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Quasi-Empirical Views of Mathematics and Mathematics Teaching

Daniel Chazan
Education Development Center, Newton, Massachusetts

The general public views mathematical knowledge as precise, rigorous, and certain. As Kline (1980) says, "Whenever someone wants an example of certitude and exactness of reasoning, he appeals to mathematics" (p. 4). This view of mathematical knowledge plays a large role in mathematics' public image: "As it is commonly perceived, mathematics is the least creative of subjects: A dead, unchanging body of facts and techniques handed down from the ancients, tolerating no room for inquiry, every question bearing one and only one answer, an answer that is already known by someone" (Goldenberg, 1989, p. 170).

These views influence school practice. In schools, mathematics is seen as different from other subjects. It is presented as the subject of certainty where there are single right answers which the teacher and text know and which students must learn how to produce.

The views of mathematics knowledge and of mathematics presented above are oversimplifications of philosophical rationalism. While they reflect certain truths about mathematics, they misrepresent mathematicians' reports of other aspects of mathematical experience — its social dynamics, creativity, and intellectual beauty. As a result, mathematicians and historians of mathematics have criticized rationalist philosophers for not accurately reflecting the uncertainty, irrationality, intuition, and exploration which characterize the everyday lives of mathematicians. Recently, some philosophers have responded with new accounts of mathematical knowledge designed to reflect more accurately the practice of mathematics.

These developments in the philosophy of mathematics are presented below as a prologue for the presentation of an innovative approach to teaching geometry which was designed with these new views of mathematics in mind.

Rationalist Views of Mathematics

Scheffler (1965) characterizes the view of mathematics which underlies the rationalist epistemological tradition.

> Mathematical truths are general and necessary, and may be established by deductive chains linking them with self-evident basic truths. Demonstration forges the chains, intuition discloses the basic truths. Intuition, moreover, guarantees each link in the chain of demonstration. Whoever understands a mathematical truth knows it to be necessary and not contingent on facts of nature. (p. 2)

Lakatos (1986) suggests that theories that are characterized this way are Euclidean theories, beyond the deductive method they begin with

> . . . an indubitable truth-injection at the top (a finite conjunction of axioms) — so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system. (p. 33)

For the rationalist, mathematical truths, both the axioms and derived truths, are *a priori*. As Scheffler (1965) explains, mathematics for the rationalist is not dependent on nature.

> A diagram may well be used to illustrate a geometrical theorem, but it cannot be construed as evidence for the theorem. Should precise measurement of the diagram show that it failed to embody the relations asserted by the theorem, the latter would not be falsified. We should rather say that the physical diagram was only an approximation or a suggestion of the truth embodied in the theorem. (p. 3)

Finally, mathematics is different from science because

> mathematicians do not need laboratories or experiments; they conduct no surveys and collect no statistics. They work with pencil and paper only, and yet they arrive at the firmest of all truths, incapable of being overthrown by experience. (p. 3)

Impact on Education and Educational Research

The rationalist view of mathematics has had a strong impact on the way teachers and students think of the mathematics taught in school. Interpretations of this view suggest: mathematics is different than other subjects; mathematics is the subject of certainty where one's common sense, one's storehouse of experiences in the world, is irrelevant; mathematics is where there are single right answers which the teacher and the textbook know to be correct beyond a shadow of a doubt.

Even the mathematics education reformer and champion of mathematical creativity, George Polya, seems to promote these views when he outlines the difference between demonstrative and plausible reasoning. In his words, "Demonstrative reasoning is safe, beyond controversy, and final. Plausible reasoning is hazardous, controversial, and provisional" (1980, p. 99).

The view that the method of deductive proof privileges mathematical knowledge with a certainty found nowhere else has also had an effect on research on students' understanding of the deductive method; it has shaped research questions and influenced the interpretation of results. In Fischbein's view, "[A] formal proof offers an absolute guarantee to a mathematical statement. Even a single practical check is superfluous" (1982, p. 17).[1] Thus, when checking high school students' understanding of deductive proofs, he presents students with a particular deductive proof and asks whether they believe that further empirical checks are necessary.

The effect of this view can also be seen in Williams's (1979) research. Without suggesting that it may be difficult to decide whether or not a given proof is valid or whether the theorem is stated in an overly general way, Williams states "the generalization principle" for deductive proofs: "If $P(x)$ is a statement function which is proven for any arbitrary but fixed value of the variable x belonging to some domain D, then $P(x)$ is proven for all x in D" (p. 45). Thus, in geometry, after seeing a proof and its associated diagram, he expects that students should be certain that the statement holds for any figure satisfying the givens. Since in his research students do not exhibit this kind of certainty, he argues that students do not understand this principle.

Challenges to Rationalist Views of Mathematics

Evolutions in mathematics — the creation or discovery of non-Euclidean geometries and the foundational crises of the 20th century — have posed serious challenges to those claiming that mathematical knowledge is certain. The severity of these challenges is indicated by the title of Morris Kline's (1980) book *Mathematics: The Loss of Certainty.*

Non-Euclidean geometries challenge the view of geometry as a Euclidean theory. No longer can the Euclidean axioms be viewed as true on the basis of intuition; contradictory sets of axioms lead to geometries that also seem to be "true" and can be used to describe the world. Thus, to use Lakatos's phrase, the development of non-Euclidean geometries deprived Euclidean geometry of its "truth-injection from the top." This development suggests conceptions that describe the truth of deduced, mathematical conclusions as contingent on the truth of the axioms. To parody Lakatos's description of a Euclidean system, since the axioms are no longer certain and a particular proof may be invalid, uncertainty inundates the whole system. Uncertainty flows down from the top through the theoretically safe and truth-preserving channels of valid inferences, which, to be realistic, are possibly invalid, to guarantee that the conclusions are not much more uncertain than the axioms.

Since the foundational crises, the certainty of mathematics has also come under attack from mathematicians who suggest that philosophers have "pretend[ed] not to notice the gap between preaching and practice" (Hersh, 1986, p. 20). These critics argue that mathematical practice creates or discovers "knowledge" which is neither precise, rigorous, nor certain.

While holding a rationalist view of proof, Polya (1954) suggests that the rationalist view of mathematics does not sufficiently emphasize the "inductive" thinking and verification or plausible reasoning that mathematicians use when deriving their conjectures and dreaming up their attempts at deductive proofs.

> To a mathematician, who is active in research, mathematics may appear sometimes as a guessing game: you have to guess a mathematical theorem before you prove it, you have to guess the idea of the [deductive] proof before you carry through the details. . . . The result of the mathematician's creative work is demonstrative reasoning, a [deductive] proof, but the [deductive] proof is discovered by plausible reasoning, by guessing. (p. 158)

Reuben Hersh (1986) enlarges the scope of Polya's critique by suggesting that philosophical descriptions of what a proof is are also insufficient. He observes that "an interpersonally verifiable notion of [a] 'correct [deductive] proof' exists at the intuitive level of the working mathematician . . . [which] is not very similar to the model of [a] formal proof in which correctness can always be verified as a mechanical procedure" (p. 20).

Hersh argues that mathematicians' attempts at deductive proofs are intuitive; to understand proofs, the reader has to supply meaning to the statements. Proofs are meant to communicate with others who share a similar background and cannot claim validity until others have checked them because "until you have checked with other people, you can never be quite sure you haven't overlooked something" (p. 19). Even then, the attempted deductive proof may not be complete, "because we have not yet seen the counterexample that would make us aware of the possibility of doubting it [the proof]" (p. 19).[2]

Gila Hanna (1983) makes a similar point about mathematical practice when she suggests that the following five criteria would all rank higher on a rank order of criteria for the admissibility of a candidate theorem than the existence of an attempt at a rigorous proof which could be verified by a mechanical procedure (an option which a philosopher might propose).

> 1. They [mathematicians] understand the [candidate] theorem, the concepts embodied in it, its logical antecedents, and its implications. There is nothing to suggest it [the candidate theorem] is not true;
> 2. The [candidate] theorem is significant enough to have implications in one or more branches of mathematics (and is thus important and useful enough to warrant detailed study and analysis);
> 3. The [candidate] theorem is consistent with the body of accepted mathematical results;
> 4. The author has an unimpeachable reputation as an expert in the subject matter of the [candidate] theorem;
> 5. There is a convincing mathematical argument for it (rigorous or otherwise), of a type they have encountered before. (p. 70)

Hersh is not content with the gap he sees between practice and the ideal. He suggests accepting this gap might mean "that we really ought to (if we only had the time and energy) write our [attempts at deductive] proofs in a form that could be checked by a computing machine" (p. 21). He finds such a proposal to be arguable, because certainty would remain elusive: "the doubtfulness of the [attempt at a deductive] proof would then be replaced by the doubtfulness of the coding and programming" (p. 21).[3]

Historians of mathematics also critique the view of mathematics as certain. On one level, this critique suggests that throughout history attempts at mathematical proofs have turned out to be mistaken and invalid, therefore mathematical knowledge is in reality not certain. Critics of this school point to articles published in mathematical journals, such as "False Lemmas in Herbrand" which appeared in the *Proceedings of the American Mathematical Society* or "Fidelity in Mathematical Discourse" published in the *American Mathematical Monthly* which contains a discussion of errors in mathematical publications (Hersh, 1986, pp. 19–20). They also point out errors which involve mathematicians with considerable reputations or problems of great notoriety which according to Davis (1986) occur about every 20 years.[4]

Perhaps the most eloquent exposition of the fact that attempted mathematical proofs have turned out to be flawed is the imaginary classroom dialogue presented in Lakatos's (1976) *Proofs and Refutations*. As the title indicates, Lakatos sees an analogy between Popper's conjectures and refutations in science and the logic of attempts at deductive proofs and refutations in mathematics. Implicit in this view is the notion that historically attempted deductive proofs turn out to be invalid or incomplete. As the footnotes to the dialogue indicate, the logic of attempts at deductive proofs and refutations does indeed characterize the development of the classification of polyhedra and Euler's conjecture about the relationship between their vertices, edges, and faces.

Yet, in an essay titled "Is Mathematical Truth Time-Dependent?", Judith Grabiner (1986) takes the argument that particular deductive proofs can turn out to be invalid or incomplete to another level. She shows that between the 18th and 19th centuries there was "a revolution in thought which changed mathematicians' views about the nature of mathematical truth" (p. 202). As a result, the definition of what constitutes a "deductive proof" changed. She argues that in the 18th century,

> the primary emphasis was on getting results. All mathematicians know many of the results from this period. . . . But the chances are good that these results were originally obtained in ways utterly different from the ways we prove them today. . . . Mathematicians [in the 18th century] placed great reliance on the power of symbols. Finite methods were routinely extended to infinite processes. Discussions of the foundations were not the basic concern. (pp. 203–205)

In contrast, during the 19th century analysts "gave rigorous, inequality-based treatments of limit, convergence, and continuity, and demanded rigorous proofs of the theorems about these concepts. We know what these proofs were like; we still use them" (p. 205).

Gila Hanna (1983) updates Grabiner's argument by showing that "there is no consensus today among mathematicians as to what constitutes an acceptable [attempt at a deductive] proof" (p. 29). She illustrates this point by presenting the controversy between intuitionists and other mathematicians about the legitimacy of non-constructive proofs for existential statements about a set of objects with an infinite number of members. Intuitionists challenge the law of the excluded middle, one of the laws of inference that other mathematicians use in their attempts to write deductive proofs.

Philosophers' definitions of a deductive proof do not specify the laws of inference to be used. Both Grabiner's and Hanna's arguments suggest that beyond the evaluation of particular attempts at a deductive proof, mathematicians have disagreed, and continue to disagree, about the use of particular laws of inference in attempts at deductive proofs. Clearly, disagreement about, and change in, the accepted laws of inference will result in disagreements in judging particular attempts at deductive proofs. Thus, their arguments provide one more reason for suggesting that, contrary to the view of the general public expressed at the beginning of this paper, mathematical knowledge should not be considered to be certain, beyond all shadow of a doubt.

Recent Philosophical Responses

Some philosophers have begun to formulate a philosophy of mathematics in which mathematical knowledge does not consist of *a priori* truths.[5] The two philosophers described below might be labelled as empiricists or pragmatists instead of rationalists; they consider mathematics to be a quasi-empirical science.

Lakatos (1986) suggests that mathematics, like the sciences, is a quasi-empirical theory. Such theories have their "crucial truth value injection" at the bottom. However,

> the important logical flow in such quasi-empirical theories is not the transmission of truth but rather the retransmission of falsity — from special theorems at the bottom ("basic statements") up towards the set of axioms . . . a quasi-empirical theory — at best — [can claim] to be well-corroborated, but always conjectural. (pp. 33–34)

Putnam (1986) also argues for a quasi-empirical view of mathematics and suggests a mechanism for the refutation of mathematical statements. He suggests that in mathematics the "basic statements" used to test theories "are themselves the product of deductive proof or calculation rather than being 'observation reports' in the usual sense" (p. 51). In his view, acceptable grounds for considering a conjecture as "verified" include "[that] extensive searches with electronic computers have failed to find a counterexample — many 'theorems' have been proved with its aid, and none of these has been disproved, the consequences of the hypothesis are plausible and of far-reaching significance, etc. . ." (pp. 51–52).

However, in his view, in mathematics, the method of deductive proof and the method of quasi-empirical verification live side by side. "[The method of deductive] proof has the great advantage of not increasing the risk of contradiction, where the introduction of new axioms or new objects does increase the risk of contradiction, at least until a relative interpretation of the new theory in some already accepted theory is found" (p. 63).

Relevance for Pedagogy

The academic critiques of rationalism raise interesting points, but it is not immediately evident that these critiques are relevant to mathematics education. After all, though mathematical knowledge in general may be uncertain, surely school mathematics — arithmetic, geometry, algebra — is certain. These fields of mathematics have been around a long time; there has been time to check results, time for counter-examples to appear. Why should schools represent mathematical knowledge as provisional or uncertain? This question can be further sharpened by thinking about the types of students found in a typical classroom. For the moment, let us assume that those preparing to be college math majors should be learning something of the practices of mathematicians, but how about those with little interest in theoretical aspects of mathematics, students who may rarely even use mathematics as a tool in their future employment. Are there reasons to present mathematical knowledge as uncertain to such students?

While this question raises many avenues for discussion, such as the importance of developing students' sceptical faculties (a traditional goal of rationalists everywhere but in mathematics), I would like to suggest one possible ramification of presenting mathematical knowledge as uncertain. Such a stance has the possibility of empowering students by changing their view of the subject, making mathematics seem less mysterious and otherworldly, an arena where one can figure something out. With the large numbers of students who are "anxious" about mathematics or find that mathematics makes little or no "sense," such possibilities seems worth exploring.

How would a change in educators' philosophy of mathematics have this effect? If a quasi-empirical view is taken, students no longer need to ignore their common sense, their experiences. Student exploration can become a central aspect of teaching. With the advent of microcomputer programs which support students' exploration of visual representations of mathematics and with a willingness to accept as provisionally true statements for which exploration reveals no counter-examples, students' creation of mathematical statements based on exploration becomes a feasible and legitimate classroom activity. Students can also demand that mathematical theory correspond with their experiences in the world. They can ask their teacher for an accounting of any differences between their views based on their experiences and accepted mathematical theories.

An Illustration: An Approach to Teaching Geometry

Even if the argument presented above is convincing, the question still remains, "Is it possible to design and implement such approaches to teaching mathematics that are practical and reasonable for teachers to use in schools?" In an effort to answer this question in the affirmative, I will outline an alternative approach to teaching high school geometry. Typically, the high school Euclidean geometry course is students' first exposure to the use of deductive proofs; it is the place currently reserved in the curriculum for an introduction to what mathematicians do. Moise (1975) argues that geometry has this role because "it seems to be the only mathematical subject that young students can understand and work with in approximately the same way as a mathematician" (p. 477). Also, software has been developed recently to enable students to explore geometric constructions.[6]

I will outline this alternative approach by highlighting four ways in which such an approach differs from traditional instruction: inclusion of exploration and conjecturing; presentation of demonstrative reasoning as explanatory; treatment of proving as a social activity; and emphasis on deductive proofs as part of an exploratory process, not its end point.

In traditional geometry instruction, students are given true statements and are asked to write proofs for these statements. It is clear to students that the statements are true (otherwise they would not be asked to prove them) and that their teacher knows how to prove them. In contrast, microcomputer software, like the Geometric Supposers, can be used in an exploratory approach where teachers pose problems to students and ask them to investigate a given geometric construction and make conjectures about all the particular drawings which can be created by this construction. These student-generated conjectures are then the statements which students are asked to prove.[7] Students' statements may or may not be true; a teacher may not necessarily be familiar with them, let alone know how to prove them. Such an approach asks students to distinguish between Polya's plausible and demonstrative reasoning and use both of these kinds of reasoning in the mathematics classroom.[8]

Traditional geometry instruction also holds that the existence of a valid deductive proof determines whether or not a statement is true. In the alternative approach that I am suggesting, students can use quasi-empirical verification, they can accept as true statements for which they can find no counter-examples. In order to present an alternative rationale for proving deductively, when deductive proofs are introduced, their explanatory role can be emphasized, that is, the insight which they provide about why a statement is true.[9] This sort of insight cannot be provided by the measurement of examples which the Supposers support. After reading a proof, students can be asked whether they now know something that they didn't know after their exploration. They can also be asked to explain in their own words why the theorem is true.

Traditional geometry instruction also downplays any social role in the determination of the validity of a proof; the teacher and textbook are the arbiters of validity. In the alternative approach that I am suggesting, one criterion for the validity of a proof is that it have no counter-examples. When deductive proofs are first introduced, there could be less of an emphasis on having students write proofs and more emphasis on critiquing proofs. Students can be asked to try to find counter-examples to textbook proofs and to expose assumptions not presented in the proof.[10]

There is another important criterion, agreement of the social group, which is emphasized in an approach which Fawcett (1938) describes. The proofs which he presented in class did not confer the status of "theorem" on a statement unless the class unanimously agreed that the proof was a good one. Students were encouraged to question the steps of a proof as well as the validity of its conclusion. Thus, to integrate this aspect with the rest of the approach which I am suggesting, when students explore a construction and begin to write their own proofs for the conjectures they have developed, the class can decide whether they think a given proof is valid or not. The role of the teacher can be downplayed. With such an approach, it is crucial that teachers are willing to admit an invalid proof or a proof of an incorrect statement, if they are unable to convince students by judiciously criticizing the proof or by providing a counter-example.

The social nature of postulates can also be emphasized with a similar strategy. Students can be encouraged to suggest postulates to the group. If a student convinces the class to accept a postulate, the statement remains as a postulate until proven from other postulates or unless students questions its truth or its usefulness in proving other statements.[11] This approach to statements which seem to be true, but which cannot be proven, can also make the concept of a "lemma" a useful one. Students can order a set of conjectures that they have developed about a construction and taking one as true, prove the rest. If later they can prove this one conjecture, it can be a lemma in their presentation of the proofs of the others. If not, they can decide to keep it as a postulate. Allowing students to introduce as many postulates as they want can also be used to raise the issue of the aesthetic of parsimony.

Finally, traditional geometry instruction presents proving as a final goal. In contrast, in the proposed alternative approach, there could be an emphasis on proof as a part of a mathematical process and not its end point. Having completed a proof, students can continue to explore. They can be asked: What have we just proven? For what geometrical objects does it hold? Can it be generalized? Are there some less general, but interesting, results that hold as a consequence of this theorem? Based on the steps of this proof, what else is true for this construction? These questions can help students raise interesting avenues for further empirical exploration.

Conclusion

The approach described above was designed by taking seriously a quasi-empirical view of mathematical practice. It asks students to learn mathematics by working like mathematicians. However, I am not suggesting that this view of mathematical practice is the correct view, and I do not believe that students should always learn a subject by working like the experts in that area. My reason for designing an approach in which students work like mathematicians (as their practice is described by one philosophical school) is a pedagogical one; I believe that many students will profit from an approach to learning mathematics which values the conclusions they arrive at based on their experiences, an approach which doesn't hide the uncertainty and social dimensions of mathematical practice. Quasi-empirical views of mathematics help me, as an educator, design coherent approaches to teaching mathematics which incorporate student exploration.

One common saw about teaching is that it is easier to teach as one has been taught than to take on a new approach which one has never seen demonstrated. Is it possible to implement the approach described above? What would it take? Clearly, such an approach requires a tremendous amount of a teacher. To carry out this approach, a teacher must be an explorer of geometry who is able to model the kind of exploration which is desired. In addition, the teacher must be familiar enough with geometry to be able to produce counterexamples to students' incorrect statements, to see connections between a list of student conjectures about a construction, and to raise provocative questions to help students continue their explorations. This approach also requires a teacher who is adept at facilitating classroom discussion, creating a feeling that the class is a community of learners, and encouraging students to take risks publicly. This teacher must also be convinced that student exploration is a pedagogically useful activity and have a coherent framework for introducing such exploration into the mathematics classroom. Together these requirements outline a formidable task for pre-service and in-service teacher training, especially since there is no well-developed lore of teaching hints and strategies for creating mathematics classes that are communities of inquirers.

But while the task is daunting, there are some positive signs. With the availability of exploratory software for the mathematics classroom, many teachers have begun to take students to a computer lab for empirical work with geometric constructions and graphs of functions and equations.[12] In some cases, classroom discussion of students' findings have become a regular part of mathematics courses.[13]

Yet, there is still a long way to go. Even some who are advocating, creating, publishing, and teaching exploratory approaches to mathematical subjects still hold views of mathematics and of the role of proof in mathematics which do not legitimize or provide a rationale for student exploration. It is important that educators continue discussion about the role of exploration in mathematics and pedagogical rationales for commitment to student exploration.

Notes

1. In the context of Fischbein's paper, it seems that he is speaking about a particular deductive proof.

2. Some of these points are made eloquently in a set of dialogues written by Davis and Hersh called "The Ideal Mathematician" (in Davis & Hersh, 1981).

3. Recent attempts at deductive proofs involving the use of computers, the four colour problem, and the non-existence of a finite projective plane of Order 10, have made this proposal relevant to actual practice. They raise the question "Is an attempt at a math proof a proof if no one can check it?" (See "Is a Math Proof," 1988.) The question can be asked on two levels, the practical and the ideal, as Will arguments of this type be accepted by mathematicians? and How is such practice related to philosophical definitions of a deductive proof?

4. In 1986, for example, there was substantial controversy over an attempt at deductive proof for Poincaré's conjecture in topology ("One of Math's," 1986) and more recently an attempted proof of Fermat's Last Theorem turned out to be incorrect ("Joy of Math," 1988).

5. While I highlight the views of Lakatos and Putnam, other philosophers of a similar bent include Quine, Wittgenstein, Lehman, Kitcher, and Peirce.

6. Examples of such software are the Geometric Supposer, developed by Education Development Center; Cabri Geometre, developed by a team at Université Joseph Fourier in Grenoble; and The Visual Sketchpad, part of Swarthmore's Visual Geometry Project.

7. For a more elaborate description of this approach, see Chazan and Houde (1989). For the role of the microcomputer software, see Schwartz and Yerushalmy (in press).

8. For a study of students' struggles with this discrimination, see Chazan (1989).

9. Hanna (1989) distinguishes between proofs that prove and proofs that explain. Most proofs in a high school geometry course are of the latter type. This does not mean that all students understand these proofs, but that in contrast to existence proofs, these proofs seek to provide information.

10. High school geometry textbook proofs frequently have shortcomings. For example, the textbook proof used by Chazan (1989, p. 89), which is the first proof presented in the text, assumes that the figures are in a plane and that the segments are connected A to D and B to C with B and C on the same side of AD. The conclusion would not hold if the given were true and the points connected in a different order. Similarly, many computer tutors for geometric proofs rely on the diagrams to convey part of the givens.

11. It may turn out that different students want to take different statements as postulates. Such alternatives could be explored to emphasize that one can choose which statements to take as postulates (and definitions as well).

12. For example, the Geometric Supposer is being used nationwide by members of the Urban Mathematics Collaboratives, by members of a group supported by Sunburst Communications, and by members of the Council of Presidential Awardees.

13. Over a four-year period, the Harvard Educational Technology Center studied a group of Boston-area teachers using such an approach with Geometric Supposers. This experience is described in a series of ETC technical reports.

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