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On Directed Graphs with an Independent Covering Set

E.C. Milner and Robert E. Woodrow*

Department of Mathematics and Statistics, The University of Calgary, 2500 University Drive, Calgary, Alberta T2N 1N4, Canada

Abstract. We prove that if a directed graph, $\mathscr{D}$, contains no odd directed cycle and, for all but finitely many vertices, EITHER the in-degrees are finite OR the out-degrees are at most one, then $\mathscr{D}$ contains an independent covering set (i.e. there is a kernel). We also give an example of a countable directed graph which has no directed cycle, each vertex has out-degree at most two, and which has no independent covering set.

1. Introduction

Throughout $\mathscr{D} = (V, E)$ denotes a simple directed graph with vertex set V and edge set $E \subseteqq (V \times V) \backslash \{(x, x): x \in V\}$. For $X \subseteqq V$ we define $\mathscr{D}(X) = \{y \in V: (x, y) \in E$ for some $x \in X\}$ and $\mathscr{D}^{-1}(X) = \{y \in V: (y, x) \in E$ for some $x \in X\}$. The *in-degree* of a vertex $x \in V$ is $|\mathscr{D}^{-1}(\{x\})|$ and the *out-degree* is $|\mathscr{D}(\{x\})|$. A subset $I \subseteqq V$ is said to be *independent* if $(I \times I) \cap E = \varnothing$. A subset $C \subseteqq V$ is a *covering set* of $\mathscr{D}$ if $V \subseteqq C \cup \mathscr{D}(C)$. If $U \subseteqq V$, the *restriction* of $\mathscr{D}$ to U is the graph $D \restriction U = (U, E \cap (U \times U))$, also we write $\mathscr{D} \backslash U = \mathscr{D} \restriction (V \backslash U)$.

This note is motivated by a question riased by P. Duchet (private communication). Von Neumann and Morgenstern [6] observed that, if $\mathscr{D}$ is finite and contains no directed cycle, then it has an independent covering set (in the terminology of [1] this is called a *kernel* of $\mathscr{D}$). This follows, by induction on $|V|$, from the fact that such a graph contains a *source* (i.e. a vertex $x \in V$ such that $\mathscr{D}^{-1}(\{x\}) = \varnothing$). Richardson [5] later showed that it is enough to assume only that the finite graph $\mathscr{D}$ contains no *odd* directed cycle. This result has been rediscovered and generalised by others (see [1, 2, 3, 4]) and we give our own proof of this fact (see Lemma 1 in §2). However, it is not true for infinite graphs. For example, denote by $\boldsymbol{\omega}$ the directed graph with vertex set ω and edge-set $E(\boldsymbol{\omega}) = \{(j, i): i < j < \omega\}$. Clearly, $\boldsymbol{\omega}$ contains no directed cycle, every verted has only finite out-degree, and the only independent subsets are the singletons and so $\boldsymbol{\omega}$ has no independent covering set. P. Duchet asked if a sufficient condition for an infinite directed graph $\mathscr{D}$ to have an independent covering set is that $\mathscr{D}$ contain no directed odd cycle and no isomorphic copy of $\boldsymbol{\omega}$.

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A fairly natural example which gives a negative answer to Duchet's question is provided by the line graph of $\boldsymbol{\omega}$, i.e., the directed graph $\mathscr{D}$ whose vertex set is the set of edges of $\boldsymbol{\omega}$, $V(\mathscr{D}) = \{(j,i): i < j < \omega\}$, and whose edge set is the set $E(\mathscr{D}) = \{((k,j),(j,i)): i < j < k < \omega\}$. Clearly $\mathscr{D}$ contains no directed cycle and no copy of $\boldsymbol{\omega}$ (it does not even contain a triangle). However, $\mathscr{D}$ does not have an independent covering set. To see this suppose, for contradiction, that $I \subseteqq V(\mathscr{D})$ is an independent covering set. For $0 < j \in \omega$ either (i) $(k,j) \in I$ for some $k > j$ or (ii) $(j,i) \in I$ for every $i < j$. Clearly (ii) cannot hold for more than one value of $j > 0$ since I is independent. So (i) holds for every large enough j, and hence there are $j < k < l$ such that (k,j) and (l,k) both belong to I, contradicting the assumption that I is independent.

The above example suggests some interesting variations of Duchet's original question. Notice that in the example, every vertex has infinite in-degree while the out-degrees are finite but unbounded. In the remaining sections we consider the questions whether, for directed graphs containing no odd cycle, a sufficient condition for the existence of an independent covering set is that either (1) the in-degrees are finite or (2) the out-degrees are uniformly bounded by some $n < \omega$.

2. All but Finitely Many Points Have Finite In-Degree

In this section we establish the following sufficient condition for the existence of an independent covering set.

Theorem 1. *Let $\mathscr{D} = (V,E)$ be a directed graph which has no directed odd cycle. If only finitely many vertices of $\mathscr{D}$ have infinite in-degree, then $\mathscr{D}$ has an independent covering set.*

The first step is to establish the result for finite graphs.

Lemma 1. *If $\mathscr{D}$ is a finite directed graph with no directed odd cycle, then $\mathscr{D}$ has an independent covering set.*

Proof. The proof is by induction on $|V|$. Since $\mathscr{D}$ is finite, there is some vertex $d \in V$ such that, whenever $h \in V$ and there is a directed path from h to d, then there is also a directed path from d to h (in the case when $\mathscr{D}$ is cycle-free, any source element has this property). Let H denote the set of all those elements $h \in V$ such that there is a directed path from h to d. Note that $d \in H \neq \emptyset$ and, for each $h \in H$, every directed path from h to d has the same length modulo 2. Let I_0 denote the set of vertices h such that there is a directed path of even length from h to d. Then I_0 is independent and $H \subseteqq I_0 \cup \mathscr{D}(I_0)$. Let $V_1 = V \setminus (I_0 \cup \mathscr{D}(I_0))$. Note that there is no directed edge between V_1 and I_0. By the induction hypothesis, there is an independent set $I_1 \subseteqq V_1$ such that $V_1 \subseteqq I_1 \cup \mathscr{D}(I_1)$. Then $I = I_0 \cup I_1$ is an independent covering set of $\mathscr{D}$. □

Next we prove the theorem for the case when every vertex has finite in-degree by a simple application of (propositional) compactness. This result was independently obtained by François Bry (unpublished) without using compactness.

Lemma 2. *If $\mathscr{D}$ is a directed graph with no odd directed cycle, and if every vertex of $\mathscr{D}$ has finite in-degree, then $\mathscr{D}$ has an independent covering set.*

Proof. For each vertex $x \in V$ introduce a propositional letter p_x. The intended interpretation is that p_x is assigned the value "true" just if x belongs to "the" independent covering set. The axioms are:

$$\neg (p_x \;\&\; p_y) \text{ for each pair } (x, y) \in E(\mathscr{D})$$

and, for each vertex $x \in V$,

$$p_x \text{ or } p_{y_1} \text{ or } p_{y_2} \text{ or} \ldots \text{or } p_{y_n},$$

where $n = n(x)$ is finite and $\{y_1, y_2, \ldots, y_n\} = \mathscr{D}^{-1}(\{x\})$. By Lemma 1 the system is finitely consistent and therefore consistent and hence there is an independent covering set. □

Proof of Theorem 1. The proof is by induction on n, the number of vertices of $\mathscr{D}$ having infinite in-degree. For $n = 0$, the result follows by Lemma 2. Suppose $n > 0$ and that x is a vertex having infinite in-degree.

For a subgraph $\mathscr{D}'$ of $\mathscr{D}$, denote by $H(\mathscr{D}')$ the set of all vertices $y \in V(\mathscr{D}')$ such that there is a directed path from y to x in $\mathscr{D}'$, also let $R(\mathscr{D}')$ denote the set of all vertices $y \in V(\mathscr{D}')$ such that there is a directed path from x to y in $\mathscr{D}'$. We will say that x is *satisfactory* for $\mathscr{D}'$ if $H(\mathscr{D}') \subseteq R(\mathscr{D}')$. In particular, x is satisfactory for $\mathscr{D}'$ if $x \notin V(\mathscr{D}')$ (since $H(\mathscr{D}') = \varnothing \subseteq R(\mathscr{D}')$).

We claim that, if x is satisfactory for a subgraph $\mathscr{D}'$, then $\mathscr{D}'$ has an independent covering set. If x has finite in-degree in $\mathscr{D}'$ this is an immediate consequence of the induction hypothesis since $\mathscr{D}'$ has fewer than n vertices with infinite in-degree. Suppose $H(\mathscr{D}') \subseteq R(\mathscr{D}')$. Then, by the same argument that was used to prove Lemma 1, the set J_0 of all those $y \in H(\mathscr{D}')$ for which there is a directed path of even length from y to x is an independent set and $H(\mathscr{D}') \subseteq J_0 \cup \mathscr{D}(J_0)$. Now, by the induction hypothesis, $\mathscr{D}'_1 = \mathscr{D}' \backslash (J_0 \cup \mathscr{D}(J_0))$ has an independent covering set J_1 (since $x \notin V(\mathscr{D}'_1)$). Clearly there is no edge from J_0 to $V(\mathscr{D}'_1)$, and there is no edge from $V(\mathscr{D}'_1)$ to $H(\mathscr{D}')$. Therefore, $J_0 \cup J_1$ is an independent covering set of $\mathscr{D}'$ and the claim is established.

Put $\mathscr{D}_0 = \mathscr{D}$, $H_0 = H(\mathscr{D}_0)$, $R_0 = R(\mathscr{D}_0)$. If x is satisfactory for $\mathscr{D}$, then, as we just observed, there is an independent covering set. Therefore, we may assume that x is not satisfactory for $\mathscr{D}_0$. In this case $H_0 \backslash R_0 \neq \varnothing$. Since $x \notin H_0 \backslash R_0$, it follows from the induction hypothesis that there is an independent covering set of $\mathscr{D} \restriction (H_0 \backslash R_0)$, say I_0. Put $\mathscr{D}_1 = \mathscr{D}_0 \backslash (I_0 \cup \mathscr{D}(I_0))$. Observe that there are no edges connecting I_0 and $V(\mathscr{D}_1)$ so that it will be enough to show that $\mathscr{D}_1$ has an independent covering set.

Generally, let α be an ordinal and suppose that we have defined subgraphs $\mathscr{D}_\beta$ and independent sets I_β for $\beta < \alpha$ such that (i) x is not satisfactory for $\mathscr{D}_\beta$, (ii) $H_\beta \backslash R_\beta \neq \varnothing$, where $H_\beta = H(\mathscr{D}_\beta)$ and $R_\beta = R(\mathscr{D}_\beta)$, (iii) $I_\beta \subseteq H_\beta \backslash R_\beta \subseteq I_\beta \cup \mathscr{D}(I_\beta)$, (iv) $\mathscr{D}_\beta = \mathscr{D} \backslash \bigcup \{I_\gamma \cup \mathscr{D}(I_\gamma) \colon \gamma < \beta\}$. Put $\mathscr{D}_\alpha = \mathscr{D} \backslash \bigcup \{I_\beta \cup \mathscr{D}(I_\beta) \colon \beta < \alpha\}$. If x is satisfactory for $\mathscr{D}_\alpha$, then there is an independent covering set, J, of $\mathscr{D}_\alpha$. Also, for $\gamma < \beta \leq \alpha$, there is no edge connecting $H_\gamma \backslash R_\gamma$ to $V(\mathscr{D}_\beta)$. Consequently, $J \cup \bigcup \{I_\beta \colon \beta < \alpha\}$ is an independent covering set for $\mathscr{D}$, and the construction terminates. If on the other hand x is not satisfactory for $\mathscr{D}_\alpha$, then $H_\alpha \backslash R_\alpha \neq \varnothing$, where $H_\alpha = H(\mathscr{D}_\alpha)$, $R_\alpha = R(\mathscr{D}_\alpha)$, and there is an independent set $I_\alpha \subseteq H_\alpha \backslash R_\alpha$ such that $H_\alpha \backslash R_\alpha \subseteq I_\alpha \cup \mathscr{D}(I_\alpha)$.

Since the sets $H_\alpha \backslash R_\alpha$ so constructed are non-empty, pairwise disjoint subsets of $H(\mathscr{D})$, it follows that, if $|H(\mathscr{D})| = \kappa$, then for some $\alpha < \kappa^+$ the construction must terminate with x being satisfactory for $\mathscr{D}_\alpha$, and we conclude that $\mathscr{D}$ has an independent covering set. □

3. Uniformly Bounded Out-Degrees

In this section we show that, if $\mathscr{D}$ contains no odd directed cycle and if the out-degree of each vertex is at most one, then again $\mathscr{D}$ has an independent covering set. In fact, slightly more is true since it is enough that all but finitely many of the vertices have out-degree at most one (Theorem 2).

Surprisingly, it is not possible to replace one by two. We give an example of a directed graph $\mathscr{D}$ with no directed cycles in which every vertex has out-degree at most two and which has no independent covering set.

Theorem 2. *If $\mathscr{D}$ is a directed graph which contains no odd directed cycle and in which the out-degree of all but finitely many vertices is at most one, then $\mathscr{D}$ has an independent covering set.*

Proof. Put $\mathscr{D}_0 = \mathscr{D}$. Define $\mathscr{D}_\alpha$ by induction on α so that $\mathscr{D}_\alpha = \mathscr{D} \backslash \bigcup \{I_\beta \cup \mathscr{D}(I_\beta): \beta < \alpha\}$, where I_β is the set of sources of $\mathscr{D}_\beta$, i.e. $I_\beta = \{x \in V(\mathscr{D}_\beta): \mathscr{D}_\beta^{-1}(\{x\}) = \varnothing\}$. There is a least ordinal α such that $I_\alpha = \varnothing$ so that $\mathscr{D}_\gamma = \mathscr{D}_\alpha$ for all $\gamma \geq \alpha$. Now $I = \bigcup\{I_\beta: \beta < \alpha\}$ is an independent set and $\mathscr{D}_\alpha = \mathscr{D} \backslash (I \cup \mathscr{D}(I))$. Since there is no edge between I and $V(\mathscr{D}_\alpha)$, it will be enough to show that $\mathscr{D}_\alpha$ has an independent covering set. In other words, without loss of generality, we may assume that $\mathscr{D}$ has no sources. Also we may assume that $\mathscr{D}$ is connected.

Suppose that $\mathscr{D}$ contains n elements which have out-degree greater than one. We prove the result by induction on n.

We first assume $n = 0$, i.e. that every vertex has out-degree at most one. Note that, in this case, the only circuits in $\mathscr{D}$ are directed cycles and therefore even. Consequently, any two (undirected) paths joining two fixed vertices x and y in $\mathscr{D}$ have the same parity. Fix a vertex $x_0 \in V$ and let I be the set of all those vertices $x \in V$ such that there is an undirected path of even length from x to x_0. Then I is an independent set and it is also a covering set since $\mathscr{D}$ has no source elements.

Now assume $n > 0$, and that $x \in V$ has out-degree greater than one. The proof from now on continues in exactly the same way as for the proof of Theorem 1 (i.e. we call x *satisfactory* for a subgraph $\mathscr{D}'$ of $\mathscr{D}$ if $H(\mathscr{D}') \subseteq R(\mathscr{D}')$ etc.). □

We conclude this section with an example of a denumerable directed graph $\mathscr{D}$ which contains no directed circuits, every vertex has out-degree at most two, and for which there is no independent covering set.

The set of vertices of our graph $\mathscr{D}$ is

$$V(\mathscr{D}) = \omega \cup \{(i,j): i + 1 < j < \omega\} \cup \{(i,j)': i + 1 < j < \omega \text{ and } j - i \text{ is even}\}.$$

We describe the edge set, $E(\mathscr{D})$, by describing $\mathscr{D}(\{x\})$ for each vertex x:

$$\mathscr{D}(\{0\}) = \varnothing; \quad \mathscr{D}(\{1\}) = \{0\}; \quad \mathscr{D}(\{j\}) = \{j-1,(j-2,j)\} \quad (\text{for } 2 \leqq j < \omega);$$

$$\mathscr{D}(\{(0,j)\}) = \{0\} \quad \text{if } j \geqq 3 \text{ is odd};$$

$$\mathscr{D}(\{(0,j)\}) = \{(0,j)'\} \quad \text{if } j \geqq 2 \text{ is even};$$

$$\mathscr{D}(\{(i,j)\}) = \{(i-1,j), i\} \quad \text{if } 2 \leqq i+1 < j \text{ and } j-i \text{ is odd};$$

$$\mathscr{D}(\{(i,j)\}) = \{(i-1,j),(i,j)'\} \quad \text{if } 2 \leqq i+1 < j \text{ and } j-i \text{ is even};$$

$$\mathscr{D}(\{(i,j)'\}) = \{i\} \quad \text{if } i+1 < j \text{ and } j-i \text{ is even}.$$

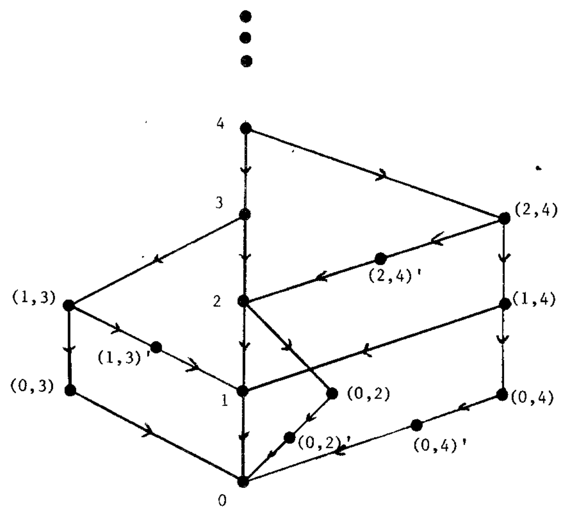

Fig. 1

Figure 1 illustrates part of the graph $\mathscr{D}$.

It is easy to see that $\mathscr{D}$ contains no directed cycle since all the edges are directed "downwards". Also $|\mathscr{D}(\{x\})| \leqq 2$ for every vertex x. We have to show that there is no independent covering set.

Suppose, for contradiction, that I is an independent covering set. Suppose that $j \in I$ for some $j \in \omega$. Then it is easily seen that $(i,j) \in I$ if $i+1 < j$ and $j-i$ is odd, and $(i,j) \notin I$ if $i+1 < j$ and $j-i$ is even. Therefore, $(i,j)' \in I$ if $i+1 < j$ and $j-i$ is even. It follows that $j-1 \notin I$ and $i \notin I$ for $i+1 < j$, i.e. $i \notin I$ for all $i < j$. Thus I contains at most one $j \in \omega$ and, without loss of generality, we may assume that $I \cap \omega = \varnothing$. It follows that $(i,j) \in I$ for $i+1 < j < \omega$ and $j-i$ even and hence that $(i,j)' \notin I$. Also $(i,j) \notin I$ for $i+1 < j < \omega$ and $j-i$ odd. Hence we see that no element of ω is covered by an element of I. This contradiction shows that $\mathscr{D}$ has no independent covering set.

4. Another Example

We have seen that, for a directed graph with no odd directed cycle, a sufficient condition for the existence of an independent covering set is that either every vertex has finite in-degree or that every vertex has out-degree at most one. In the example given in §3, each vertex $x \in \omega$ has infinite in-degree and (for $x > 1$) out-degree 2. This leaves open the possibility that a sufficient condition for the existence of an independent covering set is that $\mathscr{D}$ should contain no odd directed cycle and satisfy the following condition:

K: *for each vertex x, either x has finite in-degree, or x has out-degree at most one.*

By a slight modification of our previous example, in which we replace the original vertices $n \in \omega$ by three vertices n, n' and n'', we obtain a directed graph $\mathscr{D}$ which has no directed cycle, satisfies condition K and has no independent covering set. We omit the formal definition of $\mathscr{D}$ but illustrate in Fig. 2 its essential features.

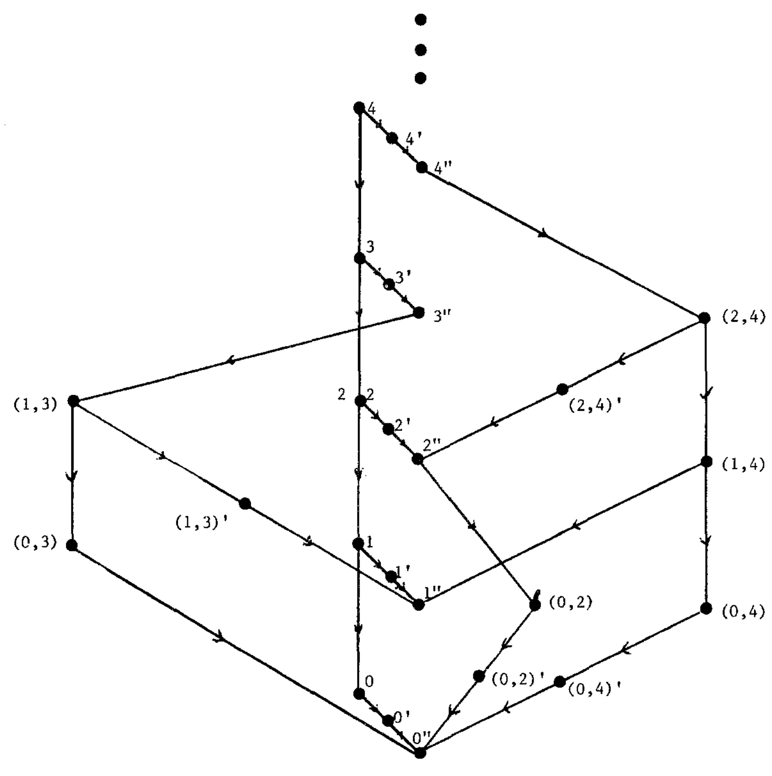

Fig. 2

The proof that there is no independent covering set is similar to that given for the example in §3, the vertex n'' ($n \in \omega$) now playing the rôle of the vertex n in that example.

In this example, the vertices n'' $(n \in \omega)$ are the only vertices with infinite in-degree, and the out-degree of n'' is at most one. Every other vertex has in-degree one and out-degree at most two.

5. Concluding Remark

We are grateful to the referee for pointing out to us the following Corollary of Theorems 1 and 2. A *semi-kernel* of a directed graph $\mathscr{D}$ is an independent set I such that $V(\mathscr{D}) = I \cup \mathscr{D}(I) \cup \mathscr{D}(\mathscr{D}(I))$ (a more descriptive term is an independent two-step covering set). The referee observed that the following corollary may be proved by the same method that we used to prove Theorems 1 and 2. In fact, it is a direct corollary of these theorems.

Corollary. *Let $\mathscr{D} = (V, E)$ be a directed graph such that EITHER there are only finitely many vertices with infinite in-degree OR there are only finitely many vertices with out-degree greater than one. Then $\mathscr{D}$ has a semi-kernel.*

Proof. Let $<$ be any linear odering of V. Consider the directed graphs $\mathscr{D}_1 = (V, E_1)$ and $\mathscr{D}_2 = (V, E_2)$, where $(x, y) \in E(\mathscr{D}_1)$ if $(x, y) \in E$ and $x < y$, and $(x, y) \in E(\mathscr{D}_2)$ if $(x, y) \in E$ and $x > y$. Both $\mathscr{D}_1$ and $\mathscr{D}_2$ are cycle-free and satisfy the same conditions as $\mathscr{D}$. Therefore by Theorems 1 and 2, there is an independent covering set (kernel) I_1 for the graph $\mathscr{D}_1$ and an independent covering set $I_2 \subseteqq I_1$ for the graph $\mathscr{D}_2 \upharpoonright I_1$. Then I_2 is a semi-kernel for $\mathscr{D}$.

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