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Generalized Definition of Coherent States and Dynamical Groups

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Abstract

It is shown that coherent states may be defined for an arbitrary dynamical (Hamiltonian) quantum system and the definition is consistent with the requirement that the Hamiltonian commutes with a Lie algebra γ, and γ can be integrated to form a Lie group G.

There are numerous important examples of features of a dynamical theory for which the use of coherent states becomes relevant: for instance, the description of a system with infinitely many degrees of freedom, in which quantum characteristics are macroscopically relevant.

On the other hand, the commonly used coherent states, based on the Fock representation, are known to be inapplicable for theories possessing nontrivial invariance groups. Only under restrictive assumptions (Rasetti, 1975) (which amount to breaking of the invariance requirement: one allows a nonstrictly translationally invariant state in the representation space) can the determination of the dynamics of the system still be handled within the framework of coherent states when the Hamiltonian includes interacting terms exhibiting wider invariance groups.

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Usual coherent states (Klauder, 1968) are therefore suitable for the description of a weakly interacting system in which low energy excitations are over a ground state in which the boson modes are highly occupied and which consequently behaves in some sense classically.

In this context they constitute a set of functional representatives of the abstract state vector of the system, where every member of which, besides being translationally invariant in the representation space, is an entire analytic function.

So, coherent states actually play the role of classical Schrödinger fields, which describe the complete many boson system just in the same way as e.g., the Maxwell field describes the classical limit of quantum electrodynamics.

Now, Bose statistics are connected through the usual commutation relations of second quantized field theoretical creation and annihilation operators, to the well known nilpotent Weyl group. The generalization of the concept of coherent state to dynamical systems not isomorphic to harmonic oscillators, and therefore to different groups is immediately appealing, and it may be formulated in abstract terms in the following way.

A Lie group G is a topological group and a differentiable manifold of class 1, differentially equivalent to an analytic manifold in which the two operations $G \times G \to G$ given by $(g, g') \to gg'$ and $G \to G$ given by $g \to g^{-1} (g, g' \in G)$ are analytic.

Any closed subgroup H of G is itself a Lie group, and the inclusion map $H \subset G$ is analytic and nonsingular. So an analytic structure is defined in the left coset space G/H in such a way that the projection p: $G \to G/H$ is analytic and of maximum rank at each point of G.

It follows that G is a bundle over G/H with respect to the projection p. Then H is a point, say, $x_0 \in M = G/H$ of the homogeneous space of the left cosets of H in G, thought of as a manifold M with a topology defined by p (actually it can be shown that M is a Hausdorff space). A local cross section of H in G is a function f mapping a neighborhood V of x_0 continuously into G and such that $pf(x) = x$ for each $x \in V$. Clearly such a function f exists, because G is a bundle over G/H.

Now, if H has a local cross section in G, then G is a fibre bundle over M as base space relative to the projection p which assigns to each g the coset gH. The fibre of the bundle is H, and the group is H acting on the fibre by left translations.

G is a group of transformations of M under the operation of left translation defined by $g \cdot x = p(g \cdot p^{-1}(x))$; transitive and it is a group of homeomorphisms of G/H. (Notice that the transformation $G \to G$ sending each g into its inverse, maps each left coset of H into a right coset and conversely: this induces a homeomorphism between the left and right coset spaces. So talking of left or right translations doesn't limit the generality.) Finding a cross section for the bundle G is just the problem of constructing in G a simply-transitive continuous family of transformations.

There exists a topological transformation group acting effectively over M, namely the factor group G/H_0 where H_0 is the intersection of all the subgroups

gHg^{-1} conjugate to H in G (so that H_0 is a closed invariant subgroup of G, and it is the largest subgroup of H which is invariant in G). Perelomov (1972) assumes as a subgroup H the subgroup of stability of any fixed point $x_0 \in M$, namely the set of transformations of G which map x_0 into x_0. Each homogeneous space is completely defined by some group of stability: actually let $g \in G$ carry x_0 into x. Then the transformations of the form hg ($h \in H$, the stability subgroup of x_0) also map x_0 into x. Thus the set of such transformations is a right coset of the stability subgroup of x_0, and there exists a mutually single-valued correspondence between the points of the homogeneous space M and the right cosets with respect to H. Note how, although the choice of x_0 is an unused degree of freedom, the space M is however uniquely defined. Actually, changing the fixed point x_0 to $x'_0 = g' \cdot x_0$ amounts to changing the subgroup of stability to $g'Hg'^{-1}$ which is a subgroup of G conjugate to H.

Let now π be a unitary representation of the topological group G on the Hilbert space $\mathscr{H}_G$. Consider the vectors $|\nu_G\rangle \in \mathscr{H}_G$. Any positive definite function $\psi \not\equiv 0$ on G corresponds to a unitary representation π of G for a suitable vector $|\nu_G\rangle$. Let $\mathscr{N}$ be the subspace of $\mathscr{H}_G$ consisting of all the vectors $|\omega_G\rangle \in \mathscr{H}_G$ whose norm is left fixed by each $\pi(h)$, $h \in H$(π is assumed to be of class 1, so that at least one such a vector exists and is not null). The subspace $\mathscr{N}$ is one dimensional (Helgason, 1968).

Provided one identifies the fiber H as a circle, the mapping of which carries $\mathscr{N}$ into x_0 (homotopy classes of such maps constituting the elements of the homotopy group of M) defines the coherent states.

In particular let G be parametrized by the set of parameters $\{\alpha\} = \{\alpha_i; i = 1, \ldots, r\} \in M$. $\mathscr{H}_G$ is defined as a space of functions $\psi(\{\alpha\})$ (a vector of which will be denoted by $|\psi_{\{\alpha\}}\rangle$ and the scalar product—linear in $|\psi_{\{\beta\}}\rangle$ and antilinear in $|\psi_{\{\alpha\}}\rangle$—by $\langle\psi_{\{\alpha\}}|\psi_{\{\beta\}}\rangle$), together with an invariant measure $d_\mu(\{\alpha\})$ such that $\psi \in \mathscr{H}_G$ if

$$\int d_\mu(\{\alpha\}) \langle\psi_{\{\alpha\}}|\psi_{\{\alpha\}}\rangle < \infty \tag{1}$$

On this manifold unitary representations may be defined in the usual way by means of right and left translations

$$T_g^{(R)}|\psi_{\{\alpha\}}\rangle = |\psi_{\{\alpha\cdot g\}}\rangle \tag{2}$$

$$T_g^{(L)}|\psi_{\{\alpha\}}\rangle = |\psi_{\{g\cdot\alpha\}}\rangle \tag{3}$$

The unitary transformations $T^{(R)}$ and $T^{(L)}$ naturally commute with each other

$$T_g^{(R)}T_{g'}^{(L)}|\psi_{\{\alpha\}}\rangle = T_g^{(R)}|\psi_{\{g'\cdot\alpha\}}\rangle = |\psi_{\{g'\cdot\alpha\cdot g\}}\rangle = T_{g'}^{(L)}T_g^{(R)}|\psi_{\{\alpha\}}\rangle \tag{4}$$

Now, if H is a circle, there exists at least one set of parameters $\{\alpha_0\}$ such that

$$T_h|\psi_{\{\alpha_0\}}\rangle = e^{i\phi(h)}|\psi_{\{\alpha_0\}}\rangle, \; h \in H \tag{5}$$

Coherent states for the Lie Group G are now defined according to the previous scheme

$$|\xi_g\rangle = |\psi_{\{pgp^{-1}\alpha\}}\rangle \tag{6}$$

It is in general possible to choose the parametrization of G in such a way that the parameters of H are restricted to the subset $\{\alpha^{(H)}\} = \{\alpha_j; j = s+1, \ldots, r\}$. If one considers now the representations of H obtained e.g., by right translations $T_h^{(R)}$, the eigenstates $|\psi_{\{\alpha\}}(k|i)\rangle$ of the Casimir operator $C_k^{(R)}$ with eigenvalues k can be factorized in the form

$$|\psi_{\{\alpha\}}(k|i)\rangle = |\phi_{\{\alpha^{(H)}\}}(k|i)\rangle\langle\{\bar{\alpha}^{(H)}\}|f_i\rangle \tag{7}$$

where $\{\bar{\alpha}^{(H)}\} = \{\alpha_j; j = 1, \ldots, s\}$ is the complementary set of $\{\alpha^{(H)}\}$ with respect to $\{\alpha\}$, and $\langle\{\bar{\alpha}^{(H)}\}|f_i\rangle$ are either arbitrary functions or members of a complete orthonormal set over the space of such functions.

The above factorization is obviously possible because $C_k^{(R)}$ depends only on $\{\alpha^{(H)}\}$. On the other hand $C_k^{(R)}$ commutes with $T_g^{(L)}$ in the same way as each $T_h^{(R)}$ does, so the effect $T_g^{(L)}$ on $|\psi_{\{\alpha\}}(k|i)\rangle$ will be given by

$$T_g^{(L)}|\psi_{\{\alpha\}}(k|i)\rangle = \sum_j [T_g^{(L)}(k)]_{i,j}|\psi_{\{\alpha\}}(k|j)\rangle \tag{8}$$

where

$$T_g^{(L)}(k) = P_k T_g^{(L)} P_k \tag{9}$$

P_k being the projection operator from $\mathscr{H}_G$ onto the subspace $\mathscr{M}_k$ which is spanned by $|\psi_{\{\alpha\}}(k|i)\rangle$ for fixed k.

For any $|\psi_{\{\alpha\}}(k)\rangle \in \mathscr{M}_k$ then one can write

$$|\psi_{\{\alpha\}}(k)\rangle = \sum_i |\phi_{\{\alpha^{(H)}\}}(k|i)\rangle\langle\{\bar{\alpha}^{(H)}\}|f_i\rangle \tag{10}$$

with

$$\langle\{\bar{\alpha}^{(H)}\}|f_i\rangle = \int d\alpha_{s+1} \ldots d\alpha_r \langle\phi_{\{\alpha^{(H)}\}}(k|i)|\psi_{\{\alpha\}}(k)\rangle \tag{11}$$

and then one has

$$T_g^{(L)}\langle\{\bar{\alpha}^{(H)}\}|f_i\rangle = \sum_j [T_g^{(L)}(k)]^{(i,j)}\langle\{\bar{\alpha}^{(H)}\}|f_i\rangle \tag{12}$$

where

$$[T_g^{(L)}(k)]^{(i,j)} = \int d\alpha_{s+1} \ldots d\alpha_r \langle\phi_{\{\alpha^{(H)}\}}(k|j)|_g T^{(L)}_{\{\alpha^{(H)}\}}(k|i)\rangle \tag{13}$$

being

$$|_g T^{(L)}_{\{\alpha^{(H)}\}}(k|i)\rangle = \int d\alpha_1 \ldots d\alpha_s \Big\{\Big(T_g^{(L)}\Big[\int d\alpha_{s+1} \ldots d\alpha_r \langle\phi_{\{\alpha^{(H)}\}} \times (k|i)|\psi_{\{\alpha\}}(k)\rangle\Big]\Big)\cdot|\psi_{\{\alpha\}}(k)\rangle\Big\} \tag{14}$$

So if the $T^{(L)}$ form a representation of G over $\mathscr{H}_{G'}$ then they will also be a representation of G over $\mathscr{M}_k$.

Now if $J_1, \ldots, J_r$ are the generators of the Lie algebra γ corresponding to G, and $T^{(L)}(J_1), \ldots, T^{(L)}(J_r)$ are their representations on $\mathscr{H}_G$ as left translations, then

$$[T^{(L)}(J), C_k^{(R)}] = 0 \tag{15}$$

and one may define the following k-dependent operators:

$$T^{(L)}(J\,|k) = P_k T^{(L)}(J) P_k \tag{16}$$

They have the usual commutation relations, and give, therefore, a representation of the Lie algebra γ. Now for any operator $\mathscr{H}$ contained in the Casimir subalgebra of the enveloping algebra of the left translations, one has then

$$[\mathscr{H}, T^{(L)}(J)] = [\mathscr{H}, C_k{}^{(R)}] = 0 \tag{17}$$

The first equation expresses the fact that the system described by $\mathscr{H}$ has the symmetry group G, while the second equation is the obvious consequence of the fact that left and right translations commute. Hence one can write

$$\mathscr{H}(k) = P_k \mathscr{H} P_k \tag{18}$$

and $\mathscr{H}(k)$ can now be assumed as the Hamiltonian for a system with coordinates $\alpha_1, \ldots, \alpha_s$ containing a constant k which can be interpreted as an interaction constant.

Conversely if the operator $\mathscr{H}(k)$ is given, as an operator on functions of the variables $\alpha_1, \ldots, \alpha_s$, one can determine that the operators $T^{(R)}(J)$ form a Lie algebra commuting with $\mathscr{H}(k)$, and show how this statement is true for all values of k. Such an algebra, however, cannot in general be integrated to form a Lie group, otherwise by reversing the argument one would find that $C_k{}^{(R)}$ had eigenvalues other than those permitted by the structure of the [compact] group H and the topology of the group manifold.

The requirement that there be a Lie algebra commuting with the Hamiltonian is of course weaker than the global symmetry requirement, namely of integrability to a Lie group of the Lie algebra.

Actually, in order that an element $s \in \gamma$ of a Lie algebra can be the generator of a one parameter subgroup $S(t)$ of a Lie group G it is necessary that the series

$$S(t) = e^{its} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} s^n \tag{19}$$

exist and converge. This means that, if s is a self-adjoint operator over the Hilbert space $\mathscr{H}_G$, then its domain $D(s)$ must be an invariance domain, i.e., if $|\psi\rangle \in D(s)$ then $s^n|\psi\rangle \in D(s)$ for all n; and that $\|S(t)|\psi\rangle\| < \infty$ for all $0 < t < t_0 \neq 0$. The necessary and sufficient condition for this to be so, is that the operator

$$\Delta = \sum_i s_i{}^2 \tag{20}$$

be essentially self-adjoint.

So the operator Δ can be only a member of the Casimir algebra for compact semisimple groups, because only in that case the metric of the Killing form can be chosen to be definite.

But this is the condition for the definition of the coherent state $|\xi_g\rangle$. Now if the space spanned by the eigenstates $|\phi_{\{\alpha^{(H)}\}}(k\,|j)\rangle$ is multidimensional, then usually the dimensions will vary with k. This means that not only k is quantized, but the dimensionality of the states $\langle\{\bar{\alpha}^{(H)}\}\,|f_i\rangle$ will depend on its value.

In other words, if the dimensionality of the vector space is fixed, then there is a symmetry group only if the interaction constant takes the value appropriate to the dimensionality of the space.

Now, in the cases when coherent states can be constructed it happens that the operator $U(k_0|k)$ exists such that

$$U(k_0|k)\mathscr{M}_k = \mathscr{M}_{k_0} \tag{21}$$

and hence

$$U(k_0|k')U(k'|k) = U(k_0|k) \tag{22}$$

If U exists, then under the mapping

$$T_g{}^{(L)}(k) \to U(k_0|k)T_g{}^{(L)}(k)U(k|k_0) = T_g{}^{(L)}(k_0) \tag{23}$$

and one has a family of representations $T_g{}^{(L)}(k_0)$ all acting in the single subspace $\mathscr{M}_{k_0}$. Only if k has the permitted value k_0 will such a representation be integrable and the coherent state be definable.

References

Helgason, S. (1968). *Differential Geometry and Symmetric Spaces,* Academic Press, New York.

Klauder, J. R., and Sudarshan, E. C. G. (1968). *Fundamentals of Quantum Optics,* Benjamin, New York.

Perelomov, A. M. (1972). *Communications in Mathematical Physics* 26, 222–236.

Rasetti, M. (1975). I.A.S. Preprint, submitted to *International Journal of Theoretical Physics.*