$$I(\tau_0, \mu) \sim \frac{L\alpha^2 k}{2\pi^2\lambda} \frac{u(\mu)}{1 - Ne^{-2k\tau_0}} \frac{e^{-k\tau_0}}{\tau_0}, \quad \tau_0 \gg 1, \quad \lambda < 1. \tag{56}$$

$$I(\tau_0, \mu) \sim \frac{L\alpha^2}{4\pi^2\tau_0^2} u(\mu), \quad \tau_0 \gg 1, \quad \lambda = 1. \tag{57}$$

which were obtained earlier by Sobolev in [3].

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STATISTICAL DESCRIPTION OF RADIATION FIELDS ON THE BASIS OF THE INVARIANCE PRINCIPLE. I. MEAN NUMBER OF SCATTERINGS IN A MEDIUM ILLUMINATED FROM WITHOUT

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A new approach to the determination of the mean number of scatterings is proposed on the basis of Ambartsumyan's invariance principle and systematic use of the method of generating functions. The average quantities found in the paper refer to the case when the medium is illuminated from without. Photons that perish in the medium during diffusion and photons that escape from the medium are considered separately. It is shown that the approach can yield the dependence of the mean number of scatterings on the characteristics of the original photon and can be used under very general assumptions about the elementary scattering event. The case of complete frequency redistribution with allowance for absorption in the continuum is studied in detail as an illustration. The ideas developed in the paper can in principle be used to determine any of the other discrete random variables giving a statistical description of a radiation field.

1. Introduction

The main problem that is usually posed in a study of photon diffusion in a medium is that of determining the radiation intensity at each point of the medium as a function of the frequency, direction, and other characteristics of the radiation. But for many reasons, quantities that give a statistical description of the scattering process are of no little interest. In our opinion, the importance of such a description is due in

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the first place to the fact that to a large degree it facilitates better understanding of the physical essence of a number of effects predicted by the mathematical solution of the problem. On the other hand, the statistical investigation of multiple scattering makes it possible to determine a number of important physical characteristics of the medium such as the mean radiation density, the mean degree of excitation of the atoms, and so forth. The theoretical significance of such an investigation is also considerable. Note also that the problem of finding the radiative regime in a medium can ultimately also be regarded as a stochastic problem requiring the determination of the statistical mean of some random variable.

Among the various quantities that give a statistical description of a radiation field, most attention in the literature has been devoted to the determination of the number of scatterings undergone by photons diffusing in a medium. Pioneering here was Ambartsumyan's work [1], in which he proposed for the mean number of scatterings per photon in some beam the formula

$$N = \lambda \partial \ln I / \partial \lambda, \tag{1}$$

where I is the radiation intensity, and λ is the probability of reemission of a photon in an elementary interaction event with atoms of the medium. The mean number of photon scatterings was subsequently estimated by many authors for different special cases, though the general treatment of the problem was given by Sobolev in a series of papers [2-5]. In particular, so far as we know, it was in these papers that the mean number of scatterings was calculated separately for photons that escape as a result of diffusion outward and photons that "perish" (i.e., undergo true absorption) in the medium during diffusion. We note here that the expression (1) is valid for estimating the mean number of scatterings only for moving photons (and not ones that have perished), so that (1) will apply to the group of photons that leave the medium.

For some cases (coherent scattering, completely incoherent scattering), Sobolev [2-5] obtained simple relations that make it possible to determine the mean number of scatterings undergone by the photons that perish in the medium as well as by all photons irrespective of their subsequent "fate." However, these relations, like the physical arguments which provide their basis, cease to hold when allowance is made for absorption and emission in the continuum. Nor has there yet been a comprehensive study of the more complicated cases when the scattering is anisotropic or subject to general laws of redistribution with respect to the frequency and direction (a first such attempt was made in the recent paper [6] of Arutyunyan and the present author). Very important too is the statistical description of scattering in its dependence on the initial characteristics of the photon, for example, the frequency, direction of motion, etc. These questions are the subject of detailed discussion in the present series of papers. But this is not the most important thing. Our guiding principle in the series is the development of a general approach to the determination of various quantities valid under fairly general assumptions about the individual scattering event, the distribution of the primary energy sources, and the geometry of the medium. Our approach is based on Ambartsumyan's invariance principle and systematic use of the method of generating and characteristic functions (according as the particular random variable is discrete or continuous), which, as is well known [7, 8], is a powerful tool in the study of probabilistic processes. In such an approach, using simple and standard procedures, we can obtain for the quantities in which we are interested equations whose study is particularly important in complicated cases in which the problem of finding the radiation field itself can be solved only numerically.

The first two papers of the series are devoted to determining the mean number of scatterings. In what follows, we shall consider a quantity that is of interest from the point of view of applications — the mean time a photon remains in the medium. The finding of this time is a separate problem, and it is only in the case of coherent scattering that it essentially reduces to determination of the mean number of scatterings. It is also intended to generalize the results to a medium of finite optical thickness. Calculations will be made that show the extent to which the statistical mean values are influenced by different redistribution laws.

2. Auxiliary Equations

We introduce the quantities needed in the following exposition. We shall be considering the fairly general case for which the medium is three dimensional and the scattering is accompanied by a redistribution with respect to the frequencies and directions.

Suppose a photon of dimensionless frequency x is infinite on a semi-infinite plane-parallel medium at angle $\cos^{-1}\eta$. We denote by $\eta'\rho(x', \eta'; x, \eta)dx'd\eta'$ the probability that as a result of multiple scatterings a photon with frequency in the interval $(x', x' + dx')$ leaves the medium in the direction η' in the solid angle $2\pi d\eta'$. We denote the analogous reflection probability, but for a photon that has undergone a definite number n of scatterings, by $\eta'\rho_n dx'd\eta'$.

Using the invariance principle for the function ρ, which is usually called the reflection function, we obtain

$$\frac{2}{\lambda}[v(x)\eta' + v(x')\eta]\rho(x', \eta'; x, \eta) = r(x', -\eta'; x, \eta) +$$
$$\eta\int_0^1 d\eta'' \int_{-\infty}^{\infty} r(x', \eta'; x'', \eta'')\rho(x'', \eta''; x, \eta)\,dx'' + \eta'\int_0^1 d\eta'' \int_{-\infty}^{\infty} \rho(x', \eta'; x'', \eta'')r(x'', \eta''; x, \eta)\,dx'' +$$
$$\eta\eta'\int_0^1 d\eta'' \int_{-\infty}^{\infty} \rho(x', \eta'; x'', \eta'')\,dx'' \int_0^1 d\eta''' \int_{-\infty}^{\infty} r(x'', \eta''; x''', -\eta''')\rho(x''', \eta'''; x, \eta)\,dx''', \tag{2}$$

where $v(x) = \alpha(x) + \beta$, $\alpha(x)$ is the absorption coefficient profile, β is the ratio of the continuum absorption coefficient to the line-center absorption coefficient, and, finally, $r(x', \eta'; x, \eta)$ is the frequency and direction redistribution function averaged over the azimuth. In all cases of astrophysical interest, the function r can be expressed as a bilinear expansion (see [9, 10]). Thus, if the frequency and direction redistribution effects are due solely to the thermal motion of the atoms, then

$$r(x', x, \gamma) = \frac{1}{\sqrt{\pi}\,\sin\gamma}\exp[-(x^2 + x'^2 - 2xx'\cos\gamma)/\sin^2\gamma] =$$
$$\sum_{k=0}^{\infty}\cos^k\gamma\,\alpha_k(x)\,\alpha_k(x'), \tag{3}$$

where γ is the scattering angle, $\alpha_k(x) = (\pi^{1/4}2^{k/2}\sqrt{k!})^{-1}\exp(-x^2)H_k(x)$, and $H_k(x)$ is the Hermite polynomial of degree k.

As was shown in [9], the redistribution function averaged over the azimuth is given in this case by

$$r(x', \eta'; x, \eta) = \frac{1}{2\pi}\int_0^{2\pi} r(x', x, \gamma)\,d\varphi = \sum_{i=0}^{\infty} r_i(x', x)\,P_i(\eta')\,P_i(\eta), \tag{4}$$

where $P_i(\eta)$ is the Legendre polynomial of degree i and

$$r_i(x', x) = \sum_{k=i}^{\infty} c_k^i\,\alpha_k(x')\,\alpha_k(x), \tag{5}$$

and $c_k^i = 0$ if k + i is odd and $c_k^i = (2i + 1)k!/(k - i)!!(k + i + 1)!!$ if k + i is even.

If we now use the expansions (4) and (5), then from (2) we obtain

$$[v(x)\eta' + v(x')\eta]\rho(x', \eta'; x, \eta) = \frac{\lambda}{2}\sum_{i=0}^{\infty}(-1)^i\sum_{k=i}^{\infty} c_k^i\,\varphi_{ik}(x', \eta')\,\varphi_{ik}(x, \eta), \tag{6}$$

where the functions $\varphi_{ik}(x, \eta)$, which are the analogs of Ambartsumyan's well-known φ function in the general theory of incoherent scattering, are determined from the system of functional equations

$$\varphi_{ik}(x, \eta) = P_i(\eta)\,\alpha_k(x) +$$

$$\frac{\lambda}{2}\eta\sum_{j=0}^{\infty}(-1)^{i+j}\sum_{m=j}^{\infty}c_m^i\int_0^1 P_i(\eta')\,d\eta'\int_{-\infty}^{\infty}\frac{\varphi_{jm}(x,\eta)\varphi_{jm}(x',\eta')}{v(x)\eta'+v(x')\eta}\alpha_k(x')\,dx'. \tag{7}$$

It should be noted that these, like the other relations given in the present section, can, although applicable to the specific case of a purely Doppler law of redistribution with respect to the frequencies and directions, still be used for other redistribution laws after some slight modifications that are not of a fundamental nature. On the other hand, these relations embrace a fairly large class of problems, since they readily permit transition to different special cases corresponding to simpler scattering mechanisms. Indeed, in many practical applications the function $r(x', \eta'; x, \eta)$ can be represented in the form

$$r(x', \eta'; x, \eta) = p^0(\eta', \eta)\, r(x', x), \tag{8}$$

where $r(x', x)$ is understood as the redistribution function averaged over the directions and $p^0(\eta', \eta)$ as the phase function averaged over the azimuth. Then, for example, to go over to the case of anisotropic coherent scattering it is sufficient to use the relation (8) and set $r(x', x) = \alpha(x)\delta(x - x')$ in the equations obtained below.

It is well known that

$$p^0(\eta', \eta) = \sum_{i=0}^{\infty}x_i P_i(\eta')P_i(\eta),$$

where x_i are the coefficients of the expansion of the phase function in Legendre polynomials. As is shown in [9-11], the frequency redistribution function averaged over the directions also admits a bilinear expansion.

Isotropic scattering in the approximation of complete frequency redistribution will be the case most frequently encountered in what follows. Then instead of (8) we have $r(x', \eta'; x, \eta) \equiv \alpha(x')\alpha_0(x)$, and the system (7) degenerates into a single equation for the function $\varphi_0(x, \eta)$:

$$\varphi_0(x,\eta) = \alpha_0(x) + \frac{\lambda}{2}\eta\varphi_0(x,\eta)\int_0^1 d\eta'\int_{-\infty}^{\infty}\frac{\varphi_0(x',\eta')}{\eta' v(x)+\eta v(x')}\alpha_0(x')\,dx'. \tag{9}$$

It is easy to show (see [12]) that the ratio $\varphi_0(x, \eta)/\alpha_0(x)$ depends only on the combination $z = \eta/v(x)$, so that, denoting $H(z) \equiv \varphi_0(x, \eta)/\alpha_0(x)$, we obtain instead of (9)

$$H(z) = 1 + \frac{\lambda}{2}zH(z)\int_0^{1/\beta} G\left(\frac{z'}{1-\beta z'}\right)\frac{H(z')}{z+z'}\,dz', \tag{10}$$

where

$$G(z) = 2A\int_{x(z)}^{\infty}\alpha^2(x')\,dx',\quad A = \pi^{-1/2},$$

$x(z) = 0$ if $z \leqslant 1$ and $x(z)$ is determined from the condition $\alpha(x(z)) = 1/z$, if $z > 1$.

Returning to the reflection function ρ introduced above, we note that on the basis of the probabilistic meaning ascribed to it we can interpret

$$R_*(x,\eta) = \int_0^1 \eta' d\eta'\int_{-\infty}^{\infty}\rho(x',\eta'; x,\eta)\,dx' \tag{11}$$

in two ways (here in all that follows, the asterisk identifies quantities corresponding to the fluxes of photons that emerge outward as a result of scatterings; the suffix 0 will be used for the analogous quantities for photons that perish in the medium). On the one hand, $R_*(x, \eta)$ is the profile of the line produced by illumination of a semi-infinite atmosphere with continuum radiation of unit intensity; on the other, it can be regarded as the probability of reflection from the medium of a photon having on incidence frequency x and moving at angle $\cos^{-1}\eta$ to the normal.

Besides the reflection function, an important function in what follows is

$Y(\tau, x', \eta'; x, \eta)$, which can be interpreted as the probability of emergence from depth τ for a photon that moves in the direction η' and has frequency x'. For the analogous probability calculated for an absorbed photon, we retain the usual literature notation $p(\tau, x', \eta'; x, \eta)$. It is assumed here, as usual, that the optical depth τ, calculated for the central frequency of the line, increases from the boundary into the medium and that the angles are measured from the direction of the outer normal to the surface of the medium.

By the reciprocity principle for optical phenomena, the function Y can also be given a somewhat different probabilistic meaning, namely, $Y(\tau, x, -\eta; x', -\eta')\, dx' d\eta'$ can be regarded as the probability that a photon incident on the medium in the direction $-\eta$ with frequency x intersects as a result of multiple scattering the plane parallel to the boundary of the medium at depth τ moving in the direction $-\eta'$ within the solid angle $2\pi d\eta'$ and having frequency in the interval $(x', x' + dx')$. In the following exposition, we shall, for convenience, use the notation $Y(\tau, x, -\eta; x', -\eta') \equiv \tilde{Y}(\tau, x, \eta; x', \eta')$ and assume that in $\tilde{Y}$ the angles are measured from the direction of the inner normal to the surface of the medium.

Application of the invariance principle leads to the following equation for the function $\tilde{Y}$:

$$\eta \frac{\partial \tilde{Y}(\tau, x, \eta; x', \eta')}{\partial \tau} + v(x) \tilde{Y}(\tau, x, \eta; x', \eta') = \int_0^1 d\eta'' \int_{-\infty}^{\infty} \alpha(x'') p(0, x'', \eta''; x, \eta) \tilde{Y}(\tau, x'', \eta''; x', \eta') dx'', \tag{12}$$

where

$$\frac{2}{\lambda} \alpha(x') p(0, x', \eta'; x, \eta) = r(x', \eta'; x, \eta) + \eta \int_0^1 d\eta'' \int_{-\infty}^{\infty} r(x', \eta'; x'', -\eta'') \rho(x'', \eta''; x, \eta) dx'', \tag{13}$$

As boundary condition we have $\tilde{Y}(0, x, \eta; x', \eta') = \delta(x - x')\delta(\eta - \eta')$, if $\eta' > 0$ and $\tilde{Y}(0, x, \eta; x', \eta') = |\eta'| \rho(x', -\eta'; x, \eta)$, if $\eta' \leqslant 0$. As in the case of the reflection function, we shall, if the event whose probability is characterized by $\tilde{Y}$ occurs after n scatterings append to it the subscript n.

3. Mean Number of Scatterings of a Photon in a Medium Illuminated from Without

We begin our study of the finding of one of the most important statistical characteristics of a radiation field, the mean number of scatterings, by considering the simpler problem in which it is assumed that the medium is illuminated from without.

Let a stream of photons of frequency x be incident on a plane-parallel semi-infinite atmosphere at angle $\cos^{-1} \eta$. We shall consider later the photons that as a result of multiple scattering undergo processes of true absorption and perish in the medium; first, we consider the photons that in the course of diffusion emerge from the medium. More precisely, we shall be interested in only a certain fraction of them, namely, the photons that are diffusely reflected from the medium after a definite number n of scatterings. This fraction, as we recall, is determined by the function ρ_n. Using the invariance principle, to find ρ_n we obtain the equations

$$\frac{2}{\lambda}[v(x)\eta' + v(x')\eta]\rho_1(x', \eta'; x, \eta) = r(x', -\eta'; x, \eta);$$

$$\frac{2}{\lambda}[v(x)\eta' + v(x')\eta]\rho_2(x', \eta'; x, \eta) = \eta \int_0^1 d\eta'' \int_{-\infty}^{\infty} r(x', \eta'; x'', \eta'') \rho_1(x'', \eta''; x, \eta) dx'' + \eta' \int_0^1 d\eta'' \int_{-\infty}^{\infty} \rho_1(x', \eta'; x'', \eta'') r(x'', \eta''; x, \eta) dx'';$$

$$\frac{2}{\lambda}[v(x)\eta' + v(x')\eta]\rho_n(x', \eta'; x, \eta) = \eta\int_0^1 d\eta''\int_{-\infty}^{\infty} r(x', \eta', x'', \eta'')\rho_{n-1}(x'', \eta''; x, \eta)\,dx'' + \eta'\int_0^1 d\eta''\int_{-\infty}^{\infty}\rho_{n-1}(x', \eta'; x''; \eta'')\,r(x'', \eta''; x, \eta)\,dx'' + \eta\eta'\sum_{k=1}^{n-2}\int_0^1 d\eta''\int_{-\infty}^{\infty}\rho_k(x', \eta'; x'', \eta')\times dx''\int_0^1 d\eta'''\int_{-\infty}^{\infty} r(x'', \eta''; x''', -\eta''')\rho_{n-k-1}(x''', \eta'''; x, \eta)\,dx'''. \quad (n > 2) \tag{15}$$

We introduce the generating function

$$W(x', \eta''; x, \eta; s) = \sum_{n=1}^{\infty}\rho_n(x', \eta'; x, \eta)\,s^n,$$

where s is a parameter. Since $\rho_n \geqslant 0$ and $\sum_{n=1}^{\infty}\rho_n = \rho$, the function W is at least defined for s satisfying $|s| \leqslant 1$. For $|s| < 1$, the generating function is infinitely differentiable with respect to s. It is also obvious that $W(x', \eta'; x, \eta; 1) = \rho(x', \eta'; x, \eta)$. Using Eqs. (14), we can readily obtain for the generating function the equation

$$\frac{2}{\lambda s}[v(x)\eta' + v(x')\eta]\,W(x', \eta'; x, \eta; s) = r(x', -\eta'; x, \eta) + \eta\int_0^1 d\eta''\int_{-\infty}^{\infty} r(x', \eta'; x'', \eta'')\,W(x'', \eta''; x, \eta; s)\,dx'' + \eta'\int_0^1 d\eta''\int_{-\infty}^{\infty} W(x', \eta'; x'', \eta''; s)\,r(x'', \eta''; x, \eta)\,dx'' + \eta\eta'\int_0^1 d\eta''\int_{-\infty}^{\infty} W(x', \eta'; x'', \eta''; s)\,dx''\int_0^1 d\eta'''\int_{-\infty}^{\infty} r(x'', \eta''; x''', -\eta''')\,W(x''', \eta'''; x, \eta; s)\,dx'''. \tag{15}$$

For the determination of the mean number of scatterings, an interesting quantity is the function $\nu(x', \eta'; x, \eta) = \partial W(x', \eta'; x', \eta; s)/\partial s|_{s=1}$. Indeed, it follows from the physical meaning of the quantities introduced above that the ratio ν/ρ gives the required mean number of scatterings for photons reflected by the medium in the direction η' within the solid angle $2\pi d\eta'$ and in the interval of frequencies $(x', x' + dx')$ under the condition that the original photon moved at angle $\cos^{-1}\eta$ and had frequency x. Differentiating (15) with respect to s and setting s = 1, we obtain for $\nu(x', \eta'; x, \eta)$ the linear equation

$$\nu(x', \eta'; x, \eta) = \rho(x', \eta'; x, \eta) + [v(x)\eta' + v(x')\eta]^{-1}\times\Big[\eta\int_0^1 d\eta''\int_{-\infty}^{\infty}\alpha(x'')\,p(0, x'', \eta''; x', \eta')\,\nu(x'', \eta''; x, \eta)\,dx'' + \eta'\int_0^1 d\eta''\int_{-\infty}^{\infty}\alpha(x'')\,p(0, x'', \eta''; x, \eta)\,\nu(x'', \eta''; x', \eta')\,dx''\Big]. \tag{16}$$

Since Eq. (15) explicitly contains a product of the parameters s and λ, it is readily concluded that Eq. (16) can also be obtained by differentiating (15) term by term with respect to λ and then multiplying both sides by λ. Therefore, when we are considering diffusely reflected photons (and only then) the two procedures involving differentiation with respect to s or λ are identical. Further, it can be concluded that to find the mean number of scatterings there is no need for a preliminary determination of the intensity of the reflected radiation, as is usually done; formal differentiation with respect to λ makes it possible to obtain a separate equation for the required quantity. What we have said acquires particular importance in complicated problems in which it is impossible to obtain a closed expression for the intensity of the emergent radiation. Thus, in the general case of incoherent scattering, when the redistribution function is represented by a bilinear expansion, we obtain from (16), taking into account (4) and (5),

$$[v(x)\eta' + v(x')\eta]\nu(x', \eta', x, \eta) =$$

$$\frac{\lambda}{2}\sum_{i=0}^{\infty}(-1)^i\sum_{k=i}^{\infty}c_k^i\varphi_{ik}(x, \eta)\varphi_{ik}(x', \eta')[1+f_{ik}(x, \eta)+f_{ik}(x', \eta')], \tag{17}$$

where $f_{ik}(x, \eta) = \lambda\partial\ln\varphi_{ik}(x, \eta)/\partial\lambda$. The determination of $\nu(x', \eta'; x, \eta)$ reduces to solving for $\psi_{ik}(x, \eta) = f_{ik}(x, \eta)\varphi_{ik}(x, \eta)$ the system of linear equations

$$\psi_{ik}(x, \eta) = \varphi_{ik}(x, \eta) - P_i(\eta)\alpha_k(x) +$$

$$\frac{\lambda}{2}\eta\sum_{i=0}^{\infty}(-1)^{i+n}\sum_{m=r}^{\infty}c_m^n\int_0^1 P_i(\eta')\,d\eta'\int_{-\infty}^{\infty}\frac{\varphi_{nm}(x, \eta)\psi_{nm}(x', \eta')+\varphi_{nm}(x', \eta')\psi_{nm}(x, \eta)}{v(x)\eta'+v(x')\eta}\alpha_k(x')\,dx'. \tag{18}$$

In the general case, Eq. (18) can be solved iteratively $\psi_{ik}(x, \eta) = \varphi_{ik}(x, \eta) - P_i(\eta)\alpha_k(x)$ being naturally chosen as the zeroth approximation. It is particularly convenient to construct the function $\nu(x', \eta'; x, \eta)$ at the same time as solving the system of functional equations (7).

In the case of complete frequency redistribution, (17) simplifies appreciably to

$$\nu(x', \eta'; x, \eta)/\rho(x', \eta'; x, \eta) = 1 + f(z) + f(z'), \tag{19}$$

where $f(z) = \lambda\partial\ln\varphi_0(x, \eta)/\partial\lambda = \lambda\partial\ln H(z)/\partial\lambda$, and $H(z)$ is the solution of Eq. (10). A relation analogous to (19) but for coherent scattering and $\beta = 0$ was obtained for the first time by Sobolev in [3]. We see that in the case considered the ratio ν/ρ is a symmetric function with respect to the pairs of arguments x, η and x', η'. At the same time, the ratio can be expressed solely in terms of the function $f(z)$ of a single variable, this function satisfying, as follows from (19), the equation

$$f(z) = H(z) - 1 + z\int_0^{1/\beta} G\left(\frac{z'}{1-\beta z'}\right)\rho(z, z')f(z'')\,dz', \tag{20}$$

where $\rho(z, z') = (\lambda/2)H(z)H(z')/(z+z')$. For $f(z)H(z) = \lambda\partial H(z)/\partial\lambda$ it is also possible to write down a singular equation obtained from the corresponding equation for $H(z)$. However, we give here the explicit expression for $f(z)$:

$$f(z) = \frac{\bar{\lambda}}{2(1-\bar{\lambda})} - \frac{\lambda}{2}\int_0^{1/\beta} F(z', \lambda, \beta)\,G\left(\frac{z'}{1-\beta z'}\right)\frac{z'\,dz'}{z+z'}, \tag{21}$$

where

$$F(z, \lambda, \beta) = \left\{[1+\lambda U(z, \beta)]^2 + \left[\lambda\frac{\pi}{2}G\left(\frac{z'}{1-\beta z'}\right)\right]^2\right\}^{-1}; \quad U(z, \beta) =$$

$$z^2\int_0^{1/2} G\left(\frac{z'}{1-\beta z'}\right)\frac{dz'}{z^2 - z'^2}; \quad \bar{\lambda} = \lambda A\int_{-\infty}^{\infty}\frac{\alpha^2(x)}{v(x)}\,dx.$$

The function $\nu(x', \eta'; x, \eta)$ contains all the information concerning the number of scatterings of photons that leave the medium. However, in practice it is often sufficient to know the quantities

$$N_*(x, \eta) = \frac{\int_0^1 \eta'\,d\eta'\int_{-\infty}^{\infty}\nu(x', \eta'; x, \eta)\,dx'}{\int_0^1 \eta'\,d\eta'\int_{-\infty}^{\infty}\rho(x', \eta'; x, \eta)\,dx'}; \quad \tilde{N}_*(x', \eta) = \frac{\int_0^1 d\eta\int_{-\infty}^{\infty}\nu(x', \eta', x, \eta)\,dx}{\int_0^1 d\eta\int_{-\infty}^{\infty}\rho(x', \eta'; x, \eta)\,dx}.$$

It is readily seen that $N_*(x, \eta)$ gives the mean number of scatterings undergone by photons of frequency x that enter the medium in the direction η and subsequently emerge from it.

On the other hand, one could have the aim of determining the mean number of scatterings that issue in photons of frequency x' emerging from the medium in the direction η'. This number is characterized by the function $\tilde{N}_*(x, \eta)$. For brevity, in what follows we give only some particular results relating to $\tilde{N}_*(x, \eta)$; we concentrate on the calculation of $N_*(x, \eta)$.

Introducing the notation

$$\nu_*(x, \eta)=\lambda\frac{\partial R_*(x, \eta)}{\partial\lambda}=\int_0^1\eta' d\eta'\int_{-\infty}^{\infty}\nu(x', \eta'; x, \eta)\, dx',$$

we obtain on the basis of Eq. (16)

$$v(x)\nu_*(x, \eta)=\int_0^1 d\eta'\int_{-\infty}^{\infty}\alpha(x')\, p(0, x', \eta'; x, \eta)\,\nu_*(x', \eta')\, dx'+l_*(x, \eta), \tag{22}$$

where

$$l_*(x, \eta)=v(x)R_*(x, \eta)+\eta\int_0^1 d\eta'\int_{-\infty}^{\infty}v(x')\rho(x', \eta'; x, \eta)\, dx'-$$
$$\eta\int_0^1 d\eta'\int_{-\infty}^{\infty}\left[\left(1-\frac{\lambda}{2}\right)\alpha(x')+\beta\right]\nu(x', \eta'; x, \eta)\, dx'+$$
$$\frac{\lambda}{2}\eta\int_0^1 d\eta'\int_{-\infty}^{\infty}\nu(x, \eta; x', \eta')\, dx'\int_0^1 d\eta''\int_{-\infty}^{\infty}r(x', -\eta'; x'', \eta'')R_*(x'', \eta'')\, dx''.$$

We shall return to (22) below, giving in the meanwhile the values of N_* and $\tilde{N}_*$ for some special cases.

In the approximation of completely incoherent scattering (see, for example, [12])

$$R_*(x, \eta)=\frac{\alpha(x)}{v(x)}\left\{1-H(z)\left[\sqrt{1-\tilde{\lambda}}+\frac{\lambda}{2}\beta\omega(z, \lambda, \beta)-\frac{\lambda}{2}\beta\gamma_{00}(\lambda, \beta)\right]\right\}, \tag{23}$$

where

$$\gamma_{00}(\lambda, \beta)=\int_0^{1/\beta}G_0\left(\frac{z}{1-\beta z}\right)H(z)\, dz;$$

$$\omega(z, \lambda, \beta)=z\int_0^{1/\beta}G_0\left(\frac{z'}{1-\beta z'}\right)\frac{H(z')}{z+z'}dz' \qquad G_0(z)=2A\int_{x(z)}^{\infty}\alpha(x)\, dx.$$

Nagirner [13] has tabulated $\gamma_{00}(\lambda, \beta)$ and $\omega(z, \lambda, \beta)$ for different values of the arguments. It is obvious that in the given case the functions N_* and $\tilde{N}_*$ will also depend only on the combination $z = \eta/v(x)$. Then, from (19) and the definition of $N_*(x, \eta)$ we obtain

$$N_*(z)=1+f(z)-\frac{1+f(z)-\dfrac{2-\tilde{\lambda}}{2\sqrt{1-\tilde{\lambda}}}H(z)+\dfrac{\lambda}{2}\beta[\bar{\omega}(z, \lambda, \beta)-\bar{\gamma}_{00}(\lambda, \beta)]}{1-H(z)\left\{\sqrt{1-\tilde{\lambda}}+\dfrac{\lambda}{2}\beta[\omega(z, \lambda, \beta)-\gamma_{00}(\lambda, \beta)]\right\}}, \tag{24}$$

where

$$\bar{\gamma}_{00}(\lambda, \beta)=\int_0^{1/\beta}G_0\left(\frac{z}{1-\beta z}\right)H(z)f(z)\, dz; \qquad \bar{\omega}(z, \lambda, \beta)=z\int_0^{1/\beta}G_0\left(\frac{z'}{1-\beta z'}\right)\frac{H(z')f(z')}{z+z'}dz'.$$

Similarly

$$\tilde{N}_*(z)=1+f(z)+\bar{\omega}(z, \lambda, \beta)/\omega(z, \lambda, \beta). \tag{25}$$

For $\beta = 0$, Eq. (24) simplifies to

$$N_*(z) = \frac{f(\infty) - f(z)}{H(\infty) - H(z)} H(z) = \frac{\int\limits_0^\infty G(z') F(z', \lambda, \beta) \frac{z' dz'}{z + z'}}{\int\limits_0^\infty G(z') H(z') \frac{z' dz'}{z + z'}}, \tag{26}$$

where we have used the equation for the H function (10), the expression (23), and have also noted that $H(\infty) = 1/\sqrt{1 - \tilde{\lambda}}$ and $f(\infty) = \lambda/2(1 - \lambda)$.

We now turn to the statistics of the number of scatterings of the photons that perish in the medium in the course of diffusion. If once more we consider photons of frequency x incident on the semi-infinite medium at angle $\cos^{-1} \eta$, then the probability of their perishing as a result of a definite number n of scatterings (the absorption event is also regarded as a scattering) will obviously be given by

$$q_n(x, \eta) = \int\limits_0^1 \frac{d\eta'}{|\eta'|} \int\limits_{-\infty}^{\infty} u(x')\, dx' \int\limits_0^\infty Y_{n-1}(\tau, x, \eta; x', \eta')\, d\tau, \tag{27}$$

where $u(x) = (1 - \lambda)\alpha(x) + \beta$.

Use of the invariance principle makes it possible to write down equations, analogous to (12), for the functions Y_n; in turn, they lead to the following equation for the generating function of the quantities $q_n(x, \eta)$:

$$v(x) Q(x, \eta; s) = \frac{\lambda}{2} s \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} r(x, \eta; x', \eta') Q(x', \eta'; s)\, dx' + s \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} u(x') W(x', \eta'; x, \eta; s)\, dx' + su(x) + \frac{\lambda}{2} s \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} Q(x', \eta'; s)\, dx' \int\limits_0^1 d\eta'' \int\limits_{-\infty}^{\infty} r(x', \eta'; x'', -\eta'') W(x'', \eta''; x, \eta; s)\, dx''. \tag{28}$$

On the basis of the probabilistic meaning of $R_0(x, \eta) = Q(x, \eta; 1)$, we can readily conclude that $R_0(x, \eta) + R_*(x, \eta) = 1$, since a photon incident from without must either be reflected by the medium or be absorbed in it. The function Y having those two interpretations, $R_0(x, \eta)$ can also be regarded as the profile of the absorption line formed in an isothermal atmosphere if the power of the primary sources is u(x). From (28) in particular there follows an equation for the function $R_0(x, \eta)$ (see also [14]):

$$v(x) R_0(x, \eta) = \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} \alpha(x') p(0, x', \eta'; x, \eta) R_0(x', \eta')\, dx' + u(x) + \eta \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} u(x') \rho(x', \eta'; x, \eta)\, dx'. \tag{29}$$

Obviously, the mean number of scatterings $N_0(x, \eta)$ of the absorbed photons can be represented in the form $N_0(x, \eta) = \nu_0(x, \eta)/R_0(x, \eta)$, where $\nu_0(x, \eta) = \partial Q(x, \eta; s)/ds|_{s=1}$. Using Eq. (28), we obtain for $\nu_0(x, \eta)$

$$v(x) \nu_0(x, \eta) = \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} \alpha(x') p(0, x', \eta'; x, \eta) \nu_0(x', \eta')\, dx' + l_0(x, \eta), \tag{30}$$

where

$$l_0(x, \eta) = v(x) R_0(x, \eta) + \eta \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} u(x') \nu(x', \eta'; x, \eta)\, dx' + \frac{\lambda}{2} \eta \int\limits_0^1 d\eta' \int\limits_{-\infty}^{\infty} \nu(x, \eta; x', \eta')\, dx' \int\limits_0^1 d\eta'' \int\limits_{-\infty}^{\infty} r(x', -\eta'; x'', \eta'') R_0(x'', \eta'')\, dx''.$$

Besides the functions $N_*(x, \eta)$ and $N_0(x, \eta)$, we introduce

$$\langle N(x, \eta) \rangle = N_*(x, \eta) R_*(x, \eta) + N_0(x, \eta) R_0(x, \eta), \tag{31}$$

which is reddily seen to be the mean number of scatterings for a photon that possesses frequency x and is incident on the medium at angle $\cos^{-1} \eta$ irrespective of whether or not it is subsequently absorbed in the medium or leaves it. Instead of finding $\nu_0(x, \eta)$, we write down an equation for the function $\langle N(x, \eta) \rangle$ whose free term is simpler than $l_0(x, \eta)$. Indeed, adding Eqs. (22) and (30), we obtain

$$v(x) \langle N(x, \eta) \rangle = \int_0^1 d\eta' \int_{-8}^{\infty} \alpha(x') p(0, x', \eta'; x, \eta) \langle N(x', \eta') \rangle dx' + v(x) + \eta \int_0^1 d\eta' \int_{-\infty}^{\infty} v(x') \rho(x', \eta'; x, \eta) dx'. \tag{32}$$

If $\rho(x', \eta'; x', \eta)$ is known, Eq. (32) can be regarded as an integral equation with kernel $\alpha(x')p(0, x', \eta'; x, \eta)$ for the function $\langle N(x, \eta) \rangle$. We have seen above that the function $R_0(x, \eta)$ also satisfies an equation of this type. As was shown in [14], the problem of finding the intensity of the emergent radiation for different distributions of the primary energy sources also reduces to the solution of an equation with kernel $\alpha(x')p(0, x', \eta'; x, \eta)$. Referring the reader to [14] for the details of the solution of equations of the type (32) for different scattering mechanisms, we mention here merely that the route to the solution proposed there is based on a representation of the kernel in the form

$$\alpha(x') p(0, x', \eta'; x, \eta) = \frac{\lambda}{2} \sum_{i=0}^{\infty} P_i(\eta') \sum_{k=i}^{\infty} c_k^i \alpha_k(x') \varphi_{ik}(x, \eta), \tag{33}$$

which follows from the bilinear expansion of the redistribution function (4) and Eqs. (6) and (13). Generally speaking, the use of the expansion (33) makes it possible to reduce the problem of solving an integral equation of the form (32) to the solution of an infinite system of algebraic equations. In some of the simplest cases, the solution, expressed in terms of φ functions, can be found in closed form.

We consider in more detail the determination of $\langle N(x, \eta) \rangle$. Once found, we can if necessary also use (31) to determine $N_0(x, \eta)$ (we assume $R_*(x, \eta)$ and $R_0(x, \eta)$ are known). However, we first draw an important conclusion by comparing Eqs. (29) and (32). We see that for $\beta = 0$ (and only then)

$$\langle N(x, \eta) \rangle = R_0(x, \eta)/(1 - \lambda), \tag{34}$$

a result that was obtained by physical arguments by Sobolev [2] in the simplest case of isotropic completely incoherent scattering.

Now suppose $\beta \neq 0$. Then by simple subtraction of (29) from (32) we readily conclude that the difference of $\langle N(x, \eta) \rangle$ and $R_0(x, \eta)$ satisfies an integral equation different from the original equations only in the free term, now $\lambda \pi^{1/4} \varphi_{00}(x, \eta)$. Using the expansion (33), to determine $\langle N(x, \eta) \rangle$ we have

$$\langle N(x, \eta) \rangle = R_0(x, \eta) + \frac{\lambda}{v(x)} \left[\pi^{1/4} \varphi_{00}(x, \eta) + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} c_k^n j_{nk} \varphi_{nk}(x, \eta) \right]. \tag{35}$$

The constants j_{nk} in (35) are determined from the system of algebraic equations

$$j_{im} = \frac{\lambda}{2} \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} c_k^n g_{im}^{nk} j_{nk} + \lambda \pi^{1/4} g_{im}^{00}, \tag{36}$$

where

$$g_{im}^{nk} = \int_0^1 P_i(\eta) d\eta \int_{-\infty}^{\infty} \varphi_{nk}(x, \eta) \frac{\alpha_m(x)}{v(x)} dx.$$

In particular, in the case of complete frequency redistribution (35) is replaced by

$$\langle N(x,\ \eta)\rangle = R_0(x,\ \eta) + \frac{\beta}{\sqrt{1-\tilde{\lambda}}}\frac{\alpha(x)}{v(x)}H(z), \tag{37}$$

or, with allowance for (23), finally

$$\langle N(x,\ \eta)\rangle = \frac{\beta}{v(x)} + \frac{\alpha(x)}{v(x)}H(z)\left\{\frac{1+\lambda\beta(\beta)}{\sqrt{1-\tilde{\lambda}}} + \frac{\lambda}{2}\beta[\omega(z,\ \lambda,\ \beta) - \gamma_{00}(\lambda,\ \beta)]\right\}. \tag{38}$$

The function

$$\delta(\beta) = A\int_{-\infty}^{\infty}\frac{\alpha(x)}{v(x)}\,dx$$

in (38) is well known in the theory of radiative transfer in a line with continuum absorption. This function has been tabulated for different profiles of the absorption coefficient.

It follows in particular from (38) that as $x\to\infty$, $\langle N(x,\ \eta)\rangle\to 1$. For $\beta = 0$, as one would expect and as is readily verified, (38) goes over into the previously obtained (34).

The expressions obtained in this paper make it possible to calculate the mean number of scatterings for photons that leave the medium and have definite frequency and direction of motion. Of course, if the external radiation sources have certain angular and spectral distributions, the obtained expressions must be averaged over these.

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