Maximizing Entropy for a Hyperbolic Flow

by

RUFUS BOWEN

Department of Mathematics
University of California
Berkeley, California 94720

Let M be a compact differentiable manifold and φ_t: $M \to M$ a differentiable one-parameter flow. A closed φ_t-invariant set $X \subset M$ containing no fixed points is called *hyperbolic* if the tangent bundle restricted to X can be written as the Whitney sum of three $D\psi_t$-invariant subbundles, $T_X M = E + E^s + E^u$, where E is the one-dimensional bundle tangent to the flow, and there are constants c, $\lambda > 0$ so that

(a) $$\|D\varphi_t(v)\| \le ce^{-\lambda t}\|v\| \text{ for } v \in E^s, t \ge 0,$$

(b) $$\|D\varphi_{-t}(u)\| \le ce^{-\lambda t}\|u\| \text{ for } u \in E^u, t \ge 0.$$

X is a *basic hyperbolic* set if, in addition,

(c) the periodic orbits contained in X are dense in X,
(d) $\varphi_t|X$ is transitive,
(e) there is an open set $U \supset X$ so that $X = \bigcap_{t \in R} \varphi_t(U)$.

These basic hyperbolic sets are the building blocks of the Axiom A flows of Smale [9]. For a transitive Anosov flow the whole manifold is one basic set.

Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X and $\mathcal{M}(\Phi, X)$ those which are φ_t-invariant for all t. For each $\mu \in \mathcal{M}(\Phi, X)$ there is a measure-theoretic entropy $h_\mu(\varphi_1)$ defined. There is also a topological entropy $h(\varphi_1)$, and a theorem of Goodwyn [10] states that $h(\varphi_1) \ge h_\mu(\varphi_1)$. We will prove the following result.

THEOREM. *Let φ_t: $X \to X$ be a basic hyperbolic set. Then there is a unique $\mu \in \mathcal{M}(\Phi, X)$ with $h_\mu(\varphi_1) = h(\varphi_1)$.*

This measure μ will be the measure μ_Φ which gives the distribution of the periodic orbits of $\varphi_t|X$ ([5] and [6]). Our result is a strengthening of a uniqueness theorem for μ_Φ proved in [6] and is the natural analogue of a theorem proved earlier for basic sets of diffeomorphisms. Our proof can be viewed as an adaptation of a proof given by Adler and Weiss [1] for a theorem of Parry [7].

We recall one definition of topological entropy [2] for a continuous map f: $Y \to Y$ on a compact metric space. For $\epsilon > 0$ and $n \ge 1$ we say a subset $E \subset Y$ is (n, ϵ)-*separated* (by f) if for distinct $x, y \in E$ there is a j with $0 \le j < n$

MATHEMATICAL SYSTEMS THEORY, Vol. 7, No. 4.

and $d(f^j(x), f^j(y)) > \epsilon$. For $K \subset Y$ compact let $s_n(\epsilon, K) < \infty$ be the maximum cardinality of any (n, ϵ)-separated set contained in K. Define

$$\bar{s}_f(\epsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \ln s_n(\epsilon, K),$$

$$h(f, K) = \lim_{\epsilon\to 0} \bar{s}_f(\epsilon, K),$$

and $h(f) = h(f, Y)$. The map f is called *h-expansive* if, for some $\epsilon > 0$,

$$h(f, \{y: d(f^j(y), f^j(x)) \leq \epsilon \text{ for all } j \geq 0\}) = 0$$

for all $x \in Y$.

Now a basic hyperbolic set either is a constant time suspension of a homeomorphism or is C-dense [5, Theorem 3.2]. Our theorem was proved earlier for the suspension case [6, Theorem 2.1]. We do not define C-density here (see [5]) but instead list some properties of a C-dense φ_t: $X \to X$.

(a) X is a finite-dimensional compact metric space.
(b) $h = h(\varphi_1) > 0$ (see [5, Theorem 4.12]).
(c) Each φ_t is an h-expansive homeomorphism of X ([3] and [5, Section 1]).
(d) Let $B_\Phi(x, \epsilon, t) = \{y \in X: d(\varphi_s(x), \varphi_s(y)) \leq \epsilon \text{ for all } s \in [0, t]\}$. There is a $\mu_\Phi \in \mathscr{M}(\Phi)$ satisfying [6, Theorem 1.3]: For each small $\epsilon > 0$ there are positive constants C_ϵ and E_ϵ so that $C_\epsilon e^{-ht} \leq \mu_\Phi(B_\Phi(x, \epsilon, t)) \leq E_\epsilon e^{-ht}$ for all $x \in X$ and all $t \geq 0$.
(e) The flow φ_t is ergodic with respect to μ_Φ [5, Theorem 5.4].

(Note: Actually (b) and (c) follow from (d)).

For f a map on X, $\mathscr{M}(f, X)$ consists of the measures $\mu \in \mathscr{M}(X)$ invariant under f; if a group G acts on X, then $\mathscr{M}(G, X)$ denotes the $\mu \in \mathscr{M}(X)$ invariant under every element of G. For $\mu \in \mathscr{M}(X)$ we let $\mathscr{B}_\mu(X)$ denote the family of finite μ-measurable partitions of X. Finally, for $\mathbf{A} \in \mathscr{B}_\mu(X)$ we write $\mathscr{A}(\mathbf{A})$ for the algebra of sets generated by $\mathbf{A}$, i.e., the collection of unions of members of $\mathbf{A}$.

We now define probabalistic entropy. For $\mathbf{A} \in \mathscr{B}_\mu(X)$ write

$$H_\mu(\mathbf{A}) = -\sum_{A\in\mathbf{A}} \mu(A) \ln \mu(A).$$

Set $\mathbf{A} \vee \mathbf{B} = \{A \cap B: A \in \mathbf{A}, B \in \mathbf{B}\}, f^{-k}\mathbf{A} = \{f^{-k}A: A \in \mathbf{A}\}$ and $\mathbf{A}_f^n = \mathbf{A} \vee f^{-1}\mathbf{A} \vee \cdots \vee f^{-n+1}\mathbf{A}$. If $\mu \in \mathscr{M}(f)$ and $\mathbf{A} \in \mathscr{B}_\mu(X)$, the limit $h_\mu(f, \mathbf{A}) = \lim_{n\to\infty} (1/n) H_\mu(\mathbf{A}_f^n)$ *exists and* $h_\mu(f, \mathbf{A}) \leq (1/n)H_\mu(\mathbf{A}_f^n)$ *for all* n (see [8]). *Finally one defines* $h_\mu(f) = \sup_{\mathbf{A}\in\mathscr{B}_\mu(X)} h_\mu(f, \mathbf{A})$.

LEMMA 1. *Suppose* f: $X \to X$ *is an h-expansive map on a compact finite-dimensional metric space. For sufficiently small* ϵ *the following is true:* $(1/n)H_\mu(\mathbf{A}) \geq h_\mu(f)$ whenever $\mu \in \mathscr{M}(f)$, $n > 0$, $j \in Z$ and $\mathbf{A} \in \mathscr{B}_\mu(X)$ satisfy $\operatorname{diam} f^k(A) < \epsilon$ for all $A \in \mathbf{A}$ and $k \in [j, j+n)$.

Proof. Since $H_\mu(\mathbf{A}) \geq h_\mu(f^n, \mathbf{A})$ and $h_\mu(f^n) = nh_\mu(f)$, it is enough to show $h_\mu(f^n, \mathbf{A}) = h_\mu(f^n)$. This statement is similar to Theorem 3.5 of [3]; in fact, for ϵ small enough to be an expansive constant for f, the proof of that theorem is easily modified to give what we want. We leave this to the reader to check.

If a group G acts on X, then for $E \subset X$ and $U \subset G$ define $\operatorname{diam}_U E = \sup_{x,y\in E} d(Ux, Uy)$. For $\mathbf{A}$ a finite partition of X let $\operatorname{diam}_U \mathbf{A} = \max_{A\in\mathbf{A}} \operatorname{diam}_U A$.

LEMMA 2. *Let G be a topological group acting continuously on a compact metric space X, $\nu \in \mathscr{M}(G)$, and U a compact subset of G containing the identity. If $B \subset X$ is ν-measurable with $gB = B$ for all $g \in G$ and $\mathbf{A}_n \in \mathscr{B}_\nu(X)$ satisfy* $\operatorname{diam}_U \mathbf{A}_n \to 0$, *then there are sets $C_n \in \mathscr{A}(\mathbf{A}_n)$ with $\nu(C_n \Delta B) \to 0$ as $n \to \infty$.*

Proof. For $\delta > 0$, pick compact sets $K_1 \subset B$ and $K_2 \subset X \setminus B$ with $\nu(B \setminus K_1) < \delta$ and $\nu((X \setminus B) \setminus K_2) < \delta$. Since UK_1 and UK_2 are disjoint compact sets (as $gB = B$ for all g), there is an $\alpha > 0$ with $d(UK_1, UK_2) > \alpha$.

If $\operatorname{diam}_U A < \alpha$, then either $A \cap K_1 = \varnothing$ or $A \cap K_2 = \varnothing$. For if $x \in A \cap K_1$ and $y \in A \cap K_2$, then

$$d(UK_1, UK_2) \leq d(Ux, Uy) \leq \operatorname{diam}_U A < \alpha.$$

For large n we have $\operatorname{diam}_U \mathbf{A}_n < \alpha$; set $C_n = \bigcup\{A \in \mathbf{A}_n\colon A \cap K_1 \neq \varnothing\}$. Then $C_n \supset K_1$ and $C_n \cap K_2 = \varnothing$. Hence

$$\begin{aligned}\nu(C_n \Delta B) &\leq \delta + \nu(C_n \Delta K_1)\\ &\leq \delta + \nu(X \setminus (K_1 \cup K_2)) \leq 3\delta.\end{aligned}$$

THEOREM. *Let $\varphi_t\colon X \to X$ be a basic hyperbolic set and $\mu \in \mathscr{M}(\Phi, X)$, Then $h_\mu(\varphi_1) = h(\varphi_1)$ if and only if $\mu = \mu_\Phi$.*

Proof. As mentioned before, we may assume $\varphi_t\colon X \to X$ is C-dense. Now $h_{\mu_\Phi}(\varphi_1) = h(\varphi_1)$ was just Theorem 5.11 of [5].

Suppose $\mu \in \mathscr{M}(\Phi)$ satisfies $h_\mu(\varphi_1) = h(\varphi_1)$ and that μ is singular with respect to μ_Φ. There is a set $B \subset X$ which is $(\mu + \mu_\Phi)$-measurable so that $\mu(B) = 1$, $\mu_\Phi(B) = 0$ and $\varphi_t B = B$ for all $t \in R$. This is well known (see the proof of Theorem 2.1 in [6] for instance).

Let $E_n \subset X$ be maximal with respect to the condition

$$B_\Phi(x, \epsilon, 2n) \cap B_\Phi(y, \epsilon, 2n) = \varnothing \text{ for } x, y \in E_n,\ x \neq y.$$

Then $X = \bigcup_{x\in E_n} B_\Phi(x, 2\epsilon, 2n)$; for were there a point in X not in this union, it could be added to E_n and without destroying the condition above. Now choose disjoint Borel sets F_x for $x \in E_n$ such that $B_\Phi(x, \epsilon, 2n) \subset F_x \subset B_\Phi(x, 2\epsilon, 2n)$ and $X = \bigcup_{x\in E_n} F_x$. Let $\mathbf{A}_n = \{\varphi_n(F_x)\colon x \in E_n\}$. Then $C_\epsilon e^{-2hn} \leq \mu_\Phi(F_x) = \mu_\Phi(\varphi_n(F_x))$. Let $G = R$ act by $tx = \varphi_t x$; let $U = [-\delta, \delta]$. Then, for small enough ϵ, $\operatorname{diam}_U \mathbf{A}_n \to 0$ as $n \to \infty$ (see 1.6 of [5]).

We now apply Lemma 2 to $\nu = \mu + \mu_\Phi$ and the B above to get $C_n \in \mathscr{A}(\mathbf{A}_n)$ with $\nu(C_n \Delta B) \to 0$. In particular, $\mu_\Phi(C_n) \to 0$. Let β_n be the number of elements of $\mathbf{A}_n$ lying in C_n; since $\mu_\Phi(C_n) \geq \beta_n C_\epsilon e^{-2hn}$, we get $\beta_n e^{-2hn} \to 0$. By Lemma 1 we have $(j = -n)$ $(1/2n) H_\mu(\mathbf{A}_n) \geq h_\mu(f) = h$ where

$$\begin{aligned}H_\mu(\mathbf{A}_n) = &- \sum_{\varphi_n(F_x) \subset C_n} \mu(F_x) \ln \mu(F_x)\\ &- \sum_{\varphi_n(F_x) \cap C_n = \varnothing} \mu(F_x) \ln \mu(F_x).\end{aligned}$$

The first sum has β_n members; the second has at most e^{2hn}/C_ϵ (this is an upper bound on card $\mathbf{A}_n$ as each member of $\mathbf{A}_n$ has measure at least $C_\epsilon e^{-2hn}$). By Jensen's inequality (if $a_1, \cdots, a_n \geq 0$ and $s = \sum a_i \leq 1$, then $-\sum a_i \ln a_i \leq s(\ln n - \ln s)$; see [8, p. 11–12]);

$$\ln e^{2hn} \leq H_\mu(\mathbf{A}_n) \leq \mu(C_n)(\ln \beta_n - \ln \mu(C_n))$$

$$+\mu(X \setminus C_n)\left(\ln \frac{1}{C_\epsilon} e^{2hn} - \ln \mu(X \setminus C_n)\right)$$

$$\leq \mu(C_n) \ln \beta_n + \mu(X \setminus C_n) \ln \frac{1}{C_\epsilon} e^{2hn} + 2k^*,$$

where k^* is the maximum value of $-x \ln x$ for $x \in [0, 1]$. Hence

$$0 \leq 2k^* + \mu(C_n) \ln \beta_n e^{-2hn} + \mu(X \setminus C_n) \ln \frac{1}{C_\epsilon}.$$

As $n \to \infty$, we have $\mu(C_n) \to 1$, $\mu(X \setminus C_n) \to 0$, and $\ln \beta_n e^{-2hn} \to -\infty$; this is a contradiction. Hence there can be no $\mu \in \mathcal{M}(\Phi)$ singular with respect to μ_Φ and $h_\mu(\varphi_1) = h(\varphi_1)$.

Now consider any $\mu \in \mathcal{M}(\Phi)$ with $h_\mu(\varphi_1) = h$. Either μ is absolutely continuous with respect to μ_Φ or $\mu = \alpha\mu_1 + \beta\mu_2$ with $\alpha > 0$, $\beta \geq 0$, $\alpha + \beta = 1$, $\mu_i \in \mathcal{M}(\Phi)$, μ_0 singular and μ_1 absolutely continuous with respect to μ_Φ. Then

$$h_\mu(\varphi_1) = \alpha h_{\mu_0}(\varphi_1) + \beta h_{\mu_1}(\varphi_1).$$

By Goodwyn's theorem (see [10] or [2]) one has $h_{\mu_i}(\varphi_1) \leq h(\varphi_1)$. The only way to have $h_\mu(\varphi_1) = h(\varphi_1)$ is to have $h_{\mu_0}(\varphi_1) = h(\varphi_1)$; but we saw already this is impossible. Hence μ is absolutely continuous with respect to μ_Φ. As μ_Φ is ergodic, it follows by the Radon-Nikodym theorem that $\mu = \mu_\Phi$.

REFERENCES

[1] R. L. ADLER and B. WEISS, Entropy, a complete metric invariant for automorphisms of the torus, *Proc. Nat. Acad. Sci. U.S.A.* **57** (1967), 1573–1576.
[2] R. BOWEN, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.* **152** (1971), 401–414.
[3] R. BOWEN, Entropy-expansive maps, *Trans. Amer. Math. Soc.* **164** (1972), 323–331.
[4] R. BOWEN, Markov partitions for Axiom A diffeomorphisms, *Amer. J. Math.* **92** (1970), 725–747.
[5] R. BOWEN, Periodic orbits for hyperbolic flows, *Amer. J. Math.* **94** (1972), 1–30.
[6] R. BOWEN, The equidistribution of closed geodesics, *Amer. J. Math.* **94** (1972), 413–423.
[7] W. PARRY, Intrinsic Markov chains, *Trans. Amer. Math. Soc.* **111** (1964), 55–66.
[8] W. PARRY, *Entropy and Generators in Ergodic Theory*, Benjamin, New York, 1969.
[9] S. SMALE, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.
[10] L. W. GOODWYN, Topological entropy bounds measure theoretic entropy, *Proc. Amer. Math. Soc.* **23** (1969), 679–688.

(*Received 15 June 1972*)