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Minimum Degree, Independence Number and Regular Factors

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Abstract. We investigate relations of the minimum degree and the independence number of a simple graph for the existence of regular factors.

1. Introduction

Sufficient conditions for the existence of regular factors in simple graphs depending on graphical parameters have been much studied (see for example [1, 5–12, 14, 15, 17–19, 25]). This topic is greatly influenced by some results or questions from hamiltonian graph theory. In this paper we present relations between the minimum degree and the independence number of a graph implying the existence of regular factors of degree at least two. Thereby, we obtain a condition, for which no corresponding statement for the existence of a hamilton cycle can hold (see the remark at the end of the paper). Moreover, we will show that our results generalize a theorem of Katerinis [10], which involves the relation of the connectivity and the independence number of a graph.

We begin with a few definitions and some notations. All graphs in this paper are simple. Let G be a graph with vertex set $V(G)$, edge set $E(G)$ and order $n = |V(G)|$. By $d_G(u)$ we denote the degree of a vertex $u \in V(G)$ in G and $\delta(G)$ denotes the minimum degree of G. A graph is called k-regular, if every vertex has degree k. A k-regular spanning subgraph of G is called a k-factor of G. The independence number $\alpha(G)$ is the cardinality of a maximum set of independent vertices of G. For the (vertex-)connectivity of G we use $\kappa(G)$. The union and the join of two disjoint graphs G and H are denoted by $G \cup H$ and $G + H$, respectively. For a positive integer p the graph pG consists of p disjoint copies of G. The complete graph of order n is denoted by K_n.

Our first result, which motivated this paper, deals with the existence of 2-factors.

Theorem 1. *Let G be a graph with $\delta(G) > \alpha(G)$. Then G has a 2-factor.*

To see that the condition of this theorem is best possible let $G_d = K_{d-1} + dK_2$ for some integer $d \geq 2$. Then it holds $\delta(G_d) = \alpha(G_d) = d$ and it is easy to see that G has no 2-factor.

In [14] we investigated relations of the minimum degree, the independence number, and the order of the graph for the existence of regular factors. In particular, for 2-factors we obtained that every graph G satisfying $\delta(G) \geq \max\{\alpha, (n+2)/3\}$ has a 2-factor (this is closely related to a result of Nash-Williams [13] stating that under the same condition every 2-connected graph is even hamiltonian). Comparing this with Theorem 1 we see that the condition $\delta(G) \geq (n+2)/3$ can be dropped, if only $\delta(G) > \alpha(G)$ instead of $\delta(G) \geq \alpha(G)$ is required. From this point of view it is interesting to investigate the case $\delta(G) = \alpha(G)$ in more detail. As an extension of Theorem 1 we prove the following

Theorem 2. *Let G be a graph with $\delta(G) \geq \alpha(G)$ having no 2-factor. Then it holds $\delta(G) = \alpha(G)$ and either*

(i) *G has a vertex that belongs to no cycle of G or*
(ii) *$G = H + \delta(G)K_2$, where H is a graph of order $\delta(G) - 1$, and $\delta(G) \geq 2$.*

Note that every graph G satisfying $\delta(G) \geq \alpha(G) \geq 2$ and having a vertex that does not belong to any cycle of G consists of $\delta(G)$ cliques, each of size at least $\delta(G) + 1$, and an additional vertex, which is joined to each clique by exactly one edge.

Now we turn to the existence of k-factors with $k \geq 3$. Here we have to distinguish depending on the parity of k. We start with the simpler case.

Theorem 3. *Let $k \geq 4$ be an even integer and let G be a graph of order at least $k + 1$. If G satisfies*

$$\delta(G) > \frac{k+2}{4}\alpha(G) + \frac{5k-3}{8} - \frac{2}{k},$$

then G has a k-factor.

The lower bound for the minimum degree is for every even integer $k \geq 4$ nearly best possible. To show this let α and δ be positive integers with

$$\delta \leq \frac{k+2}{4}\alpha + \frac{k}{2} - \frac{2}{k}.$$

Then we have $\alpha(G) = \alpha$ and $\delta(G) = \delta$ for the graph

$$G = K_{\delta-k/2} + \alpha K_{(k+2)/2}.$$

To see that G has no k-factor one can choose $(D, S) = (V(K_{\delta-k/2}), V(\alpha K_{(k+2)/2}))$ in the k-factor-theorem (Theorem 7) below. If α is small compared to k, then there exist similar graphs having the minimum degree even closer to the bound of the theorem.

Before we give our result on k-factors with k odd, we show that some additional hypotheses will be necessary in this case. Clearly, a graph of odd order has no regular factor of odd degree, and thus we consider only graphs of even order.

But apart from this obvious restriction, we need another condition, since for every integer $\alpha \geq 2$ there exist graphs without k-factor having independence number equal to α and arbitrary large minimum degree. To give examples for this remark, we consider the graphs $K_b + \alpha K_d$, where $b \geq 0$ and $d \geq 1$ are integers with $\alpha \geq bk + 2$, d odd and $b + \alpha d$ even. These graphs do not contain a k-factor (choose $(D, S) = (V(K_b), \emptyset)$ in the k-factor-theorem below) and have minimum degree $d + b - 1$, where d can be made arbitrarily large.

Choosing an additional condition we were led by the proof technique of the case with k even, where it is possible to assume that a $(k-2)$-factor exists.

Theorem 4. *Let $k \geq 3$ and l be odd integers with $1 \leq l \leq k$, and let G be a graph of even order at least $k + 1$ having an l-factor. If G satisfies*

$$\delta(G) > \frac{(k+1)^2}{4k}\alpha(G) + \frac{5k-4}{8} - \frac{2}{k},$$

then G has a k-factor.

The lower bound for the minimum degree is for every $k \geq 3$ again nearly best possible. Let therefore α, δ, l be positive integers with $l < k$ and

$$\max\left\{\alpha + \frac{k-3}{2}, \frac{(l+1)^2}{4l}\alpha + \frac{5l-4}{8} - \frac{2}{l}\right\} < \delta \leq \frac{(k+1)^2}{4k}\alpha + \frac{k-1}{2} - \frac{2}{k},$$

and let

$$\delta - \frac{k-1}{2} + \alpha\frac{k+1}{2}$$

be even. Then the graph

$$G = K_{\delta-(k-1)/2} + \alpha K_{(k+1)/2}$$

is of even order with $\alpha(G) = \alpha$ and $\delta(G) = \delta$. Since $\delta - (k-1)/2 \geq \alpha$, it is easy to see that G has a 1-factor. Thus by Theorem 4 we obtain that G has a l'-factor for every odd l' satisfying $1 \leq l' \leq l$. Finally, one can see that G does not contain a k-factor by choosing $(D, S) = (V(K_{\delta-(k-1)/2}), V(\alpha K_{(k+1)/2}))$ in the k-factor-theorem (Theorem 7) below.

From the preceding results it is easy to deduce the following

Corollary 5. *Let k be a positive integer and let G be a graph of order n with $n \geq k + 1$ and kn even. If G satisfies*

$$\kappa(G) > \begin{cases} \dfrac{k+2}{4}\alpha(G) + \dfrac{5k-3}{8} - \dfrac{2}{k} & \text{if } k \text{ is even} \\[2ex] \dfrac{(k+1)^2}{4k}\alpha(G) + \dfrac{5k-4}{8} - \dfrac{2}{k} & \text{if } k \text{ is odd,} \end{cases} \tag{1}$$

then G has a k-factor.

The first result of this type is the following theorem of Katerinis, which was motivated by a well-known result for hamiltonicity due to Chvátal and Erdös [4].

Theorem 6 [10]. *Let k be a positive integer and let G be a graph of order n with $n \geq k+1$ and kn even. If G satisfies*

$$\kappa(G) > \frac{(k+1)^2}{4k}\alpha(G) + \frac{5k-4}{8} - \frac{2}{k},$$

then G has a k-factor.

Corollary 5 is for even k a slight improvement of Theorem 6. A closely related result is due to Nishimura [18], who showed that the terms on the right-hand-side of (1) not depending on $\alpha(G)$ can be removed, if the connectivity is at least $\lfloor (k+1)^2/4 \rfloor$. This result cannot be derived from our results.

2. Proofs

We need some further notation. Let G be a graph and let $S \subseteq V(G)$ be non-empty. For convenience we write $d_G(S)$ instead of $\sum_{x \in S} d_G(x)$. By $G[S]$ we denote the subgraph of G induced by S. If $u \in V(G) - S$, then $e_G(u, S)$ denotes the number of edges joining u to a vertex in S. If $T \subseteq V(G) - S$, then we write $e_G(T, S)$ instead of $\sum_{u \in T} e_G(u, S)$.

Our proofs depend on a special case of *Tutte's f-factor theorem* [21], which was first proved by Belck [3] and characterizes those graphs that do not have a k-factor, where k is a non-negative integer. Let D, S be disjoint subsets of $V(G)$. We call a component of $G - (D \cup S)$ an *odd component* (of G with respect to (D, S, k)), if $k|V(C)| + e_G(V(C), S)$ is odd, and by $q_G(D, S, k)$ we denote the number of odd components. Let $\Theta_G(D, S, k) = k|D| - k|S| + d_{G-D}(S) - q_G(D, S, k)$.

Theorem 7 (k-factor-theorem). *Let G be a graph of order n and let k be a non-negative integer with kn even. Then the following statements hold.*

(i) [21] *$\Theta_G(D, S, k)$ is even for all disjoint sets $D, S \subseteq V(G)$;*
(ii) [3, 21] *G does not have a k-factor if and only if G has a k-*Tutte-pair, *that is a pair of disjoint subsets (D, S) of $V(G)$ with $\Theta_G(D, S, k) \leq -2$.*

The k-factor-theorem is often difficult to apply. One of the known approaches to obtain additional information depends on suitable choices of the k-Tutte-pairs. Especially, two choices, which have been made first by Katerinis [9] and Katerinis and Woodall [11] and by Enomoto, Jackson, Katerinis and Saito [7], respectively, have been used frequently. Here we need a partial combination of those. (Note that all choices of k-Tutte-pairs can be seen as consequences of the *transfer principle* described by Tutte [22, 23]. Another approach, which uses two k-Tutte-pairs simultaneously, was developed in [16].)

Lemma 8. *Let k be a positive integer and let G be a graph of order n with kn even. If (D, S) is a k-Tutte-pair of G such that $|S| - |D|$ is minimal, then the following two statements hold:*

a) $e_G(u, S) \le k - 1$ *for every* $u \in V(G) - (D \cup S)$,
b) $d_{G-D}(x) \le k - 2 + c(x) \le k - 2 + q_G(D, S, k)$ *for every* $x \in S$, *where* $c(x) = c(x, D, S, k)$ *denotes the number of odd components of* G *with respect to* (D, S, k) *containing a neighbor of* x.

Proof. a) If $u \in V(G) - (D \cup S)$, then $(D \cup \{u\}, S)$ is not a k-Tutte-pair of G by the choice of (D, S). Hence by Theorem 7

$$\begin{aligned} -2 &\ge \Theta_G(D, S, k) - \Theta_G(D \cup \{u\}, S, k) \\ &= -k + e_G(u, S) - q_G(D, S, k) + q_G(D \cup \{u\}, S, k) \\ &\ge -k + e_G(u, S) - 1, \end{aligned}$$

and thus $e_G(u, S) \le k - 1$.

b) Similar to a) we have that $(D, S - \{x\})$ is not a k-Tutte-pair for every $x \in S$. Now it follows together with $q_G(D, S - \{x\}, k) \ge q_G(D, S, K) - c(x)$

$$\begin{aligned} -2 &\ge \Theta_G(D, S, k) - \Theta_G(D, S - \{x\}, k) \\ &= -k + d_{G-D}(x) - q_G(D, S, k) + q_G(D, S - \{x\}, k) \\ &\ge -k + d_{G-D}(x) - c(x), \end{aligned}$$

and so $d_{G-D}(x) \le k - 2 + c(x) \le k - 2 + q_G(D, S, k)$. □

The following lemma will be very useful.

Lemma 9. *Let* G *be a graph without* k*-factor, where* $k \ge 2$ *is an integer. If* G *has a* $(k-2)$*-factor, then it holds* $|S| \ge |D| + 1$ *for every* k*-Tutte-pair* (D, S) *of* G.

Proof. Let (D, S) be a k-Tutte-pair of G. Then we have $\Theta_G(D, S, k) \le -2$ and $\Theta_G(D, S, k-2) \ge 0$ by Theorem 7. Since it holds $q_G(D, S, k) = q_G(D, S, k-2)$, we obtain

$$-2 \ge \Theta_G(D, S, k) - \Theta_G(D, S, k-2) = 2|D| - 2|S|,$$

and therefore $|S| \ge |D| + 1$. □

Proofs of Theorem 1 and Theorem 2. Let G be a graph with $\delta(G) \ge \alpha(G)$ having no 2-factor. Then G has a 2-Tutte-pair (D, S) by Theorem 7. Since G has a 0-factor, it holds by Lemma 9

$$|S| \ge |D| + 1 \tag{2}$$

for every 2-Tutte-pair (D, S) of G. Let (D, S) be chosen such that $|S| - |D|$ is minimal. Then we have $d_{G-D}(x) \le c(x)$ for every $x \in S$ by Lemma 8 b), and therefore S is an independent set and the neighbors of every $x \in S$ in $G - D$ belong to different components of $G - (D \cup S)$.

Suppose first that $|S| \le 1$. Then we obtain $|S| = 1$ and $|D| = 0$ by (2). Let v denote the vertex in S. Since the neighbors of v belong to different components of $G - v$, v is not contained in any cycle of G. Moreover, we have $\alpha(G) \ge \omega(G - v) \ge d_G(v) \ge \delta(G) \ge \alpha(G)$, and thus $\delta(G) = \alpha(G)$. So, the theorems are proved, if $|S| \le 1$.

Let now $|S| \geq 2$. For convenience we set $d = \min\{d_{G-D}(x) | x \in S\}$ and $q = q_G(D, S, 2)$. Obviously, it holds $|D| \geq \delta(G) - d$. Furthermore, by ω we denote the number of components of $G - (D \cup S)$.

Case A. $\delta(G) > \alpha(G)$.
With (2) we obtain

$$\delta(G) - 1 \geq \alpha(G) \geq |S| \geq |D| + 1 \geq \delta(G) - d + 1,$$

and hence $d \geq 2$. Since (D, S) is a 2-Tutte-pair, we have now

$$\begin{aligned}\alpha(G) &\geq q_G(D, S, 2) \geq 2|D| - 2|S| + d_{G-D}(S) + 2 \\ &\geq 2|D| + |S|(d-2) + 2 \geq 2(\delta(G) - d) + 2(d-2) + 2 \\ &= 2\delta(G) - 2 \geq 2\alpha(G).\end{aligned}$$

So, this case cannot occur. Note that thereby Theorem 1 is already proved.

Case B. $\delta(G) = \alpha(G)$.
We may assume that G has at least three vertices, since otherwise $G = K_2$ and so G has a vertex that belongs to no cycle of G. Especially, we have therefore $\alpha(G) \geq 2$, since G cannot be complete. Now let us verify

$$d = 1, \tag{3}$$

$$\delta(G) = \alpha(G) = |S| = |D| + 1, \tag{4}$$

and

$$q = \omega = d_{G-D}(S) = |S|. \tag{5}$$

Suppose, contrarily to (3), that $d \neq 1$. If $d = 0$, then we obtain the contradiction $\delta(G) = \alpha(G) \geq |S| \geq |D| + 1 \geq \delta(G) + 1$. If $d \geq 2$, then we get similar to Case A

$$\begin{aligned}\alpha(G) &\geq \omega \geq q \geq 2|D| - 2|S| + d_{G-D}(S) + 2 \\ &\geq 2|D| + |S|(d-2) + 2 \geq 2(\delta(G) - d) + 2(d-2) + 2 \\ &= 2\delta(G) - 2 = 2\alpha(G) - 2.\end{aligned}$$

By $\alpha(G) \geq 2$ we have therefore $\alpha(G) = 2$, and equality must hold in all estimations above. This yields $\alpha(G) = \delta(G) = \omega = q = d = |S| = 2$ and $|D| = 0$. Since it is easy to see that no graph can satisfy these equalities, we obtain again a contradiction.

With (2) and (3) we get (4) by

$$\delta(G) = \alpha(G) \geq |S| \geq |D| + 1 \geq \delta(G) - d + 1 = \delta(G).$$

Next we observe that with (4) it holds

$$q \geq 2|D| - 2|S| + d_{G-D}(S) + 2 = d_{G-D}(S).$$

Thus we obtain with (4) and (3)

$$q \leq \omega \leq \alpha(G) = |S| = d|S| \leq d_{G-D}(S) \leq q,$$

from which (5) follows.

By (3), (4) and (5) we can easily complete the proof. Since S is an independent set of $G - D$ with $|S| = \alpha(G)$, every vertex of $V(G) - (D \cup S)$ is joined by at least one edge to a vertex of S. So we have $|S| = d_{G-D}(S) \geq |V(G) - (D \cup S)| \geq q = |S|$, impling that $V(G) - (D \cup S)$ is an independent set of vertices, where each vertex has degree one in $G - D$. Since S is an independent set, the same statement holds for S because of $d = 1$ and $d_{G-D}(S) = |S|$. Thus $G - D = |S|K_2 = \delta(G)K_2$. Finally, $|D| = \delta(G) - 1$ yields $G = H + \delta(G)K_2$, where H is a graph of order $\delta(G) - 1$. This completes the proof of Theorem 2. □

In the next proof we need the following lower bound for the independence number due to Wei [24], which is often also attributed to Y. Caro (unpublished).

Theorem 10. *For every graph G it holds*

$$\alpha(G) \geq \sum_{x \in V(G)} \frac{1}{1 + d_G(x)}.$$

Proofs of Theorem 3 and 4. The proof is by contradiction. We suppose that an integer $k \geq 3$ and a graph G exist, which satisfy the hypotheses of Theorem 3 or Theorem 4 (depending on the parity of k) such that G has no k-factor. Without loss of generality we may assume that k is chosen minimal with respect to these properties. Then G has a $(k - 2)$-factor. This follows for $k = 4$ by Theorem 1 and for $k \neq 4$ by the choice of k.

By Lemma 9 we have

$$|S| \geq |D| + 1 \geq 1 \tag{6}$$

for every k-Tutte-pair (D, S) of G. We choose a k-Tutte-pair (D, S) of G, which is minimal with respect to $|S| - |D|$. By Lemma 8 we have

$$e_G(u, S) \leq k - 1 \qquad \text{for every } u \in V(G) - (D \cup S) \tag{7}$$

and

$$d_{G-D}(x) \leq k - 2 + q_G(D, S, k) \qquad \text{for every } x \in S. \tag{8}$$

It is easy to verify that the lower bounds for the minimum degree in the hypotheses imply

$$\delta(G) > \frac{\alpha(G) - 3}{2} + k \tag{9}$$

and

$$\delta(G) > \frac{\alpha(G) - 2}{k} + k. \tag{10}$$

This is left to reader, who should notice therefore that $\alpha(G) \geq 2$, since G cannot be complete because of $n \geq k + 1$ and nk even.

Let $q = q_G(D, S, k)$ and $d = \min\{d_{G-D}(x) | x \in V(G) - D\}$.

We now consider three cases.

Case 1. $d \le k - 1$.

Here we use the ideas of the proof of Theorem 6 from [10]. We only improve some of the estimations in the case where k is even.

Let $U = \{u_1, \ldots, u_q\}$ be a set containing exactly one vertex of every odd component of $G - (D \cup S)$ with respect to (D, S). Then let $H = G[S \cup U]$. For $i = 0, 1, \ldots, \Delta$, where $\Delta = \Delta(G)$ is the maximum degree of G, we set $S_i = \{x \in S \mid d_{G-D}(x) = i\}$ and $s_i = |S_i|$.

By Theorem 10 and (7) we obtain

$$\alpha(G) \ge \alpha(H) \ge \sum_{x \in S} \frac{1}{1 + d_H(x)} + \sum_{j=1}^{q} \frac{1}{1 + d_H(u_j)}$$

$$\ge \sum_{x \in S} \frac{1}{1 + d_{G-D}(x)} + \sum_{j=1}^{q} \frac{1}{1 + e_G(u_j, S)} \ge \sum_{i=d}^{\Delta} \frac{s_i}{1 + i} + \frac{q}{k}.$$

Thus

$$\alpha(G) \ge \sum_{i=d}^{\Delta} \frac{s_i}{1 + i} + \frac{q}{k}. \tag{11}$$

Let first k be even. Then it holds $k(k + 2)/4 \ge (k - i)(1 + i)$ for every integer i. So, if we multiply (11) by $k(k + 2)/4$, we obtain with $k \ge 2$

$$\frac{k(k + 2)}{4}\alpha(G) \ge \sum_{i=d}^{\Delta} \frac{k(k + 2)s_i}{4(1 + i)} + \frac{k + 2}{4} q \ge \sum_{i=d}^{\Delta} (k - i)s_i + q.$$

The right-hand-side of this inequality is equal to $k|S| - d_{G-D}(S) + q$, and therefore at least $k|D| + 2$, since (D, S) is a k-Tutte-pair of G. Together with $|D| \ge \delta(G) - d$ and the lower bound for the minimum degree we obtain

$$\frac{k(k + 2)}{4}\alpha(G) \ge k|D| + 2 \ge k\delta(G) - kd + 2$$

$$> \frac{k(k + 2)}{4}\alpha(G) + \frac{k(5k - 3)}{8} - kd.$$

Rearranging yields $d > (5k - 3)/8$.

If k is odd, then we multiply (11) by $(k + 1)^2/4$ and obtain analogously $d > (5k - 4)/8$.

Next we are going to multiply (11) by $(k - d)(1 + d)$. Therefore we notice that for $i \ge d > (5k - 4)/8$ it holds

$$(k - d)(1 + d) \ge (k - i)(1 + i),$$

since $(k - i)(1 + i)$ is monotone decreasing for $i \ge (k - 1)/2$. Moreover, we have $(k - d)(1 + d) \ge k$ for $0 \le d \le k - 1$. So we get as above

$$(k - d)(1 + d)\alpha(G) \ge \sum_{i=d}^{\Delta} (k - i)s_i + q = k|S| - d_{G-D}(S) + q$$

$$\ge k|D| + 2 \ge k\delta(G) - kd + 2.$$

Using the lower bounds for the minimum degree from the hypotheses we obtain after rearranging

$$\frac{k(5k-3)}{8} < \left[(k-d)(1+d) - \frac{k(k+2)}{4}\right]\alpha(G) + kd,$$

if k is even, and

$$\frac{k(5k-4)}{8} < \left[(k-d)(1+d) - \frac{(k+1)^2}{4}\right]\alpha(G) + kd,$$

if k is odd. Since the terms in the brackets are less than or equal to 0 for every integer d and since $\alpha(G) \geq 2$, we may replace $\alpha(G)$ by 2 in these inequalities. Thereafter rearranging yields

$$9k^2 + 13k < 8(3k - 2d)(1 + d),$$

if k is even, and

$$9k^2 + 12k + 4 < 8(3k - 2d)(1 + d),$$

if k is odd. Next we observe that $8(3k - 2d)(1 + d)$ becomes maximum for $d = (3k - 2)/4$, and thus $8(3k - 2d)(1 + d) \leq 9k^2 + 12k + 4$. Thereby we have already a contradiction if k is odd, and for even k we obtain $k < 4$, a contradiction.

Case 2. $d = k$.
Since (D, S) is a k-Tutte-pair of G, we obtain with $\alpha(G) \geq q$ and $|D| \geq \delta(G) - d$

$$\alpha(G) \geq q \geq k|D| - k|S| + d_{G-D}(S) + 2 \geq k|D| + 2 \geq k\delta(G) - k^2 + 2.$$

This yields $\delta(G) \leq (\alpha(G) - 2)/k + k$, contradicting (10).

Case 3. $d \geq k + 1$.
Since (D, S) is a k-Tutte-pair, we have

$$q \geq k|D| - k|S| + d_{G-D}(S) + 2 \geq |D| + |S| + 2. \tag{12}$$

We have now two subcases.

Case 3.1. $|S| \leq k - 1$.
Since S is non-empty by (6), we can choose a vertex $x \in S$.

First we consider the situation with $k = 3$. By (6) we have $0 \leq |D| \leq 1$, where $|S| = 2$, if $|D| = 1$. By (8) and the lower bounds for the minimum degree we obtain

$$1 + \alpha(G) \geq 1 + q \geq d_{G-D}(x) \geq \delta(G) - |D| > \frac{4}{3}\alpha(G) + \frac{17}{24} - |D|.$$

So, if $|D| = 0$, we get already the contradiction $\alpha(G) < 1$. If $|D| = 1$, we have only $\alpha(G) < 4$. A contradiction is then obtained with (12) by

$$4 \geq 1 + \alpha(G) \geq 1 + q \geq 1 + |D| + |S| + 2 \geq 6.$$

Let now $k \geq 4$. By (12) and (6) it holds $\alpha(G) \geq q \geq |D| + |S| + 2 \geq 2|D| + 3$. Thus it follows with (8)

$$k - 2 + \alpha(G) \geq k - 2 + q \geq d_{G-D}(x)$$
$$\geq \delta(G) - |D| \geq \delta(G) - \frac{\alpha(G) - 3}{2}.$$

Hence

$$\frac{2k - 7 + 3\alpha(G)}{2} \geq \delta(G). \tag{13}$$

Now we use the lower bounds for the minimum degree from the hypotheses in (13). For even k we get with $\alpha(G) \geq 2$ after rearranging

$$\frac{3k - 25}{8} + \frac{2}{k} > \frac{k - 4}{4}\alpha(G) \geq \frac{k - 4}{2}.$$

Now it is easy to see that this is impossible for $k \geq 4$. For odd $k \geq 5$ we get analogously

$$\frac{3k - 24}{8} + \frac{2}{k} > \frac{k^2 - 4k + 1}{4k}\alpha(G) \geq \frac{k^2 - 4k + 1}{2k},$$

which is impossible also. Thus we have obtained a contradiction for $k \geq 4$.

Case 3.2. $|S| \geq k$.
By (12) and (6) it follows $\alpha(G) \geq q \geq |D| + |S| + 2 \geq 2|D| + 3$, and therefore $|D| \leq (\alpha(G) - 3)/2$. Together with (9) we have

$$\begin{aligned}
\alpha(G) \geq q &\geq k|D| - k|S| + d_{G-D}(S) + 2 \\
&\geq k|D| - k|S| + |S|(\delta(G) - |D|) + 2 \\
&= (k - |S|)|D| + |S|(\delta(G) - k) + 2 \\
&\geq (k - |S|)\left(\frac{\alpha(G) - 3}{2}\right) + |S|(\delta(G) - k) + 2 \\
&= k\frac{\alpha(G) - 3}{2} + |S|\left(\delta(G) - k - \frac{\alpha(G) - 3}{2}\right) + 2 \\
&\geq k\frac{\alpha(G) - 3}{2} + k\left(\delta(G) - k - \frac{\alpha(G) - 3}{2}\right) + 2 \\
&= k(\delta(G) - k) + 2.
\end{aligned}$$

Thus

$$\delta(G) \leq \frac{\alpha(G) - 2}{k} + k,$$

contradicting (10). □

Proof of Corollary 5. Clearly, $\delta(G) \geq \kappa(G)$ holds for every graph G. So, for even k the statement follows directly from Theorem 2 and Theorem 3. If k is odd, it will suffice by Theorem 4 to prove that G has a 1-factor. Suppose therefore that G has no 1-factor. Then by Tutte's 1-factor theorem (see [20]) there exists a set $D \subset V(G)$ such that $o(G - D) > |D|$, where $o(G - D)$ denotes the number of components of odd order of $G - D$. Since G is of even order, we have $o(G - D) \geq |D| + 2$. This implies in particular that D is a cutset of G and so $|D| \geq \kappa(G)$. Now it follows

$$\begin{aligned} o(G - D) - 2 \geq |D| \geq \kappa(G) &> \frac{(k+1)^2}{4k}\alpha(G) + \frac{5k-4}{8} - \frac{2}{k} \\ &\geq \alpha(G) - 2 \geq o(G - D) - 2. \end{aligned}$$

This contradiction completes the proof of the corollary. □

Remark. As we mentioned in the introduction, no relation of the minimum degree and the independence number can be sufficient for hamiltonicity. This is obvious, since such relations do not imply that a graph is 2-connected or 1-tough. Moreover, there exist some well-known examples of 1-tough, non-hamiltonian graphs having independence number equal to three and arbitrarily large minimum degree (for example graphs consisting of three disjoint cliques of size at least three and the edge-sets of two vertex-disjoint triangles containing exactly one vertex of every clique). The well-known examples have connectivity equal to two. Recently, Bauer, Broersma, van den Heuvel and Veldman [2] gave two families of graphs with arbitrarily large connectivity. To describe one of these families let l and m be positive integers. Denote by $H_1, \ldots, H_{2m+1}$ disjoint copies of K_l and let T be a copy of K_{2m+1} disjoint from $H_1, \ldots, H_{2m+1}$ with $V(T) = \{u_1, \ldots, u_{2m+1}\}$. Form $H_{l,m}$ by joining the vertex u_i to all vertices of H_i for $i = 1, \ldots, 2m + 1$. Now let $G_{l,m} = K_m + H_{l,m}$. Then $G_{l,m}$ is a non-hamiltonian graph with $\delta(G_{l,m}) = l + m$, $\alpha(G_{l,m}) = 2m + 1$, $\kappa(G_{l,m}) = m + 1$ and toughness equal to $3m/(2m + 1)$.

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