Degenerate Parabolic Equations and Harnack Inequality (*).

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Sunto. – *Viene risolto il problema di Cauchy Dirichlet relativo all'operatore parabolico degenere* $\partial u/\partial t - \partial/\partial x_i(a_{ij}(x,t)\,\partial u/\partial x_j)$, *in opportune ipotesi di integrabilità per gli autovalori di* $a_{ij}(x,t)$. *Vengono inoltre forniti controesempi circa l'impossibilità di risultati di regolarità per le soluzioni deboli mostrando in tal modo che operatori parabolici degeneri hanno un comportamento radicalmente differente da quello dei corrispondenti operatori ellittici degeneri.*

Introduction.

Degenerate elliptic and parabolic partial differential equations have been extensively studied in the last 10-15 years.

In particular, for elliptic operators of the form:

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),\qquad \lambda^{-1}\omega(x)|\xi|^2\leqq a_{ij}(x)\,\xi_i\xi_j\leqq\lambda\omega(x)|\xi|^2\,,\qquad \forall\xi\in R^m \tag{0.1}$$

it was clear, since 10 years ago, that some local assumptions on $\omega(x)$ (as the ones given in [T_1] or [T_2] and more or less implicitely assumed in [M-S] (see also [M-S]$_{\text{bis}}$)) were needed in order to get local Hölder continuity of the solutions. More precisely these authors assume that:

$$\sup_C\left(\frac{1}{|C|}\int_C[\omega(x)]^s\,dx\right)^{1/s}\left(\frac{1}{|C|}\int_C\frac{1}{[\omega(x)]^t}\,dx\right)^{1/t}\leqq K<+\infty \tag{0.2}$$

where C is any cube in R^m and $s, t>0$, $1/s+1/t<2/m$.

In the recent paper [F-K-S] these assumptions have been significantly weakened. In this work the weight $\omega(x)$ giving the degeneracy of the equation can be assumed only to satisfy an A_2 condition, that is:

$$\sup_C\left(\frac{1}{|C|}\int_C\omega(x)\,dx\right)\left(\frac{1}{|C|}\int_C\frac{1}{\omega(x)}\,dx\right)\leqq K<+\infty \tag{0.3}$$

C cube any in R^m.

The theory appears considerably less advanced for parabolic degenerate equations and in fact results such as the Harnack inequality are in general false *on the*

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usual parabolic cylinders, $Q_\varrho = \{(x,t)\colon |x-x_0|<\varrho,\ |t-t_0|<\varrho^2\}$, with a constant independent of ϱ (for more precise statements and examples see next section).

The parabolic degenerate operators we consider are of the form:

$$\frac{\partial}{\partial t}-\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right); \qquad \lambda^{-1}\omega(x,t)|\xi|^2\leq a_{ij}\xi_i\xi_j\leq\lambda\omega(x,t)|\xi|^2, \qquad \forall\xi\in R^m . \tag{0.4}$$

Let us briefly recall some previous results for these degenerate parabolic equations.

The first we wish to mention is the work by KRUZHKOV and KOLODII [K-K], in which a Harnack inequality is proven on the usual parabolic cylinders. The constant in this inequality depends on the sum of two averages like the ones appearing in (0.2) (the averages are this time on a parabolic cylinder) so that if one wants to get the inequality on *all* the parabolic cylinders (this has to be done e.g. in order to deduce from it the local Hölder continuity of solutions) one actually needs the *non-degeneracy* of the equation. In the work of KRUZHKOV and KOLODII no attention is dedicated to the study of the existence of solutions and moreover the regularity is studied assuming the solutions have a square integrable time derivative.

In his papers [Iv-1], [Iv-2], [Iv-3] A. V. IVANOV studied degenerate parabolic equations from the point of view of existence theory and also considered questions of regularity such as local Hölder continuity of solutions.

In the first a Harnack inequality is stated for solutions of a degenerate equation that have a strong L^2 derivative with respect to time. However under his assumption on the weight his statement is incorrect as our examples (see the next section) indicate. In [Iv-2] an existence theorem for the Cauchy-Dirichlet problem is shown but the continuity in t of these solutions in the L^2 norm (i.e. the continuity of the application $t\to\int_\Omega u^2(x,t)\,dx$ for $t\in[0,T]$) is not, in general, proven.

In the third paper (containing only the statements of the theorems) an hypothesis on the weight is assumed that implies in fact both the maximum and minimum eigenvalue can be supposed to be time independent. Under this hypothesis the strong L^2 continuity of the solutions of the first boundary problem is obtained; a Harnack inequality and local Hölder continuity result for solutions is also stated but again these regularity results are incorrectly stated.

In the end we would like to quote the papers [N-1], [N-2] in which more general boundary problems are studied for the complete parabolic equation. In these papers the assumption is that the weight is of a particular form, namely, $\omega(x,t)=\omega_1(x)\omega_2(t)$, $\omega_2(t)$ increasing and in the study of the strong L^2 continuity of the solutions $\omega_2(t)$ must be supposed to be bounded away from zero.

Trying now to summarize, the state of art for parabolic divergence form degenerate equations with non smooth coefficients (at least for what we know) is as follows:

1) no strong L^2 continuity is known for solutions of the Cauchy-Dirichlet problem for really time dependent and time degenerate equations;

2) no Harnack (with cylinder independent constant) or local Hölder continuity result is known for these equations for *any* kind of degeneracy.

In our study of parabolic degenerate equations our aim was to extend as far as possible the results in [F-K-S] to the parabolic case.

We are not aiming to the largest generality so that we will restrain ourselves, for the time being, to the equation:

$$(*) \qquad -\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right)+\frac{\partial u}{\partial t}=f$$

where we assume: $\lambda^{-1}\omega(x,t)|\xi|^2\leqq a_{ij}(x,t)\xi_i\xi_j\leqq\lambda\omega(x,t)|\xi|^2$, $\forall\xi\in R^m$, a.e. in a cylinder $Q=\Omega\times]0,T[$, Ω bounded open set $\subseteq R^m$. For this equation we will study only the Cauchy-Dirichlet problem.

The assumption we make on the weight $\omega(x,t)$ is an A_2 condition in the space variable uniformly with respect to time and the same in the time variable uniformly with respect to space (for precise statements see sec. 2).

This implies an A_2 global condition but, of course, is stronger than such a condition. What happens is that if one wants to use the Steklov averages ($S_h u = 1/h\int_t^{t+h}u(x,\tau)\,d\tau$), a very convenient device to study weak solutions of parabolic equations, an A_2 condition in time uniformly with respect to x turns out to be necessary.

On the other side under these assumptions we succeded to extend the global theory to the parabolic degenerate equation which includes the L^2 continuity of solutions to the Cauchy-Dirichlet problem but we couldn't get any kind of local regularity results. But in fact we found (see sec. 1) that even local boundedness is false under our hypothesis. Moreover, as we already mentioned, we found that also improving the assumptions on the weight it is impossible to get the «usual» Harnack inequality.

This paper is divided in three sections.

In sect. 1 some counterexamples related to the impossibility of local L^∞ estimates for our equation and the impossibility of the Harnack inequality are collected.

In sect. 2 we introduce various functional spaces and study the global properties of solutions of degenerate parabolic equations.

Sect. 3 is devoted to the proofs of a Sobolev-like embedding theorem and of a denseness result useful to achieve the L^2 continuity of the solutions.

In a following paper we will prove, under stronger integrability hypothesis for $1/\omega$, the continuity of the solutions of $(*)$ using a variant of the usual Harnack inequality.

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1. – Counterexamples.

The counterexamples in this section concern the solutions of the parabolic equation $(*)$ in the introduction with $f = 0$. The assumption on the coefficients is the same as in $(*)$ if not otherwise stated. A feature of this equation is the lack of local L^∞ estimates while under the same hypothesis in the elliptic case they hold true (see [F-K-S]). This is shown by the following simple example:

EXAMPLE 1.1. – Let

$$Q = B_1(0)\times(0,1)\ (^1)\,, \qquad \omega(x,t)\equiv\omega(x) = |x|^2\,, \qquad u(x,t) = \exp\left[-\alpha(m-\alpha)t\right]|x|^{-\alpha}$$

where α is a small positive number. Then, for convenient α, $|\nabla u|^2\omega\in L^1(Q)$ and u solves: $\operatorname{div}\big(\omega(x)\nabla u\big) = u_t$ in Q (see sec. 2 for precise definitions). Furthermore $|x|^2$ is an A_2 weight in R^m if $m\geqq 3$. Obviously $u(x,t)$ is unbounded in Q and, for small values of α, $u(x,0)\in L^p(\Omega)$ for any fixed value of p.

The bad behavior of equation $(*)$ forced us to abandon the hope of proving any kind of local regularity result. However we conjectured that bounded (resp. continuous) initial data u_0 would give continuous (resp. Hölder continuous) solutions.

This too turned out to be false as it is shown by the following examples:

EXAMPLE 1.2 (*Bounded u_0 doesn't imply continuous u*). – Let

$$Q = B_1(0)\times(0,1)\,, \qquad \omega(x) = |x|^2\,, \qquad u(x,t) = \frac{\sum_{i=1}^m x_i}{|x|}\exp\left[(1-m)t\right].$$

Then $|\nabla u|^2\omega\in L^1(Q)$ and u is a solution of $\operatorname{div}\big(\omega(x)\nabla u\big) = u_t$, bounded for any t, but it is not continuous in Q.

EXAMPLE 1.3 (*Continuous u_0 doesn't imply Hölder continuous u*). – Let

$$Q = B_{\frac12}(0)\times(0,1)\,, \qquad \omega(x) = -\lambda|x|^{2-m}(\log|x|)^2\int_0^{|x|} s^{m-1}(\log s)^{-1}\,ds \quad (\lambda>0)\,,$$

$$u(x,t) = -\frac{\exp[\lambda t]}{\log|x|}\,.$$

Then $\omega(x)$ is an A_2 weight, $|\nabla u|^2\omega\in L^1(Q)$ and u solves $\operatorname{div}\big(\omega(x)\nabla u\big) = u_t$.

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(1) $B_r(x_0)$ is the open ball: $\{x\in R^m\colon |x-x_0|<r\}$.

Moreover $u(x, 0)$ is continuous but u isn't Hölder continuous.

Let us now give some further counterexamples concerning the Harnack inequality. The first refers to the impossibility of a Harnack inequality for degenerate operators without local assumptions on the weight. Let us make precise what we mean with « Harnack inequality on the parabolic cylinders » (H.I.P.C.).

We consider solutions u of the equation:

$$-\sum_{ij}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right)+u_t=0 \quad \text{in a cylinder} \quad Q=\Omega\times(0,T)\,.$$

$u(x,t)$ is supposed to be non negative in a cylinder $Q_\varrho(x_0,t_0)=B_\varrho(x_0)\times(t_0-\varrho^2, t_0+\varrho^2)$. Let $Q_\varrho^+=B_{\varrho/2}(x_0)\times(t_0+\alpha\varrho^2, t_0+\varrho^2)$ and $Q_\varrho^-=B_{\varrho/2}(x_0)\times(t_0-\beta\varrho^2, t_0-\gamma\varrho^2)$ where: $0<\alpha<1$, $0<\gamma<\beta<1$ are fixed numbers.

Then we will say that a H.I.P.C. holds if it exists a constant K *independent* from ϱ (and, obviously, from u) such that:

$$\operatorname*{ess\,sup}_{Q_\varrho^-} u(x,t) \leqq K \operatorname*{ess\,inf}_{Q_\varrho^+} u(x,t)\,. \tag{1.1}$$

EXAMPLE 1.4. - Let $\omega(x,t)=1$ for $x_1\in(-1,0]$, $|x_i|<1$, $i=2,\dots,m$, $t\in(-1,1)$; $\omega(x,t)=x_1^\sigma$ for $x_1\in(0,1)$, $|x_i|<1$, $i=2,\dots,m$, $t\in(-1,1)$; σ a given number in $(0,1)$.

We consider in $Q=B_1(0)\times(0,1)$ the degenerate parabolic equation:

$$-\frac{\partial}{\partial x_i}\left(\omega(x,t)\frac{\partial u}{\partial x_j}\right)+u_t=0\,. \tag{1.2}$$

Given any $p\geqq 1$, σ can be chosen so small that both ω and $\omega^{-1}\in L^p(Q)$.

Given any ϱ greater than zero consider:

$$u(x,t)=u(x_1)=\int_{-\varrho}^{x_1}\frac{ds}{\omega(s)}\,.$$

This function is a solution of (1.2), non negative in $Q_\varrho(0,0)$ and such that

$$\operatorname*{ess\,inf}_{Q_\varrho^+} u(x,t)=\frac{\varrho}{2}, \qquad \operatorname*{ess\,sup}_{Q_\varrho^-} u(x,t)=\left(\frac{\varrho}{2}\right)^{1-\sigma}\cdot\frac{1}{1-\sigma}+\varrho$$

so that (1.1) cannot be true for all the values of ϱ.

The interest of this example is in the fact that it is an essentially elliptic one so that it somehow proves that for both elliptic and parabolic degenerate operators integrability conditions only are not enough for the validity of (1.1) and some kind of « local » condition is needed.

The next counterexample shows the impossiblity of an H.I.P.C. for any kind of degenerate parabolic equations.

EXAMPLE 1.5 (*No Harnack*). - Let $\omega(x, t) = -\log|t|$ in $Q = (-1, 1)^m \times (-\frac{1}{2}, \frac{1}{2})$; $\omega(x, t)$ and $\omega^{-1} \in L^p(Q)$ for any $p \in [1, +\infty)$. However, unlike the weight in Example 1.4, $\omega(x, t)$ satisfies any kind of local conditions like the A_p conditions (see sec. 2) or the Murthy-Stampacchia-Trudinger conditions (see [M-S]$_{\text{bis}}$, [Tr 1], [Tr 2]) that we quoted in the Introduction.

Consider $Q_\varrho(0, 0)$ and $u = \exp[t(\log|t| - 1)]\cos x - \exp[-\varrho^2(1 - \log\varrho^2)]\cos\varrho$. u is a solution of (1.2) for $\omega(x, t) = -\log|t|$ in Q, non negative in Q_ϱ.

Now (1.1) reads:

$$\exp[\beta\varrho^2(1 - \log\beta\varrho^2)] - \exp[-\varrho^2(1 - \log\varrho^2)]\cos\varrho \leq K \cdot$$
$$\cdot\left[\exp[-\varrho^2(1 - \log\varrho^2)]\cos\frac{\varrho}{2} - \exp[-\varrho^2(1 - \log\varrho^2)]\cos\varrho\right]$$

that cannot be true for all the values of ϱ.

In this example we started from a solution of the heat equation changing the t variable, so that an Harnack inequality is still true but on cylinders that are no longer of the form « (ϱ, ϱ^2) ».

With some more effort, but basically with the same idea in mind, one can construct an example of a time independent weight such that the non negative solutions of the corresponding equation don't satisfy H.I.P.C. The outline of the construction follows.

EXAMPLE 1.6 (*No Harnack, time independent*). - Consider the equation: $u_t = \Delta u + V \cdot \nabla u$, where V is a vector function whose components V_i are assumed conveniently integrable. For such an equation it is well known that the fundamental solution's behavior is the same as for the equation $u_t = \Delta u$. From this one can show that only on the usual parabolic cylinders an Harnack inequality can be true.

This assumed we consider the (time independent) transformation $\mathcal{T}x = x/|x|^\alpha$ where α is a number conveniently close to zero.

By this change of space variables our equation is transformed in:

$$|J| u_t = \operatorname{div}(\mathcal{A}\nabla u) + |J|\tilde{V} \cdot J^{-1} \cdot \nabla u \,. \tag{1.3}$$

Here: J is the jacobian matrix of $\mathcal{T}$ whose determinant is $|J|$; $\mathcal{A} = |J|(J^{-1})^*(J^{-1})$ ($(J^{-1})^*$ denotes the transpose of J^{-1}) and $\tilde{V}(x) = V(Tx)$. (1.3) can be written as:

$$u_t = \operatorname{div}\frac{\mathcal{A}}{|J|} \cdot \nabla u - \nabla(|J|^{-1})\,\mathcal{A} \cdot \nabla u + \tilde{V} \cdot J^{-1} \cdot \nabla u \,. \tag{1.4}$$

Now $\mathcal{A}/|J|$ is a definite positive matrix whose eigenvalues are of the order of $|x|^{2\alpha}$.

We look for a $\tilde{V}(x)$ of the form $\varphi(|x|)x$ such that the coefficient of ∇u is zero.

This φ can be found so that the original $V(x)$ is of the form: const $\cdot\, |x|^{\alpha^2 - 2}x$ which, for $|\alpha|$ close to zero, can be taken arbitrarily integrable.

With this choice of V the solutions of (1.4) satisfy an Harnack inequality *only* on the cylinders corresponding through the $\boldsymbol{T}$ to the standard ones.

2. – The degenerate parabolic equation.

We start giving some definitions and introducing some function spaces we will need in the following.

Let Ω be an open set in R^m, $(a, b) \subseteq R$. Let $Q = \Omega \times (a, b)$. We will say that a real, measurable, non negative function defined in Q is an $A_p(1 < p < +\infty)$ weight in Ω, uniformly with respect to t in (a, b), if:

$$(2.1)\qquad \operatorname*{esssup}_{t\in(a,b)} \sup_{C} \left(\frac{1}{C}\int_C \omega(x, t)\, dx\right)\left(\frac{1}{|C|}\int_C \omega(x, t)^{-1/(p-1)}\, dx\right)^{p-1} = M < +\infty .$$

Here the supremum is taken on all the m-dimensional cubes C contained in Ω. A_p weights in (a, b) uniformly with respect to x in Ω are defined in an analogous way. (2.1) is the usual definition of A_p weights for time independent weights.

Two fundamental properties of A_p weights are given by the following theorems:

THEOREM 2.1 (Reversed Hölder Inequality; [C-F] Theor. IV, Lemma 2). – *Let ω belong to A_p in R^m. Then the inequality*:

$$(2.2)\qquad \left(\frac{1}{|C|}\int_C \omega(x)^{1+\delta}\, dx\right)^{1/(1+\delta)} \leq K_0\left(\frac{1}{|C|}\int_C \omega(x)\, dx\right)$$

holds for all cubes C, with constants K_0, $\delta > 0$ dependent only on the A_p constant of $\omega(x)$.

From this follows that if $\omega(x) \in A_p$ then $\omega \in A_{p-\varepsilon}$ for some positive ε.

Let us now consider $f \in L^1_{\text{loc}}(R^m)$. We will denote:

$$M[f(x)] = \sup_{\{C:\, x\in C\}} \frac{1}{|C|}\int_C |f(y)|\, dy$$

its maximal function. Then the following result holds:

THEOREM 2.2 ([M]; [C.F.]). – *Suppose $\omega(x) \in L^1_{\text{loc}}(R^m)$ is a non negative function, $1 < p < +\infty$. The inequality*:

$$(2.3)\qquad \left(\int_{R^m} [M(f(x))]^p \omega(x)\, dx\right)^{1/p} \leq K_1\left(\int_{R^m} (f(x))^p \omega(x)\, dx\right)^{1/p}$$

holds for any $f \in L^p(R^m; \omega)$ (2), *if and only if* $\omega(x)$ *is an* A_p *weight.* K_1 *depends only on the* A_p *constant of* $\omega(x)$.

REMARK 2.1. – Let $\omega(x, t)$ be an A_p weight in (a, b) uniformly with respect to $x \in R^m$. Then it is easy to prove that the extended weight:

$$\tilde{\omega}(x, t) = \omega(x, t)\,, \quad t \in (a, b); \qquad \tilde{\omega}(x, t) = \omega(x, -t + 2a)\,, \quad t \in (2a - b, a);$$

$$\omega(x, t) = (x, -t + 2b)\,, \quad t \in (b, 2b - a); \qquad \text{etc.}$$

is still an A_p weight in R uniformly with respect to $x \in R^m$. The A_p constants of ω and $\tilde{\omega}$ are comparable.

Let T be a positive number and $\omega(x, t)$ an A_2 weight in R^m, uniformly with respect to t in $(0, T)$ and an A_2 weight in $(0, T)$ uniformly with respect to x in R^m. In all what follows we will assume that ω has been extended as in the above remark. We will denote:

$V_\omega(Q)$ *the space of the functions* $u \in L^2(Q; \omega)$ *whose distributional derivatives with respect to the space variables* $x_1, \ldots, x_m$ *belong to* $L^2(Q; \omega)$.

Endowed with the scalar product: $(u, v)_{V_\omega} = \int_Q \{uv + u_{x_i} v_{x_i}\}\omega\, dx\, dt$ V_ω is a Hilbert space.

$\mathring{V}_\omega(Q)$ *is the closure of* $\mathcal{D}(Q)$ *in* $V_\omega(Q)$. If Ω is bounded the inner product: $(u, v)_{\mathring{V}_\omega(Q)} = \int_Q u_{x_i} v_{x_i} \omega\, dx\, dt$, turns $\mathring{V}_\omega(Q)$ into a Hilbert space whose norm is equivalent to $\|v\|_{V_\omega(Q)}$. Let us remark that if $v(x, t) \in \mathring{V}_\omega(Q)$ then for a.e. $t \in (0, T)$ it belongs to $H_0^1(\Omega; \omega(\cdot, t))$ and

$$\|v\|_{\mathring{V}_\omega(Q)} = \left(\int_0^T \|v(\cdot, t)\|^2_{H_0^1(\Omega; \omega(\cdot, t))}\, dt \right)^{\frac{1}{2}} \text{ (3)}.$$

$V'_\omega(Q)$ *is the dual space of* $\mathring{V}_\omega(Q)$. $V'_\omega(Q)$ is a Hilbert space and a subspace of $\mathcal{D}'(Q)$. Let us observe that $F \in \mathcal{D}'(Q)$ is in $V'_\omega(Q)$ iff $f_1, \ldots, f_m$ exist, $f_i \in L^2(Q; 1/\omega)$

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(2) If Ω is an open set in R^m, ω is a non negative function, $\omega \in L^1_{\text{loc}}(\Omega)$, we will denote $L^p(\Omega; \omega)$ the space of (the classes of) real, Lebesgue measurable functions, such that: $\int_\Omega |f(x)|^p \omega(x)\, dx < +\infty$. Endowed with the norm: $\|f\|_{L^p(\Omega;\, \omega)} = \left(\int_\Omega |f|^p \omega(x)\, dx \right)^{1/p}$, $L^p(\Omega; \omega)$ is a Banach space.

(3) For all values t such that $\omega(\cdot, t)$ is an A_2 weight in Ω, $H_0^1(\Omega; \omega(\cdot, t))$ is the completion of $\mathcal{D}(\Omega)$ with respect to the norm:

$$\|v\|_{H_0^1(\Omega;\, \omega(\cdot, t))} = \left(\int_\Omega v_{x_i}^2\, \omega(x, t)\, dx \right)^{\frac{1}{2}}.$$

such that: $F = -\,\partial f_i/\partial x_i$ is in $\mathcal{D}'(Q)$. In fact if $F \in V'_\omega(Q)$, by the Riesz representation theorem, an $f \in \mathring{V}_\omega(Q)$ exists such that $\langle F, \varphi\rangle = (f, \varphi)_{\mathring{V}_\omega(Q)}$, $\forall \varphi \in \mathcal{D}(Q)$. Then:

$$\langle F, \varphi\rangle = -\left\langle \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\,\omega\right), \varphi\right\rangle, \quad \forall \varphi \in \mathcal{D}(Q)\,.$$

Letting $f_i = (\partial f/\partial x_i)\omega$ one gets the conclusion. Also $\|F\|_{V'_\omega(Q)} = \|f\|_{\mathring{V}_\omega(Q)}$. The distribution (on Ω)

$$F(t) = \frac{\partial}{\partial x_i}\left(\frac{\partial f(\cdot, t)}{\partial x_i}\,\omega(\cdot, t)\right) \in H^{-1}\left(\Omega; \frac{1}{\omega(\cdot, t)}\right) \ (^4) \quad \text{for a.e. } t \in (0, T)$$

and it is easy to see that: $\langle F, \varphi\rangle = \int_0^T {}_{-1}\langle F(t), \varphi(t)\rangle_1\, dt$, $\forall F \in V'_\omega(Q)$, $\forall \varphi \in \mathring{V}_\omega(Q)$ and:

$$\|F\|^2_{V'_\omega} = \int_0^T \|F(t)\|^2_{H^{-1}(\Omega;\,1/\omega(\cdot, t))}\, dt\,.$$

$W(Q)$ is the space of the functions $u \in \mathring{V}_\omega(Q)$ s.t. u_t (in $\mathcal{D}'(Q)$) belongs to $V'_\omega(Q)$. Endowed with the scalar product, $(u, v)_W = (u, v)_{\mathring{V}_\omega} + (u_t, v_t)_{V'_\omega}$: W is a Hilbert space.

REMARK 2.2. – If $u \in W(Q)$ it is possible to find an extension $\tilde{u}$ of u s.t. $\tilde{u} \in W(\Omega \times R)$ (this being considered with respect to the weight extended as in the Remark 2.1).

PROOF. – Let $\varphi \in C_0^\infty(\Omega \times R)$, $\varphi = 1$ in $\Omega \times [0, T]$, $\varphi = 0$ in $\Omega \times \{R \setminus (-T/2, \frac{3}{2}T)\}$, $0 \leq \varphi \leq 1$. Define: $u_1(x, t) = u(x, -t)$ in $(-T, 0)$, $u_1(x, t) = u(x, 2T - t)$ in $(T, 2T)$, etc. Let $\tilde{u} = u_1\varphi$. Then $\tilde{u} \in W(\Omega \times R)$, $\operatorname{supp} \tilde{u} \subset \bar{\Omega} \times [-T/2, \frac{3}{2}T]$ and

$$\|\tilde{u}(x, t)\|_{W(\Omega \times R)} \leq K(T)\|u\|_{W(Q)}\,.$$

Two essential, but somewhat technical, lemmas are now stated. Their proof will be given in Sec. 3.

LEMMA 2.1. – *The subspace of the* $C^\infty(\tilde{Q})$ (5) *functions with compact carrier in* Ω *for any* $t \in R$ *is dense in* $W(\tilde{Q})$. *This implies*: $W(Q) \hookrightarrow C^0([0, T]; L^2(\Omega))$.

LEMMA 2.2. – *Let* $u \in \mathring{V}_\omega(Q) \cap C^0([0, T]; L^2(\Omega))$. *Then exist constants* $K > 0$ *and* $l > 1$ *(*K *depending only on the* A_2 *constant in the space variable,* Ω *and* m; l *depending*

(4) $H^{-1}(\Omega; 1/\omega(\cdot, t))$ is the dual space of $H^1_0(\Omega; \omega(\cdot, t))$ for all the t's for which the latter exists. We will denote: ${}_{-1}\langle\cdot,\cdot\rangle_1$ the duality between $H^{-1}(\Omega; 1/\omega(\cdot, t))$ and $H^1_0(\Omega; \omega(\cdot, t))$.

(5) $\tilde{Q} = \Omega \times R$.

on the A_2 constant in the space variable and m) such that:

$$\|u\|_{L^{2}(Q;\,\omega)} \leqq \left(K \max_{[0,T]}\left(\int_{\Omega} u^2\,dx\right)^{\frac{1}{2}} + \|u\|_{\mathring{V}_{\omega}(Q)}\right).$$

Finally: $W_*(Q)$ *is the space of the* $u \in \mathring{V}_{\omega}(Q)$ *such that* $u_t \in L^2(Q; 1/\omega)$.

$W_*(Q)$ is a Hilbert space with the inner product: $(u, v)_W = (u, v)_{\mathring{V}_\omega} + (u_t, v_t)_{L^2(Q;1/\omega)}$. Lemma 2.1 still holds for $W_*(Q)$. Also: if $u \in W_*$ and $t_0 \in [0, T]$ exists s.t. $u(x, t_0) = 0$ then $u \in L^2(Q; 1/\omega)$. In fact:

$$\iint_Q u^2 \frac{1}{\omega}\,dx\,dt \leqq \int_\Omega \left(\int_0^T (u_t(x,t))^2 \frac{1}{\omega}\,dt\right)\left(\int_0^T \omega(x,t)\,dt\right)\left(\int_0^T \frac{1}{\omega(x,t)}\,dt\right)dx\,.$$

Before turning to the study of the equation let us introduce for $u \in L^1_{\mathrm{loc}}(Q)$ the Steklov averages:

$$S_h u \equiv u_h \equiv \frac{1}{h}\int_t^{t+h} u(x,s)\,ds \qquad \text{and} \qquad S_{\bar h} u \equiv u_{\bar h} \equiv \frac{1}{h}\int_{t-h}^{t} u(x,s)\,ds \ \ (^6)\,.$$

These averages are needed in the proof of some basic estimates and of Lemma 2.1. Many important continuity properties of these regularizing operators are stated and proved in Sec. 3.

We now consider in $Q = \Omega \times (0, T)$ (Ω bounded open set in R^m, $T > 0$) the divergence form parabolic equation:

$$Lu - u_t \equiv \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) - \frac{\partial u}{\partial t} = f\,. \tag{2.4}$$

We assume that the coefficients $a_{ij}(x, t)$ are measurable functions a.e. defined in Q and fulfill the following:

$$\begin{cases} a_{ij}(x,t) = a_{ji}(x,t)\,, \quad i, j = 1, \ldots, m \\ \exists \lambda > 0 \ \text{such that:}\ \dfrac{1}{\lambda}\,\omega(x,t)|\xi|^2 \leqq a_{ij}(x,t)\xi_i\xi_j \leqq \lambda\omega(x,t)|\xi|^2\,, \\ \hfill \text{a.e. in } Q\,, \quad \forall \xi \in R^m \end{cases} \tag{2.5}$$

where $\omega(x, t)$ is an A_2 weight in R^m uniformly with respect to t in $(0, T)$ and an A_2 weight in $(0, T)$ uniformly with respect to x in Ω. We also assume that $f \in V'_\omega(Q)$.

---

(6) Now and in what follows we assume that ω and u have been extended to the whole $\Omega \times R$ according to Remarks 2.1-2.2.

DEFINITION 2.1. – *We say that* $u \in V_\omega(Q)$ *is a solution of* (2.4) *in* Q *if*:

$$\int_Q \left\{ a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} - u \frac{\partial \varphi}{\partial t} \right\} dx\, dt = -\langle f, \varphi \rangle , \qquad \forall \varphi \in W_*(Q),\ \varphi(0) = \varphi(T) = 0 . \tag{2.6}$$

DEFINITION 2.2. – *Given* $u_0(x) \in L^2(\Omega)$ *we say that* $u \in \overset{\circ}{V}_\omega(Q)$ *is a solution to the Cauchy-Dirichlet problem* (*with* u_0 *Cauchy data, and homogeneous Dirichlet data*) *for the equation* (2.4) *if*:

$$\int_Q \left\{ a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} - u \frac{\partial \varphi}{\partial t} \right\} dx\, dt = -\langle f, \varphi \rangle + \int_\Omega u_0(x) \varphi(x, t)\, dx , \qquad \forall \varphi \in W,\ \varphi(T) = 0 . \tag{2.7}$$ [7]

We can now state the following:

THEOREM 2.3. – *Assume* (2.5) *holds. Let* $u_0(x) \in L^2(\Omega)$ *and* $f \in V'(Q)$. *Then there exists a unique solution* u *to the problem* (2.7). *Moreover* $u \in C^0([0, T]; L^2(\Omega))$ *and* $\lim_{t \to 0} u(x, t) = u_0(x)$ *in* $L^2(\Omega)$.

Let us remark that, once Theorem 2.3 has been proved, it is also possible to solve the Cauchy-Dirichlet problem with non-homogeneous Dirichlet data. More precisely we have:

COROLLARY 2.1. – *Assume* (2.5) *holds. Let* $u_0(x) \in L^2(\Omega)$ *and* $f \in V'_\omega$. *Moreover let* $g(x, t) \in V_\omega$, $g_t(x, t) \in V'_\omega$ *and* $g \in C^0([0, T]; L^2(\Omega))$.

Then there exists a unique $u \in V_\omega$ *s.t.*:

i) u is a solution of (2.4) (Definition 2.1);

ii) $u(\cdot, t) - g(\cdot, t) \in H^1_0(\Omega; \omega(\cdot, t))$ a.e. in $(0, T)$;

iii) u is $C^0([0, T]; L^2(\Omega))$ and $\lim_{t \to 0} u(x, t) = u_0(x)$ in $L^2(\Omega)$.

PROOF OF THEOREM 2.3. – Let us consider $\varphi(x, t) \in C^\infty(\overline{Q})$, $\varphi(x, t)$ compactly supported in Ω for all t. For such φ we have:

$$2 \cdot {}_{V'_\omega}\langle \varphi_t(x, t), \varphi(x, t) \rangle_{\overset{\circ}{V}_\omega} = \int_0^T \left(\frac{d}{dt} \int_\Omega \varphi^2(x, t)\, dx \right) dt = \int_\Omega \varphi^2(x, T)\, dx - \int_\Omega \varphi^2(x, 0)\, dx . \tag{2.8}$$

The above class of functions φ forms a dense subset of W, the space where the solution u of the problem (2.7) is to belong.

---

[7] We should write: $\langle \varphi_t, u \rangle$ in the place of $\int_Q u \varphi_t\, dx\, dt$, the braces standing for the duality between $\overset{\circ}{V}_\omega(Q)$ and $V'_\omega(Q)$.

So that we can get an equality as (2.8) for u.

Let us define in $W\times W$ the two forms:

$$\mathcal{U}(u, v) = -\langle u, v_t\rangle + \int_0^T \int_\Omega a_{ij}(x, t) u_{x_i}(x, t) v_{x_j}(x, t)\, dx\, dt$$

and:

$$\mathcal{U}'(u, v) = \langle u_t, v\rangle + \int_0^T \int_\Omega a_{ij}(x, t) u_{x_i}(x, t) v_{x_j}(x, t)\, dx\, dt\,.$$

We can now derive using (2.5) and (2.8):

$$\frac{2}{\lambda} \|u(x, t)\|_{\overset{\circ}{V}_\omega(Q)} + \int_\Omega u^2(x, 0)\, dx \leqq 2\mathcal{U}(u, u) + \int_\Omega u^2(x, T)\, dx$$

$$\frac{2}{\lambda} \|u(x, t)\|_{\overset{\circ}{V}_\omega(Q)} + \int_\Omega u^2(x, T)\, dx \leqq 2\mathcal{U}'(u, u) + \int_\Omega u^2(x, 0)\, dx$$

for all $u \in W$.

The conclusion is obtained using standard techniques (see [T], p. 402 ff) from a theorem of J. L. LIONS [L].

Let us remark that the continuity result in Theorem 2.3 follows from the belonging of u (the solution of the Cauchy-Dirichlet problem (2.7)) to the class W recalling Lemma 2.1. For more details and the proof of the Lemma 2.1 see Sec. 3.

We now study the boundedness properties of the solutions of (2.4)

THEOREM 2.4. - *Let $u(x, t)$ be a solution of problem* (2.7) *and assume that*:

$$\operatorname*{ess\,sup}_\Omega |u_0(x)| = \tilde{K} < +\infty\,.$$

Let us also assume that $f \equiv -(f_i)_{x_i}$ be s.t. $f_i/\omega \in L^r(Q; \omega)$ with $r > 2l/(l-1)$, l being the same as in Lemma 2.2.

Then

$$\operatorname*{ess\,sup}_Q |u(x, t)| \leqq \operatorname*{ess\,sup}_\Omega |u_0(x)| + C \left\|\frac{f}{\omega}\right\|_{L^r(Q;\,\omega)}$$

C being a constant depending on Ω, T, the A_2 constants of ω and r.

PROOF. - Let $\tilde{\varphi} \in W_*(\Omega\times]-h, T[)$, $\tilde{\varphi}(x, t) = 0$ for $t \leqq 0$ and $t \geqq T - h$ $(h > 0)$. Let us consider $S_{\bar{h}}\tilde{\varphi}$. $S_{\bar{h}}\tilde{\varphi} \in W_*(Q)$ (see Sec. 3) and $S_{\bar{h}}\tilde{\varphi}(x, 0) = S_{\bar{h}}\tilde{\varphi}(x, T) = 0$.

Then, by the Fubini's theorem,

$$-\int_Q u(S_{\bar h}\tilde\varphi)_t\,dx\,dt=-\int_Q (S_h u)\tilde\varphi_t\,dx\,dt=\int_Q (S_h u)_t\tilde\varphi\,dx\,dt \tag{2.9}$$

so that we get from (2.6)

$$\int_Q\{S_h(a_{ij}u_{x_i})\tilde\varphi_{x_j}+(S_h u)_t\tilde\varphi\}\,dx\,dt=\int_Q (S_h f_i)\tilde\varphi_{x_i}\,dx\,dt\,. \tag{2.10}$$

It is easy to check (see e.g. [L.S.U.], p. 142) that (2.10) holds for any $\varphi\in\mathring{V}_\omega$ that vanishes for $t>\tau$, where τ is any number $\leqq T-h$.

Hence we have

$$\int_{Q_\tau}\{S_h(a_{ij}u_{x_i})\varphi_{x_j}+(S_h u)_t\varphi\}\,dx\,dt=\int_{Q_\tau}(S_h f_i)\varphi_{x_i}\,dx\,dt \;(^8) \tag{2.11}$$

for any $\varphi\in\mathring{V}_\omega(Q_\tau)$.

Let us now take $\varphi=(S_h u)^{(K)}=\max\{S_h u(x,t)-K,0\}$ for $K\geqq\tilde K$.

Since

$$\int_{Q_\tau}(S_h u(x,t))_t(S_h u(x,t))^{(K)}\,dx\,dt=\frac12\Big[\int_\Omega\{[S_h u(x,t)]^{(K)}\}^2\,dx\Big]_{t=0}^{t=\tau}$$

we can write (2.11) as follows:

$$\int_{Q_\tau}\{S_h(a_{ij}u_{x_j})(S_h u)^{(K)}_{x_i}\}\,dx\,dt+\frac12\Big[\int_\Omega\{[S_h u(x,t)]^{(K)}\}^2\,dx\Big]_{t=0}^{t=\tau}=\int_{Q_\tau}(S_h f_i)(S_h u)^{(K)}_{x_i}\,dx\,dt\,. \tag{2.12}$$

Now let h approach zero. This is possible for the first term since $a_{ij}u_{x_i}\in L^2(Q;1/\omega)$ and S_h acts continuously on this space (see Sec. 3) and $(S_h u)^{(K)}_{x_i}$ converges in $L^2(Q;\omega)$ to $u^{(K)}_{x_i}$. The same is true for the right side while the $\int_\Omega([S_h u(x,\tau)]^{(K)})^2\,dx$ converges, for all τ's, to $\int_\Omega[u^{(K)}(x,\tau)]^2\,dx$ because $u\in C^0([0,T];L^2(\Omega))$.

From (2.12) using the boundedness assumption on $u_0(x)$ it follows that

$$\int_{Q_\tau}a_{ij}u_{x_j}u^{(K)}_{x_i}\,dx\,dt+\frac12\int_\Omega(u^{(K)}(x,\tau))^2\,dx=\int_{Q_\tau}f_i u^{(K)}_{x_i}\,dx\,dt \tag{2.13}$$

for all $K\geqq\tilde K$.

(8) $Q_\tau=\Omega\times(0,\tau)$.

From (2.13) using standard techniques we deduce that the

$$\max_{[0,T]}\int_{\Omega}[u^{(K)}(x,t)]^2\,dx+\int_{Q}[u^{(K)}_{x_i}(x,t)]^2\omega(x,t)\,dx\,dt\leqq C_1\int_{A_K}f_i^2(x,t)\,\frac{1}{\omega(x,t)}\,dx\,dt$$

where A_K denotes the set: $\{(x,t)\in Q\colon u(x,t)\geqq K\}$.

By Lemma 2.2 and the above it follows that

$$\left(\int_0^T\int_{\Omega}u_K^{2l}(x,t)\,\omega(x,t)\,dx\,dt\right)^{1/l}\leqq C_2\left(\int_0^T\int_{\Omega}\left(\frac{f}{\omega}\right)^r\omega\,dx\,dt\right)^{2/r}\left(\int_{A_K}\omega(x,t)\,dx\,dt\right)^{1-2/r}.$$

Now, considering $h>K$, we get

$$(h-K)^2\left(\int_{A_h}\omega\,dx\,dt\right)^{1/l}\leqq C_2\left\|\frac{f}{\omega}\right\|^2_{L^r(Q;\,\omega)}\left(\int_{A_K}\omega\,dx\,dt\right)^{(r-2)/r}.$$

Letting, for $h>\tilde{K}$, $\varphi(h)=\int_{A_h}\omega(x,t)\,dx\,dt$, $\varphi(h)$ is a non negative, nondecreasing function such that the following inequality holds:

$$\varphi(h)\leqq\frac{C_2^l\|f/\omega\|^{2l}_{L^r(Q;\,\omega)}}{(h-K)^{2l}}\,[\varphi(K)]^{\beta}\,,\qquad\beta=l\cdot\frac{r-2}{r}>1$$

for all $h>K\geqq\tilde{K}$.

The theorem now follows by G. STAMPACCHIA's Lemma (see [STA], page 93).

We would like to point out that no really new technique has been used in this section; in fact all the proofs followed quite closely the ones given in the [L.S.U.] treatise or any other work on parabolic divergence form equations. However, it is the A_2 assumption in the time variable that allowed us the use of such standard technical devices as the Steklov averages; dropping this assumption to get also the most basic inequalities we should have assumed much stronger hypotheses on the weight (e.g. substantial time independence as in [IV-3] or assumptions on the form of the function $\omega(x,t)$ as in [N_1], etc.).

Given this we will not give a detailed proof of the following theorem that can be carried out in very similar way to [STA], Cor. 5.2, p. 141.

THEOREM 2.5. – *Suppose u is a solution of* (2.4) *with $f\equiv 0$ such that $u\in C^0([0,T];$ $L^2(\Omega))$ (this is true e.g. if $u\in\mathring{V}_\omega(Q)$ or if some smoothness assumption is made on $\partial\Omega$). Suppose that $u_0(x)=\lim_{t\to 0}u(x,t)$ is in $L^\infty_{\mathrm{loc}}(\Omega)$. Then:*

$$\operatorname*{ess\,sup}_{B_R(x_0)\times(0,T)}|u(x,t)|\leqq\operatorname*{ess\,sup}_{B_R(x_0)}|u_0(x)|+K\int_{B_R(x_0)\times(0,T)}[u^{(M_0)}]^2\omega\,dx\,dt\cdot\left(\int_0^T\int_{A_{M_0,R}}\omega(x,t)\,dx\,dt\right)^{(\theta-1)/2},$$

where: $B_R(x_0)$ is the open ball centered at $x_0 \in \Omega$ with radius R ([9]), M_0 is the $\operatorname{ess\,sup}_{B_R(x_0)} |u_0(x)|$, $u^{(M_0)} = \max\{u(x,t) - M_0, 0\}$, $A_{M_0,R}$ is the subset of $B_R(x_0) \times (0, T)$ where $u(x,t) \geqq M_0$, θ is a number larger than 1 and K depends on the A_2 constants of ω and R (it is unbounded for $R \to 0$).

REMARK 2.3. – The theorem above is the best possible (under our assumptions) « local boundedness » result for solutions of (2.4). These solutions, as we have seen in Sec. 1, are not in general locally bounded.

3. – Machinery. Proofs of Lemmas 2.1, 2.2.

We begin with some facts about the Steklov averages introduced in Sec. 2. We assume ω and u have been extended to $\Omega \times R$ as in the Remark 2.1 and 2.2.

LEMMA 3.1. – *For any $u \in \overset{\circ}{V}_\omega(Q)$ $\|S_h u\|_{\overset{\circ}{V}_\omega(Q)} \leqq K \|u\|_{\overset{\circ}{V}_\omega(Q)}$. An analogous inequality holds for $S_{\bar{h}} u$. Here K depends only on the A_2 constant of ω in the time variable.*

PROOF. – For a.e. x in Ω, $\omega(x,t)$ is an A_2 weight with respect to t. Hence

$$\int_{-\infty}^{+\infty} (S_h u)_{x_i}^2 \, \omega(x,t)\, dt \leqq \int_{-\infty}^{+\infty} [S_h(|\nabla u|)]^2 \omega(x,t)\, dt \leqq$$
$$\leqq K \int_{-\infty}^{+\infty} [M_t(|\nabla u|)]^2 \omega(x,t) \leqq K \int_{-\infty}^{+\infty} |\nabla u|^2 \, \omega(x,t)\, dt\,.$$

Here $M_t(f)$ is the maximal function of f with respect to the t variable, ∇ is the space gradient and we have used Theorem 2.2. Integrating in Ω and using the continuity in $\overset{\circ}{V}_\omega$ of the extension of u to $\Omega \times R$, the conclusion follows.

REMARK 3.1. – As we have just shown, an A_2 condition in t, uniform with respect to $x \in \Omega$, is sufficient to get the continuity of S_h from $L^2(Q; \omega)$ to $L^2(Q; \omega)$. On the other hand, as we will check immediately, the same condition is also necessary for the continuity of all the S_h and $S_{\bar{h}}$ $(h > 0)$ from $L^2(Q; \omega)$ to $L^2(Q; \omega)$. Indeed, assume that for a fixed $h > 0$ both the following inequalities hold:

$$\tag{3.1} \left\{ \begin{aligned} &\int_Q \left(\frac{1}{h} \int_t^{t+h} v(x,s)\, ds\right)^2 \omega(x,t)\, dx\, dt \leqq K \int_Q v^2(x,t)\, \omega(x,t)\, dx\, dt\,, \\ &\int_Q \left(\frac{1}{h} \int_{t-h}^{t} v(x,s)\, ds\right)^2 \omega(x,t)\, dx\, dt \leqq K \int_Q v^2(x,t)\, \omega(x,t)\, dx\, dt \end{aligned} \right.$$

([9]) Here we take R so small that $B_R(x_0) \subseteq \Omega$.

for any non negative function $v \in L^2(Q;\omega)$. Then

$$\int_Q \left(\frac{1}{2h}\int_{t-h}^{t+h} v(x,s)\,ds\right)^2 \omega(x,t)\,dx\,dt \leqq K\int_Q v^2(x,t)\,\omega(x,t)\,dx\,dt\,.$$

Now let $v(x,t) = u(x,s)\chi_{[a,b]}(s)$, for any $[a,b] \subseteq (0,T)$ with $b-a=h$ and any $u \in L^2(Q;\omega)$, $u \geqq 0$.

Then

$$\int_a^b\int_\Omega \left(\frac{1}{b-a}\int_a^b u(x,s)\,ds\right)^2 \omega(x,t)\,dx\,dt \leqq K\int_a^b\int_\Omega u^2(x,t)\,\omega(x,t)\,dx\,dt\,.$$

Now let

$$u(x,t) = \frac{\sqrt{\alpha(x)}}{\omega(x,t)} \qquad \text{where} \qquad \alpha(x) = \beta(x)\left(\int_a^b \frac{dt}{\omega(x,t)}\right)^{-1}$$

and $\beta(x)$ is any non negative continuous function with compact support in Ω, to get

$$\int_\Omega \beta(x)\left(\frac{1}{b-a}\int_a^b \omega(x,t)\,dt\right)\left(\frac{1}{b-a}\int_a^b \frac{1}{\omega(x,t)}\,dt\right)dx \leqq K\int_\Omega \beta(x)\,dx\,.$$

Therefore

$$\left(\frac{1}{b-a}\int_a^b \omega(x,t)\,dt\right)\left(\frac{1}{b-a}\int_a^b \frac{1}{\omega(x,t)}\,dt\right) \leqq K \qquad \text{a.e. in } \Omega$$

for all the $[a,b] \subseteq (0,T)$ with $b-a=h$.

So that if we assume the validity of (3.1) for any $h>0$ $\omega(x,t)$ must satisfy an A_2 condition in $(0,T)$ uniformly with respect to x in Ω.

LEMMA 3.2. – *For any* $u \in W$, $\|S_h u\|_W \leqq K\|u\|_W$. *The same is true for* $S_{\bar h}$. K *is as in Lemma* 3.1.

It is enough to consider $(S_h u)_t$. For any $\varphi \in \mathcal{D}(Q)$:

$$|\langle (S_h u)_t, \varphi\rangle| = |\langle S_h u, \varphi_t\rangle| = \left|\int_{\tilde Q} \frac{1}{h}\int_t^{t+h} u(x,s)\,ds\,\varphi_t(x,t)\,dx\,dt\right| = {}^{(10)}$$

$$= \left|\int_{\tilde Q} u(x,t)\frac{1}{h}\int_{t-h}^{t} \varphi_t(x,s)\,ds\,dx\,dt\right| = \left|\int_{\tilde Q} u(x,t)\left(\frac{1}{h}\int_{t-h}^{t}\varphi(x,s)\,ds\right)_t dx\,dt\right| =$$

$$= |\langle u(x,t), (S_{\bar h}\omega)_t\rangle| = |\langle u_t, S_{\bar h}\varphi\rangle| \leqq \|u_t\|_{V'_\omega}\|S_{\bar h}\varphi\|_{\mathring V_\omega} \leqq K\|u_t\|_{V'_\omega}\|\varphi\|_{\mathring V_\omega}\,.$$

---

(10) $\tilde Q = \Omega \times R$.

If we define $\langle S_h F, \varphi\rangle = \langle F, S_{\bar h}\varphi\rangle$, $\forall\varphi \in \mathfrak{D}(Q)$ for $F \in V'_\omega(Q)$ then, for $u \in W$, $(S_h u)_t = S_h(u_t)$.

Furthermore for $F \in V'_\omega \cap L^1_{\mathrm{loc}}$ the new definition agrees with the old one.

LEMMA 3.3. – *Let $u \in W$. Then $S_h u$ converges to u in W as $h \to 0$.*

Follows from Lemma 3.1, Lemma 3.2 and denseness of $\mathfrak{D}(Q)$ in $\mathring{V}_\omega(Q)$ and $V'_\omega(Q)$

Let us remark that, in virtue of Theorem 2.2, it is possible to obtain the same conclusions as in the above lemmas for $u * \varrho_\varepsilon(t)$ where $\varrho_\varepsilon(t)$ is some family of $C_0^\infty(R)$ convolution kernels.

We are now able to give the

PROOF OF LEMMA 2.1. – Because of the remark immediately above we can assume u to be in $C^\infty(R)$ for a.e. $x \in \Omega$. Let us also remark that all the derivatives with respect to t of u can be assumed to belong to $\mathring{V}_\omega$.

Hence we can assume that $u_t \in \mathring{V}_\omega \cap V'_\omega$.

For fixed $\varphi(x) \in \mathfrak{D}(\Omega)$ the function:

$$F(s) = \|u_t(\cdot, s) - \varphi(\cdot)\|_{H^{-1}(\Omega;\, 1/\omega(\cdot, s))} + \|u_t(\cdot, s) - \varphi(\cdot)\|_{H_0^1(\Omega;\, \omega(\cdot, s))},$$

is an $L^1(R)$ function (see the remarks following the definition of V'_ω).

Hence by Lebesgue theorem:

$$\lim_{h\to 0} \frac{1}{2h} \int_{z-h}^{z+h} F(s)\, ds = F(z) \qquad \text{a.e. in } R. \tag{3.2}$$

We point out that the exceptional set in (3.2) can be taken independent of φ for φ belonging to a convenient, still dense, countable subset of $\mathfrak{D}(\Omega)$.

For any given $\sigma > 0$ and z such that the space $H_0^1(\Omega; 1/\omega(x, z))$ is well defined let $\varphi_z(x)$, belonging to a countable dense subset of $\mathfrak{D}(\Omega)$, satisfy

$$\|u_t(x, t) - \varphi_z(x)\|^2_{H^{-1}(\Omega;\, 1/\omega(x, s))} + \|u_t(x, z) - \varphi_z(x)\|^2_{H_0^1(\Omega;\, \omega(x, z))} < \sigma \ ^{(11)}.$$

Now, for all the z such that (3.2) holds, consider all the $h_z > 0$ such that:

$$\frac{1}{2h_z} \int_{z-h_z}^{z+h_z} [\|u_t(x, s) - \varphi_z(x)\|^2_{H^{-1}(\Omega;\, 1/\omega(x, s))} + \|u_t(x, s) - \varphi_z(x)\|^2_{H_0^1(\Omega;\, \omega(x, s))}]\, ds < \sigma.$$

---

(11) $\mathfrak{D}(\Omega)$ is dense in $H_0^1(\Omega; \omega(x, z)) \cap H^{-1}(\Omega; 1/\omega(x, z))$ with the usual intersection norm.

The family F of intervals: $(z-h_z, z+h_z)$ is a covering of R with the possible exception of a null set. In particular it is a covering of $(-T/2, \frac{3}{2}T)$ (we recall that u is supported in $\Omega\times(-T/2, \frac{3}{2}T)$) up to a null subset that we call E_0.

Moreover, letting $E=(-T/2, \frac{3}{2}T)\setminus E_0$, F is a Vitali covering of E (see, e.g. [STE], p. 24).

Then there exists a null set E_1 and a countable disjoint subfamily $\tilde F$ of F ($\tilde F = \{(z_n - h_{z_n}, z_n + h_{z_n})\}$) such that:

$$\bigcup_{n=1}^{+\infty}(z_n - h_{z_n}, z_n + h_{z_n}) = \left(-\frac{T}{2}, \frac{3T}{2}\right)\setminus(E_0\cup E_1)\,.$$

Define a function $\psi(x,t)$ in $\tilde Q$ letting $\psi(x,t)=\varphi_{z_n}(x)$ for $t\in(z_n-h_{z_n}, z_n+h_{z_n})$ and $\psi(x,t)=0$ elsewhere.

ψ is a measurable function in $\tilde Q$ and for all t its carrier is a compact subset of Ω. Moreover, $\psi\in \mathring{V}_\omega$, $\psi\in V'_\omega$ and

$$\|\psi(x,t)\|_{\mathring{V}_\omega} = \left(\int_{-T/2}^{3T/2}\|\psi(\cdot,t)\|^2_{H_0^1(\Omega;\,\omega(x,t))}\,dt\right)^{\frac12};\quad \|\psi(x,t)\|_{V'_\omega} = \left(\int_{-T/2}^{3T/2}\|\psi(\cdot,t)\|^2_{H^{-1}(\Omega;\,1/\omega(x,t))}\,dt\right)^{\frac12}.$$

Let us remark that

$$\|u_t(x,t)-\psi(x,t)\|^2_{V'_\omega} = \int_{-T/2}^{3T/2}\|u_t(\cdot,t)-\psi(\cdot,t)\|^2_{H^{-1}(\Omega;\,1/\omega(x,t))}\,dt =$$
$$= \sum_{n=1}^{+\infty}\int_{z_n-h_{z_n}}^{n+h_{z_n}}\|u_t(\cdot,t)-\varphi_{z_n}(\cdot)\|^2_{H^{-1}(\Omega;\,1/\omega(\cdot,t))}\,dt < \sigma\sum_{n=1}^{+\infty}2h_{z_n} = 2\sigma T\,,$$

so that ψ can be taken «close» to u_t in V'_ω.

Let now

$$\chi(x,t) = u\left(x,-\frac{T}{2}\right) + \int_{-T/2}^{t}\psi(x,s)\,ds = \int_{-T/2}^{t}\psi(x,s)\,ds\,.$$

$$\chi\in\mathring{V}_\omega \quad\text{and}\quad \|u(x,t)-\chi(x,t)\|^2_{\mathring{V}_\omega} = \left\|\int_{-T/2}^{t}u_t(x,s)\,ds - \int_{-T/2}^{t}\psi(x,s)\,ds\right\|^2_{\mathring{V}_\omega} =$$
$$= \int_{-T/2}^{\frac32 T}\int_\Omega\left[\left(\int_{-T/2}^{t}(u_t(x,s)-\psi(x,s))\,ds\right)_{x_i}\right]^2\omega(x,t)\,dx\,dt \leq$$
$$\leq \int_{-T/2}^{\frac32 T}\int_\Omega\left(\int_{-T/2}^{\frac32 T}[(u_t-\psi)_{x_i}]^2\omega\,ds\right)\left(\int_{-T/}^{\frac32 T}\frac{ds}{\omega(x,s)}\right)\omega(x,t)\,dx\,dt =$$

$$= \int_{\Omega}\left(\int_{-T/2}^{3T/2}[(u_t-\psi)_{x_i}]^2\omega\,ds\right)\left(\int_{-T/2}^{3T/2}\frac{1}{\omega(x,s)}\,ds\right)\left(\int_{-T/2}^{3T/2}\omega(x,s)\,ds\right)\leqq$$

$$\leqslant K\int_{-T/2}^{\frac{3}{2}T}\|u_t(\cdot,t)-\psi(\cdot,t)\|^2_{\mathring{H}^1(\Omega;\,\omega(\cdot,t))}\,dt < K\sigma\sum_{n=1}^{+\infty}2h_{z_n} = K\sigma\,.$$

This implies that $\chi(x,t)$ can be taken close to u in $\mathring{V}_\omega$ and in W. One more regularization in the time variable $(\chi * \varrho_\varepsilon(t))$ and the first part of the lemma follows.

The second part is fairly standard (see eg. [T]). The idea is to write a relation like (2.7) for a function $\varphi \in C^\infty(\Omega\times R)$ compactly supported in Ω for any t. From that at once follows

$$\max_{[0,T]}\|\varphi(\cdot,t)\|_{L^2(\Omega)}\leqq K\|\varphi\|_{W(Q)}\,.$$

The conclusion follows from the denseness result of the first part.

The purpose of the last part of this section is to prove the Sobolev like Lemma 2.2.

The proof is divided into some steps. The first will be to obtain the following variant of the Sobolev weighted inequalities obtained in [F.K.S.].

LEMMA 3.4. – *Given $\omega(x,t)\in A_2$ in R^m, uniformly with respect to the time variable, there exist constants $K,\mu,\delta=\delta(\mu)>0$ s.t. for any ball B_R, all $u\in C_0^\infty(B_R)$ and all numbers h satisfying $1\leqq h\leqq m/(m-1)+\delta$*

$$\text{(3.3)}\qquad \frac{1}{\omega^{1+\mu}(B_R;t)}\int_{B_R}|u|^{2h}\omega^{1+\mu}\,dx\Bigg)^{1/2h}\leqq KR\left(\frac{1}{\omega(B_R;t)}\int|\nabla u|^2\omega\,dx\right)^{\frac12}\ \text{(12)}\,,$$

Here K is dependent on m and the A_2 constant of ω in the space variable. Let us also point out that it will be possible to choose $h=1+\mu$ obtaining

$$\left(\frac{1}{\omega^{1+\mu}(B_R;t)}\int_{B_R}|u|^{2(1+\mu)}\omega^{1+\mu}\,dx\right)^{1/2(1+\mu)}\leqq KR\left(\frac{1}{\omega(B_R;t)}\int_{B_R}|\nabla u|^2\omega\,dx\right)^{\frac12}.$$

To prove Lemma 3.4 we will need the following:

LEMMA 3.5. – *Set*

$$Tf(x)=\sup_{0<s}\frac{1}{s^{m-1}}\int_{B_s(x)}|f(y)|\,dy$$

(12) $\omega(B_R;t)=\int_{B_R}\omega(x,t)\,dx$; ∇u is the gradient in the *space* variable.

where f is measurable and supported in a ball B_R of radius R. If $\omega(x,t)$ is as in Lemma 3.4 there exist positive constants K, δ and μ s.t. for all the numbers h, $1 \leqq h \leqq \leqq m/(m-1)+\delta$, and for all $f \in L^2(B_R;\omega)$ we have

$$\left(\frac{1}{\omega^{1+\mu}(B_R;t)}\int_{B_R}(Tf)^{2h}\omega^{1+\mu}(x,t)\,dx\right)^{1/2h} \leqq kR\left(\frac{1}{\omega(B_R;t)}\int_{B_R}(f)^2\omega(x,t)\,dx\right)^{\frac{1}{2}}. \tag{3.4}$$

PROOF. – From Theorem 2.1 it follows that ω, $1/\omega \in A_{2-\varepsilon}$ for some $\varepsilon > 0$; then, for any μ in the range: $0 < \mu \leqq 1/(1-\varepsilon)-1$ it follows that $\omega^{1+\mu}(\cdot,t), 1/\omega^{1+\mu}(\cdot,t) \in A_2$.

From now on we will use the notation $\tilde{\omega}(\cdot,t) = [\omega(\cdot,t)]^{1+\mu}$. The proof will follow the one of Lemma 1.1 of [F.K.S.]. We may suppose that f is non negative. For any positive number λ set $E_\lambda = \{x \in B_R\colon Tf(x) > \lambda\}$.

For each $x \in E_\lambda$ there is a ball, $B_{r(x)}(x)$ such that

$$\frac{1}{[r(x)]^{m-1}}\int_{B_{r(x)}(x)} f(y)\,dy > \lambda\,.$$

We can always take $r(x) < 2R$ since the function

$$s \to \frac{1}{s^{m-1}}\int_{B_s(x)} f(y)\,dy$$

is decreasing for $s > 2R$ and $x \in B_R$.

From the Vitali covering lemma ([STE], p. 9) we can select a subsequence of disjoint balls $B_j = B_{r(x_j)}(x_j)$ from the above family so that $E_\lambda \subseteq U_j B_j^*$, where $B_j^* = B_{5r(x_j)}(x_j)$.

From what we have previously indicated all the balls B_j are contained in B_{3R}.

We want now to estimate $\tilde{\omega}(E_\lambda;t)$. Recalling that when $\omega \in A_p$ then $\omega(B_j^*) \leqq \leqq c\omega(B_j)$ (the «doubling condition») we have for any $h \geqq 1$ and for any p larger than $2-\varepsilon$.

$$\begin{aligned}\tilde{\omega}(E_\lambda;t) &\leqq K\sum_j \tilde{\omega}(B_j;t) \leqq K\sum_j \tilde{\omega}(B_j;t)\,\lambda^{-ph}|B_j|^{-(1-1/m)ph}\left(\int_{B_j} f\,dx\right)^{ph} \leqq \\ &\leqq \frac{c}{\lambda^{ph}}\sum_j \tilde{\omega}(B_j;t)|B_j|^{ph/m}[\omega(B_j;t)]^{-h}\left(\int_{B_j} f^p\omega(x,t)\,dx\right)^{h}.\end{aligned} \tag{3.5}$$

Now observe that it is possible to choose $q < 2$ and close enough to 2 and $\eta > 0$ and small enough so that $\tilde{\omega}(\cdot,t)$ belongs to A_q and $\omega(\cdot,t) \in A_{(q-1)/(1+\eta)+1}$; with this choice of q and η the following inequalities follow from the Hölder's inequality and the A_p condition:

$$\begin{aligned}[\omega(B_j;t)]^{-h} &\leqq |B_j|^{-(1+(q-1)/(\eta+1))}\left(\int_{B_j}\left(\frac{1}{\tilde{\omega}(x,t)}\right)^{1/(q-1)}dx\right)^{((q-1)/(1+\eta))h} \leqq \\ &\leqq K[\omega(B_j;t)]^{-h}\end{aligned} \tag{3.6}$$

and

$$\tilde{\omega}(B_j;t) \leqq K|B_j|^q \left(\int_{B_j} \left(\frac{1}{\tilde{\omega}(x,t)}\right)^{1/(q-1)} dx\right)^{-(q-1)} \leqq c\tilde{\omega}(B_j;t)\,. \tag{3.7}$$

substituting in (3.5) we get

$$\tilde{\omega}(E_\lambda;t) \leqq \frac{K}{\lambda^{ph}} \sum_j |B_j|^{q+ph/m-h(1+(q-1)/(\eta+1))} \left(\int_{B_j} \left(\frac{1}{\tilde{\omega}(x,t)}\right)^{1/(q-1)} dx\right)^{-(q-1)+h((q-1)/(\eta+1))} \cdot$$
$$\cdot\left(\int_{B_j} f^p \omega(x,t)\,dx\right)^h.$$

We now choose h so that $q + ph/m - h(1 + (q-1)/(\eta+1)) \geqq 0$; this is equivalent to $h \leqq m/(m-1) + \delta$ where $\delta = \delta(m, q, p, \eta) > 0$.

With this choice of h we get

$$\tilde{\omega}(E_\lambda;t) \leqq \frac{K}{\lambda^{ph}} R^{-mh((q-1)/(\eta+1)+1)+qm+ph} \left(\int_{B_{3R}} \left(\frac{1}{\tilde{\omega}(x,t)}\right)^{1/(q-1)} dx\right)^{h((q-1)/(\eta+1))-(q-1)} \cdot$$
$$\cdot\left(\int_{B_R} f^p \omega(x,t)\,dx\right)^h.$$

Once again using the two relations (3.6) and (3.7)

$$[\tilde{\omega}(E_\lambda;t)]^{1/ph} \leqq \frac{K}{\lambda} R[\tilde{\omega}(B;t)]^{1/ph}[\omega(B;t)]^{-1/p}\left(\int_{B_R} f^p \omega\,dx\right)^{1/p}. \tag{3.8}$$

This final inequality states that the operator T is continuous from $L^p(B_R; \omega(\cdot, t))$ into the weak $L^{ph}(B_R; \omega^{1+\eta}(\cdot, t))$ provided that $1 \leqq h \leqq m/(m-1) + \delta$ and for any $p \geqq 2 - \varepsilon$. The Marcinkiewicz interpolation theorem implies the boundedness of T from $L^2(B_R; \omega(\cdot, t))$ into $L^{2h}(B_R; \omega^{1+\eta}(\cdot, t))$. Moreover from the previous inequality (3.8) we obtain the conclusion of Lemma 3.5.

Lemma 3.4 will now follow from Lemma 3.5 and the following theorem [M-W]:

THEOREM 3.1. – *Given* $p \in (1, +\infty)$ *and* $\omega \in A_\infty$ (13) *there exists a constant* $K > 0$ *such that for all measurable non negative functions* f

$$\left\| \int_{R^m} \frac{f(y)}{|x-y|^{m-1}}\,dy \right\|_{L^p(R^m;\,\omega)} \leqq K \|Tf\|_{L^p(R^m;\,\omega)}\,.$$

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(13) i.e. there exist positive constants $K, \delta > 0$ so that given any cube C and any measurable subset $E \subseteq C$, $\omega(E)/\omega(C) \leqq K(|E|/|C|)^\delta$.

Now returning to the proof of Lemma 3.4

$$|u(x)| \leqq K\int \frac{|\nabla u(y)|}{|x-y|^{m-1}}\, dy; \qquad \text{since } \omega^{1+\eta} \in A_2$$

$$\|u\|_{L^{2h}(B_R;\, \omega^{1+\eta}(\cdot,t))} \leqq K\left\| \int_{B_R} \frac{|\nabla u(y)|}{|x-y|^{m-1}}\, dy \right\|_{L^{2h}(R^m;\, \omega^{1+\eta}(\cdot,t))} \leqq K\|T(|\nabla u|)\|_{L^2(R^m;\, \omega^{1+\eta}(\cdot,t))}\,.$$

LEMMA 3.6. – *Let $u(x,t) \in C^\infty(B_R \times [0,T])$, satisfy for all $t \in [0,T]$, $u(x,t)$ is compactly supported in B_R, and assume $\omega(x,t) \in A_2$ in R^m uniformly in t. Then there are constants K, $\eta > 0$ such that if $l = (1+2\eta)/(1+\eta) > 1$ we have:*

$$\left(\int_a^b \int_{B_R} u^{2l}\omega\, dx\, dt\right)^{1/2l} \leqq K\left\{\left(\max_{[a,b]} \frac{1}{|B_R|}\int_{B_R} u^2\, dx\right)^{\frac12} + R\left(\int_a^b \int_{B_R} |\nabla u|^2 \omega\, dx\, dt\right)\right\}^{\frac12}$$

for all $(a,b) \subseteq (0,T)$.

PROOF. – Let η be the same as in Theorem 2.1. Then, from the Hölder inequality,

$$\left(\int_{B_R} u^{2l}\omega(x,t)\, dx\right)^{1/2l} \leqq \left(\int_{B_R} u^{2(1+\eta)}[\omega(x,t)]^{1+\eta}\, dx\right)^{1/2l(1+\eta)} \left(\int_{B_R} u^2\, dx\right)^{(1/2l)(1-1/(1+\eta))}.$$

Since $[\omega(x,t)]^{1+\eta} \in A_2$

$$\omega(B_R;t) \geqq K[\omega^{1+\eta}(B_R;t)]^{1/(1+\eta)}|R_R|^{1-1/(1+\eta)}\,.$$

From (3.9), (3.10) and Lemma 3.4 it follows that

$$\begin{aligned}
\left(\frac{1}{\omega(B_R;t)}\int_{B_R} u^{2l}\omega(x,t)\, dx\right)^{(1/2l)} &\leqq K\left(\frac{1}{\omega^{1+\eta}(B_R;t)}\int_{B_R} u^{2(1+\eta)}\omega^{1+\eta}(x,t)\, dx\right)^{(1/2l)(1/(1+\eta))}\cdot \\
&\quad \cdot\left(\frac{1}{|B_R|}\int_{B_R} u^2\, dx\right)^{(1/2l)(1/(1+\eta))} \leqq \\
&\leqq K\left(\max_{[a,b]} \frac{1}{|B_R|}\int_{B_R} u^2\, dx\right)^{\frac12(\eta/(1+2\eta))}\left(\frac{1}{\omega^{1+\eta}(B_R;t)}\int_{B_R} u^{2(1+\eta)}\omega^{1+\eta}(x,t)\, dx\right)^{\frac12(1/(1+2\eta))} \leqq \\
&\leqq KR^{1/l}\left(\max_{a,b} \frac{1}{|B_R|}\int_{B_R} u^2\, dx\right)^{\frac12(\eta/(1+2\eta))}\left(\frac{1}{\omega(B_R;t)}\int_{B_R} |\nabla u|^2\omega(x,t)\, dx\right)^{1/2l}.
\end{aligned}$$

Now, taking the $2l$ power, multiplying by $\omega(B_R; t)$ and integrating in $[a, b]$ we get:

$$\left(\int_a^b \int_{B_R} u^{2l}\omega\, dx\, dt\right)^{1/2l} \leq KR^{1/l}\left(\max_{[a,b]} \frac{1}{|B_R|} \int u^2\, dx\right)^{(l-1)/2l}\left(\int_0^T \int_{B_R} |\nabla u|^2 \omega\, dx\, dt\right)^{1/2l} \leq$$

$$\leq K\left\{\left(\max_{[a,b]} \frac{1}{|B_R|} \int_{B_R} u^2\, dx\right)^{\frac{1}{2}} + R\left(\int_{B_R} |\nabla u|^2 \omega\, dx\, dt\right)^{\frac{1}{2}}\right\}.$$

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