Graphs and Combinatorics 3, 141–143 (1987)

Graphs and Combinatorics

On a Theorem of Ganley

Dieter Jungnickel

Mathematisches Institut, Justus-Liebig-Universität Gießen, Arndstr. 2, 6300 Gießen, F.R. Germany

Abstract. Let D be a relative difference with parameters $(n, n, n, 1)$ in an abelian group G of even order n^2. By a result of Ganley [3], n is necessarily a power of 2 and G is a direct sum of copies of $\mathbb{Z}_4$. We present a simple (and much shorter) alternative proof of this result, based on a geometric argument and a simple characterisation of the groups in question.

Recall that a relative difference set with parameters $(n, n, n, 1)$ is an n-subset D of a group G of order n^2 with a normal subgroup N of order n such that an element $g \neq 0$ of G has a (necessarily unique) representation $g = d - d'$ $(d, d' \in D)$ if and only if $g \notin N$. In [3] Ganley proved the following result.

Theorem 1. *Let D be a relative difference set with parameters $(n, n, n, 1)$ where n is even and G is abelian. Then n is a power of 2, G is isomorphic to a direct sum of copies of $\mathbb{Z}_4$, and N is elementary abelian.*

Theorem 1 may be interpreted in terms of quasiregular collineation groups of projective planes (cf. p. 181 [1]): A relative difference set with parameters $(n, n, n, 1)$ in G is equivalent to a projective plane of order n with G as a quasiregular collineation group of type c (see p. 182 [1] and [3]). Ganley's proof of Theorem 1 makes use of this equivalence and proceeds via coordinatizing the corresponding projective plane by a certain type of cartesian group and studying the absolute points of certain polarities; this rests on results of [2]. It is the aim of this note to provide an alternative proof of Ganley's theorem which avoids coordinatizing the plane and is both simpler and considerably shorter. We shall use D to (implicitly) produce an oval in the associated projective plane; this is similar to the approach to planar abelian difference sets used in [8]. This approach enables us to reduce the problem to a combinatorial characterization of the groups described in Theorem 1:

Lemma 2. *Let D be a relative difference set with parameters $(n, n, n, 1)$ where n is even and G is abelian. Assume $0 \in D$; then the following properties hold:*
(i) $2D = N$;
(ii) *D is a system of coset representatives for N;*
(iii) *N contains all involutions of G.*

Proof. Statements (ii) and (iii) are clear from the definition. To prove statement (i) we define an incidence structure $\mathbf{D} = (G, \{D + x : x \in G\}, \in)$. Then for two distinct points $g, h \in G$, it holds that if $g - h \notin N$ then there exists exactly one line $D + x$ such that $g, h \in D + x$, and if $g - h \in N$ then there is no such line. Now suppose that a line $D + x$ intersects $-D$ in two points, say $c + x = -d$ and $c' + x = -d'$. Then $d' - d = c - c'$, and so either $d' = d$ or $d' = c = -(d + x)$. This means that a line $D + x$ intersects $-D$ in at most two points, and it intersects $-D$ in exactly one point if and only if $x = -2d$ for some $d \in D$. (Such a line is referred to as a *tangent* of $-D$.) Now pick $g \in G$ with $g \notin -D$. By (ii), there exists exactly one point $b \in -D$ such that $g - b \in N$. That is, the lines of D passing through g cover all points in $-D$ except for b. Since $|-D| = n$ is even, there exists a tangent that passes through g. Note that for $d \in D$, $D - 2d$ is a tangent at $-d$; thus every $g \in G$ is on a tangent of $-D$. Hence $\bigcup_{d \in D}(D - 2d) = G$. Since $|D| = n$ and $|G| = n^2$, this implies that $(D - 2c) \cap (D - 2d) = \varnothing$ for all $c, d \in D$ with $c \neq d$. Using $0 \in D$, we see that $N = 2D$.

It is now almost trivial to finish the proof of Theorem 1. For let G be a group satisfying conditions (i), (ii), (iii) of Lemma 2. Suppose that there exists $b \in N$ with $2b \neq 0$, and choose $d \in D$ such that $2d = 2b$. If $d \notin N$, then $2(d - b) = 0$ contradicts (iii). Thus $d \in N$; but then $2x \neq 0$ for all $x \in D \setminus \{d\}$ by (ii) and (iii), and thus $2d = 0$ by (i) which is absurd. Hence $2N = 0$. Since $2G = N$ by (i) and (ii), we have $4G = 0$, and the desired conclusion follows from (iii).

We remark that the incidence structure $\mathbf{D}$ in the proof of Lemma 2 is a divisible design which yields an affine plane $\mathbf{A}$ of order n by adjoining the cosets $N + x$ as new lines (cf. [6]). Then $-A$ is an arc of $\mathbf{D}$ which extends to an oval (and then, for n even) to a hyperoval of the projective extension $\mathbf{P}$ of $\mathbf{A}$ (see [7]). Using properties of ovals (cf. [5]), our proof of Lemma 2 might be further shortened.

It may be helpful to briefly discuss the known examples of relative difference sets with parameters $(n, n, n, 1)$. These all belong to projective planes of order n which are coordinatized by a semi-field K (or "division ring" in the terminology of [5]), and the group G consists of the translations with direction the special point on the line at infinity plus shears. One always obtains a "Singer group" G in this way from which one then may derive a relative difference set with the desired parameters. To guarantee that G is abelian one has to assume that K is commutative. In this case, G will be elementary abelian if n is odd, and as in Theorem 1 if n is even. (In particular, one may always use the Desarguesian planes to construct examples for all prime power values of n.) All these facts are basically due to Hughes [4]; see also [6] where the language of divisible designs is used more explicitly. As far as the author knows the general structure of projective planes admitting this type of quasiregular group has not been determined up to now; in particular it seems to be unknown if they have to be translation planes.

We finally remark that the groups of Theorem 1 admit other examples of sets D satisfying the conclusion of Lemma 2 than relative difference sets: We may always take $D = \{0, 1\}^k$ in $G = \mathbb{Z}_4^k$. Clearly then $2D = \{0, 2\}^k \cong \mathbb{Z}_2^k$; also, $d - d'$ has coordinates 0, 1 and 3 only, showing that D satisfies the conclusion of Lemma 2. But D is not a relative difference set, since e.g. $(1, 1, 0, \ldots, 0)$-$(1, 0, \ldots, 0) = (0, 1, 0, \ldots, 0)$-$(0, \ldots, 0)$.

A more general investigation of the role of arcs corresponding to relative difference sets (also with other parameters) is undertaken in [7].

Acknowledgement. The author would like to thank the referee for his detailed suggestions which substantially improved the presentation of this note.

References

1. Dembowski, P.: Finite Geometries. Berlin-Heidelberg-New York: Springer-Verlag 1968
2. Dembowski, P., Ostrom, T.G.: Planes of order n with collineation groups of order n^2. Math. Z. **103**, 239–258 (1968)
3. Ganley, M.J.: On a paper of Dembowski and Ostrom. Arch. Math. **27**, 93–98 (1976)
4. Hughes D.R.: Partial difference sets. Amer. J. Math. **78**, 650–674 (1956)
5. Hughes, D.R., Piper, F.C.: Projective Planes. Berlin-Heidelberg-New York: Springer-Verlag 1973
6. Jungnickel, D.: On automorphism groups of divisible designs. Canad. J. Math. **34**, 257–297 (1982)
7. Jungnickel, D.: Divisible semiplanes, arcs and relative difference sets. Canadian J. Math. (to appear)
8. Jungnickel, D., Vedder, K.: On the geometry of planar difference sets. Europ. J. Comb. **5**, 143–148 (1984)

Received: June 6, 1985
Revised: April 12, 1987