manifold, defining the Lie-algebroid structure of its cotangent bundle, was defined by Fuchssteiner in an article of 1982 [6] which is not often cited, though it is certainly one of the first papers to mention this important definition. Thirdly, as A. Weinstein has shown [18, 19], Sophus Lie's book [11] contains a comprehensive theory of Poisson manifolds under the name of function groups, including, among many results, a proof of the contravariant form of the Jacobi identity, a proof of the duality between Lie algebra structures and linear Poisson structures on vector spaces, the notions of distinguished functions (Casimir functions) and polar groups (dual pairs), and the existence of canonical coordinates. Moreover, Carathéodory, in his book [2], proves explicitly the tensorial character of the Poisson bivector and gives a rather complete account of this theory, based on a short article by Lie [10] that appeared even earlier than the famous *Theorie der Transformationsgruppen*, Part II, of 1890.

1. Lie Algebroids, Schouten Brackets and Differentials

It is well known that whenever a vector bundle $\pi\colon A \to M$ is a Lie algebroid (see, e.g., [16, 13]), the following structures are defined:

(i) a differential d on the graded vector space $\Gamma(\Lambda A^*)$ of sections of the exterior algebra bundle ΛA^* of the dual vector bundle of A (by this, we mean that the linear map d is a derivation of degree 1 and of square 0 of the associative, graded commutative algebra $(\Gamma(\Lambda A^*), \wedge)$),

(ii) a graded Lie bracket, called the Schouten bracket, on the graded vector space of sections of the exterior algebra bundle ΛA.

We recall that, by definition, a Gerstenhaber algebra $(A = \oplus_{i\in\mathbf{Z}} A^i, \wedge, [\ ,\ ])$ is an associative, graded commutative algebra, with a graded Lie bracket, with respect to the grading shifted by 1 such that, for each element a in A^i, $[a,\ ]$ is a derivation of degree $i+1$ of the graded commutative algebra $(A = \oplus_{i\in\mathbf{Z}} A^i, \wedge)$.

The Schouten bracket is the unique Gerstenhaber bracket on $\Gamma(\Lambda A)$ extending the Lie bracket, $[\ ,\ ]$, on ΓA defined by the Lie algebroid structure of A, and we denote it by the same symbol.

EXAMPLE 1.1. For any smooth manifold M, the tangent bundle TM, equipped with the Lie bracket of vector fields, is a Lie algebroid (with the identity mapping as anchor). The bracket on $\Gamma(\Lambda(TM))$ is the Schouten bracket of fields of multivectors (whence, the name in the more general situation). The associated differential on $\Gamma(\Lambda(T^*M))$ is the de Rham differential of forms.

EXAMPLE 1.2. Let P be a Poisson structure on a manifold M, i.e., a field of bivectors such that $[P,P]=0$. We denote by the same letter the vector-bundle morphism $P\colon T^*M \to TM$, defined by $\langle\beta, P\alpha\rangle = P(\alpha,\beta)$ for all differential 1-forms α and β. To each Poisson structure P on M there corresponds a Lie algebroid structure [3] on the cotangent bundle T^*M of M, with anchor P, and a Lie bracket, $[\ ,\ ]_P$, satisfying

$$[\mathrm{d}f, \mathrm{d}g]_P = \mathrm{d}\{f,g\}, \tag{1.1}$$

for all functions f and g on M, where $\{\ ,\ \}$ is the Poisson bracket of functions defined by the Poisson structure P. The associated bracket on $\Gamma(\Lambda(T^*M))$ is the Koszul bracket [9] of differential forms, and the associated differential on $\Gamma(\Lambda(TM))$ is the Lichnerowicz–Poisson differential, $\mathrm{d}_P = [P,\ ]$, where $[\ ,\ ]$ denotes the Schouten bracket of fields of multivectors, defined on any smooth manifold. (See [8] and references therein.) Moreover, P is a Lie algebroid morphism from $(T^*M, [\ ,\ ]_P)$ to $(TM, [\ ,\ ])$.

EXAMPLE 1.3. Now let M be a manifold with a Nijenhuis structure, N, i.e., there is a field, N, of (1,1)-tensors on M with vanishing Nijenhuis torsion. Recall that the Nijenhuis torsion of a (1,1)-tensor N is the vector-valued 2-form $T(N)$ defined by

$$T(N)(X,Y) = [NX,NY] - N([NX,Y] + [X,NY]) + N^2[X,Y]. \quad (1.2)$$

It was proved in [8] that a Nijenhuis structure defines a new Lie algebroid structure on TM, with anchor $N\colon TM \to TM$ and bracket $[\ ,\ ]_N$, defined by

$$[X,Y]_N = [NX,Y] + [X,NY] - N[X,Y], \quad (1.3)$$

for all vector fields X and Y. Moreover, N is a Lie algebroid morphism from $(TM, [\ ,\ ]_N)$ to $(TM, [\ ,\ ])$. We shall denote by ${}^tN$ the transpose of N, which is a vector-bundle morphism of T^*M into itself.

The associated differential on $\Gamma(\Lambda(T^*M))$ is

$$\mathrm{d}_N = [i_N, \mathrm{d}], \quad (1.4)$$

where $[\ ,\ ]$ is the graded commutator, d is the de Rham differential, and i_N denotes the derivation of degree 0 defined by

$$(i_N\alpha)(X_1,\ldots,X_p) = \sum_{i=1}^{p} \alpha(X_1,\ldots,NX_i,\ldots,X_p), \quad (1.5)$$

for a differential p-form α.

Our aim is to show that, when the Poisson structure P and the Nijenhuis structure N satisfy the compatibility condition of a Poisson–Nijenhuis structure in the sense of [15] and [8], the Lie algebroid structures thus defined on TM and T^*M constitute in fact a Lie bialgebroid structure in the sense of [14] and that, moreover, the converse holds. We thus obtain a formulation of the compatibility of a Poisson and a Nijenhuis structure that is the simplest that we have found in the literature [1, 17], and had eluded previous attempts to characterize *PN*-structures.

2. Lie Bialgebroids

We now assume that dual vector bundles E and E^* are both Lie algebroids. On $\Gamma(\Lambda E)$ are defined both the Schouten bracket $[\ ,\ ]$ associated with the Lie algebroid structure of E and the differential d_* associated with the Lie algebroid structure of E^*. Dually, on $\Gamma(\Lambda E^*)$ are defined both the Schouten bracket $[\ ,\ ]_*$ associated with the Lie algebroid structure of E^* and the differential d associated with the Lie algebroid structure of E. It is natural to impose, as a compatibility condition for the Lie algebroid structures of E and E^*, that d_* be a derivation of the graded Lie algebra $(\Gamma(\Lambda E), [\ ,\ ])$ In fact, if E and E^* are Lie algebras (a particular case of Lie algebroids, where the base manifold reduces to a point), this condition reduces to Drinfeld's cocycle condition for the pair (E, E^*) to be a Lie bialgebra [5]. In this case, the compatibility condition is known to be self-dual, and this fact is true in the more general situation of Lie algebroids. In fact,

PROPOSITION 2.1. *Let (E, E^*) be a pair of Lie algebroids in duality. Then the differential* d *is a derivation of* $(\Gamma(\Lambda E^*), [\ ,\ ]_*)$ *if and only if the differential* d_* *is a derivation of* $(\Gamma(\Lambda E), [\ ,\ ])$.

This proposition justifies the following definition, which we proposed in [7] where we also proved that it is equivalent to the original definition of Mackenzie and Xu [14].

DEFINITION 2.2. *A Lie bialgebroid is a pair (E, E^*) of Lie algebroids in duality such that the differential* d_* *is a derivation of* $(\Gamma(\Lambda E), [\ ,\ ])$.

Mackenzie and Xu [14] have shown that the Lie bialgebroids are the infinitesimal objects of Poisson groupoids.

EXAMPLE 2.3. If (M, P) is a Poisson manifold, then (TM, T^*M), where TM is equipped with the Lie bracket and T^*M is equipped with the Lie algebroid bracket $[\ ,\ ]_P$ (Example 1.2) is a Lie bialgebroid. In fact, we know that the differential d_*, which we have denoted d_P above, is equal to $[P,\ ]$ and therefore is a derivation of the Schouten bracket of fields of multivectors. It is also well known [9] that d is a derivation of the bracket $[\ ,\ ]_P$.

3. Another Example of Lie Bialgebroids: PN Manifolds

We shall now show that Poisson–Nijenhuis manifolds constitute another class of examples of Lie bialgebroids. We first recall various formulas from Poisson geometry. Let (M, P) be a Poisson manifold. Then for $f \in C^\infty(M), \alpha \in \Gamma(T^*M)$,

$$[\alpha, f]_P = \mathcal{L}_{P\alpha} f = \langle \mathrm{d}f, P\alpha \rangle = P(\alpha, \mathrm{d}f), \tag{3.1}$$

$$[\alpha, \mathrm{d}f]_P = -[\mathrm{d}\alpha, f]_P + \mathrm{d}[\alpha, f]_P, \tag{3.2}$$

$$[\mathrm{d}\alpha, \mathrm{d}f]_P = \mathrm{d}[\alpha, \mathrm{d}f]_P = -\mathrm{d}[\mathrm{d}\alpha, f]_P. \tag{3.3}$$

As a particular case of either (3.1) or (3.2), we recover

$$[\mathrm{d}f, \mathrm{d}g]_P = \mathrm{d}(P(\mathrm{d}f, \mathrm{d}g)) = \mathrm{d}\{f, g\}. \qquad (1.1)$$

If we now assume that (M, N) is a Nijenhuis manifold, then

$$\mathrm{d}_N f = {}^t N \, \mathrm{d}f, \qquad (3.4)$$

$$\mathrm{d}_N \alpha = i_N \, \mathrm{d}\alpha - \mathrm{d}({}^t N \alpha), \qquad (3.5)$$

$$\mathrm{d}_N(\mathrm{d}f) = \mathrm{d}({}^t N \, \mathrm{d}f). \qquad (3.6)$$

DEFINITION 3.1. A Poisson structure P and a Nijenhuis structure N on a manifold are called compatible if

$$NP = P\,{}^t N \qquad (3.7)$$

and $C(P, N) = 0$, where

$$C(P, N)(\alpha, \beta) = [\alpha, \beta]_{NP} - ([{}^t N\alpha, \beta]_P + [\alpha, {}^t N\beta]_P - {}^t N[\alpha, \beta]_P), \qquad (3.8)$$

for all $\alpha, \beta \in \Gamma(T^*M)$. When M is a manifold equipped with a Poisson structure, P, and a Nijenhuis structure, N, which are compatible, the manifold is called a Poisson–Nijenhuis manifold, or a PN-manifold.

Our main result is the following.

PROPOSITION 3.2. *Let M be a manifold equipped with a Poisson structure, P, and a Nijenhuis structure, N. The following properties are equivalent:*

- (i) *P and N are compatible,*
- (ii) *the Lichnerowicz–Poisson differential d_P is a derivation of the graded Lie bracket $[\,,\,]_N$ on $\Gamma(\Lambda(TM))$,*
- (iii) *the differential d_N is a derivation of the graded Lie bracket $[\,,\,]_P$ on $\Gamma(\Lambda(T^*M))$,*
- (iv) *the vector bundle TM equipped with the Lie algebroid bracket $[\,,\,]_N$ and the vector bundle T^*M equipped with the Lie algebroid bracket $[\,,\,]_P$ constitute a Lie bialgebroid.*

Proof. The equivalence of (ii) and (iii) follows from Proposition 2.1, while the equivalence of (ii) and (iv) is precisely the definition of a Lie bialgebroid. We shall prove that (i) and (iii) are equivalent. For α a differential p-form and β a differential form on M, let us set

$$A(\alpha, \beta) = \mathrm{d}_N[\alpha, \beta]_P - [\mathrm{d}_N\alpha, \beta]_P - (-1)^{p+1}[\alpha, \mathrm{d}_N\beta]_P. \qquad (3.9)$$

We have to prove that A vanishes if and only if P and N are compatible. We shall first consider the case where α and β are both functions, which we denote by f and g. Then, by (3.4) and (3.1),

$$\begin{aligned}A(f,g) &= \mathrm{d}_N[f,g]_P - [\mathrm{d}_N f,g]_P + [f,\mathrm{d}_N g]_P \\ &= -[{}^tN\,\mathrm{d}f,g]_P + [f,{}^tN\,\mathrm{d}g]_P \\ &= -\langle \mathrm{d}g, P\,{}^tN\,\mathrm{d}f\rangle - \langle \mathrm{d}f, P\,{}^tN\,\mathrm{d}g\rangle \\ &= \langle \mathrm{d}f, (NP - P\,{}^tN)\,\mathrm{d}g\rangle.\end{aligned}$$

Thus, the vanishing of A on functions is equivalent to condition (3.7). Let us now compute $A(\mathrm{d}f,g)$ using (3.4), (3.5), and again (3.1), (1.1) and the fact that d is a derivation of $[\ ,\ ]_P$.

$$\begin{aligned}A(\mathrm{d}f,g) &= \mathrm{d}_N[\mathrm{d}f,g]_P - [\mathrm{d}_N\mathrm{d}f,g]_P - [\mathrm{d}f,\mathrm{d}_N g]_P \\ &= {}^tN[\mathrm{d}f,\mathrm{d}g]_P + [\mathrm{d}({}^tN\mathrm{d}f),g]_P - [\mathrm{d}f,{}^tN\,\mathrm{d}g]_P \\ &= {}^tN[\mathrm{d}f,\mathrm{d}g]_P - [{}^tN\,\mathrm{d}f,\mathrm{d}g]_P - [\mathrm{d}f,{}^tN\,\mathrm{d}g]_P + \mathrm{d}[{}^tN\,\mathrm{d}f,g]_P \\ &= C(P,N)(\mathrm{d}f,\mathrm{d}g) - [\mathrm{d}f,\mathrm{d}g]_{NP} + \mathrm{d}\langle \mathrm{d}g, P\,{}^tN\,\mathrm{d}f\rangle \\ &= C(P,N)(\mathrm{d}f,\mathrm{d}g) - \mathrm{d}(\langle \mathrm{d}g, NP\,\mathrm{d}f\rangle - \langle \mathrm{d}g, P\,{}^tN\,\mathrm{d}f\rangle) \\ &= C(P,N)(\mathrm{d}f,\mathrm{d}g) - \mathrm{d}\langle \mathrm{d}g, (NP - P\,{}^tN)\,\mathrm{d}f\rangle \\ &= C(P,N)(\mathrm{d}f,\mathrm{d}g) + \mathrm{d}(A(f,g)).\end{aligned}$$

We now consider A evaluated on exact 1-forms $\mathrm{d}f, \mathrm{d}g$. By (1.1), (3.5) and (3.3), we find that

$$\begin{aligned}A(\mathrm{d}f,\mathrm{d}g) &= \mathrm{d}_N[\mathrm{d}f,\mathrm{d}g]_P - [\mathrm{d}_N\,\mathrm{d}f,\mathrm{d}g]_P - [\mathrm{d}f,\mathrm{d}_N\,\mathrm{d}g]_P \\ &= \mathrm{d}({}^tN[\mathrm{d}f,\mathrm{d}g]_P - [\mathrm{d}{}^tN\,\mathrm{d}f,\mathrm{d}g]_P - [\mathrm{d}f,\mathrm{d}{}^tN\,\mathrm{d}g]_P \\ &= -\mathrm{d}({}^tN[\mathrm{d}f,\mathrm{d}g]_P) + \mathrm{d}[{}^tN\,\mathrm{d}f,\mathrm{d}g]_P + \mathrm{d}[\mathrm{d}f,{}^tN\,\mathrm{d}g]_P \\ &= -\mathrm{d}(C(P,N)(\mathrm{d}f,\mathrm{d}g)) - \mathrm{d}[df,\mathrm{d}g]_{NP} \\ &= -\mathrm{d}(C(P,N)(\mathrm{d}f,\mathrm{d}g)),\end{aligned}$$

since $[\mathrm{d}f,\mathrm{d}g]_{NP} = \mathrm{d}\langle \mathrm{d}g, NP\,\mathrm{d}f\rangle$. To conclude, we need only prove that

$$A(\mathrm{d}f, h\,\mathrm{d}g) = hA(\mathrm{d}f,\mathrm{d}g) + A(\mathrm{d}f,h)\wedge \mathrm{d}g,$$

for all functions f, g and h. This is proved by a direct computation, and we have shown that $A = 0$ if and only if N and P are compatible.

CONSEQUENCES 3.3. Once the compatibility of a Poisson and a Nijenhuis structure has been expressed in terms of a Lie bialgebroid structure, we obtain new proofs of the properties of PN-manifolds to be found in [15, 8, 17].

COROLLARY 3.4. *If the Poisson bivector P and the Nijenhuis tensor N are compatible, then NP is a Poisson bivector.*

Proof. We use Proposition 3.4 of [7] (see also Proposition 3.6 of [14]) which states that there is a Poisson bracket on M, induced by the Lie bialgebroid structure of (TM, T^*M), given by $\langle \mathrm{d}_N f, \mathrm{d}_P g\rangle$, for functions f and g on M. To conclude, we only have to remark that the bracket $\{\ ,\ \}_{NP}$ defined by the bivector NP coincides with this bracket, since

$$\begin{aligned}\langle \mathrm{d}_N f, \mathrm{d}_P g\rangle &= [\mathrm{d}_N f, g]_P = [{}^tN\,\mathrm{d}f, g]_P = \langle \mathrm{d}g, P\,{}^tN\,\mathrm{d}f\rangle \\ &= NP(\mathrm{d}f, \mathrm{d}g) = \{f, g\}_{NP}.\end{aligned}$$

Similarly, as a consequence of Corollary 3.5 in [14] or of formula (6) in [7], we obtain

$$\mathrm{d}_N\{f, g\}_{NP} = [\mathrm{d}_N f, \mathrm{d}_N g]_P,$$

whence

$${}^tN[\mathrm{d}f, \mathrm{d}g]_{NP} = [{}^tN\,\mathrm{d}f,\ {}^tN\,\mathrm{d}g]_P,$$

where $[\ ,\ ]_{NP}$ is the Koszul bracket of differential forms defined by the Poisson bivector NP. This property implies that ${}^tN$ is a Lie algebra morphism, mapping bracket $[\ ,\ ]_{NP}$ of differential 1-forms into bracket $[\ ,\ ]_P$.

Also, from formula (6*) in [7], we obtain

$$\mathrm{d}_P\{f, g\}_{NP} = -[\mathrm{d}_P f, \mathrm{d}_P g]_N,$$

and therefore, using $\mathrm{d}_P f = -P(\mathrm{d}f)$,

$$P[\mathrm{d}f, \mathrm{d}g]_{NP} = [P\,\mathrm{d}f, P\,\mathrm{d}g]_N,$$

which implies that P is a Lie algebra morphism, mapping bracket $[\ ,\ ]_{NP}$ of differential 1-forms to bracket $[\ ,\ ]_N$ of vector fields.

References

1. Beltrán, J. V. and Monterde, J.: Poisson–Nijenhuis structures and the Vinogradov bracket, *Ann. Global Anal. Geom.* **12** (1994), 65–78.
2. Carathéodory, C.: *Variationsrechnung und Partielle Differentialgleichungen Erster Ordnung*, Teubner, Berlin 1935; English translation: *Calculus of Variations and Partial Differential Equations of the First Order*, vol. 1, Holden-Day, 1965, chapter 9.
3. Coste, A., Dazord, P. and Weinstein, A.: Groupoïdes symplectiques, Publ. Univ. Lyon-I, 2-A (1987).
4. Dorfman, I. Ya.: *Dirac Structures and Integrability of Nonlinear Evolution Equations*, Wiley, New York, 1993.
5. Drinfeld, V. G.: Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations, *Soviet Math. Dokl.* **27**(1) (1983), 68–71.
6. Fuchssteiner, B.: The Lie algebra structure of degenerate Hamiltonian and bi-Hamiltonian systems, *Progr. Theoret. Phys.* **68** (1982), 1082–1104.

7. Kosmann-Schwarzbach, Y.: Exact Gerstenhaber algebras and Lie bialgebroids, *Acta Appl. Math.* **41** (1995), 153–165.
8. Kosmann-Schwarzbach Y. and Magri, F.: Poisson–Nijenhuis structures, *Ann. Inst. Henri Poincaré, Phys. Théor.* **53** (1990), 35–81.
9. Koszul, J.-L.: Crochet de Schouten–Nijenhuis et cohomologie, in: *Elie Cartan et les mathématiques d'aujourd'hui*, Astérisque, hors série, Soc. Math. France, 1985.
10. Lie, S.: Zur Theorie der Transformationsgruppen, *Christ. Forh. Aar.*, No. 13 (1888), reprinted in *Ges. Abh., V*, XXIII, 553–557.
11. Lie, S.: *Theorie der Transformationsgruppen*, Zweitner Abschnitt, Teubner, Leipzig 1890.
12. Liu, Zhang-Ju, Weinstein, A. and Xu, Ping: Manin triples for Lie bialgebroids, Preprint, 1995.
13. Mackenzie, K.: *Lie Groupoids and Lie Algebroids in Differential Geometry*, London Math. Soc. Lecture Notes Series 124, Cambridge Univ. Press, 1987.
14. Mackenzie, K. and Xu, Ping: Lie bialgebroids and Poisson groupoids, *Duke Math. J.* **73** (1994), 415–452.
15. Magri, F. and Morosi, C.: A Geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, Quaderno S19, University of Milan, 1984.
16. Pradines, J.: Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, *C.R. Acad. Sci. Paris A* **264** (1967), 245–248.
17. Vaisman, I.: Poisson-Nijenhuis manifolds revisited, *Rend. Sem. Mat. Univ. Politec. Torino* **52**(4) (1994), 377–394.
18. Weinstein, A.: Sophus Lie and symplectic geometry, *Expo. Math.* **1**, 95–96 (1983).
19. Weinstein, A.: The local structure of Poison manifolds, *J. Differential Geom.* **18** (1983), 523–557.