Annali di Matematica pura ed applicata
(IV), Vol. CLXXV (1998), pp. 141-164

Partial Regularity of Minimizers of Quasiconvex Integrals with Subquadratic Growth (*).

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Abstract. – *We prove partial regularity for minimizers of quasiconvex integrals of the form* $\int_\Omega F(Du(x))\,dx$ *where the integrand* $F(\xi)$ *has subquadratic growth, i.e.* $|F(\xi)| \leq L(1+|\xi|^p)$, *with* $1<p<2$.

1. – Introduction.

In this paper we study the partial regularity of minimizers of the functional

$$I(u) = \int_\Omega F(Du(x))\,dx\,,$$

where Ω is a bounded open subset of $\mathbb{R}^n$, u is a $W^{1,p}(\Omega;\mathbb{R}^N)$ function , with $p>1$, and $F(\xi)$: $\mathbb{R}^{nN}\to\mathbb{R}$ is a C^2 uniformly strict quasiconvex function i.e.

$$\int_\Omega F(\xi + D\phi(x))\,dx \geq \int_\Omega [F(\xi) + \nu(1+|\xi|^2 + |D\phi(x)|^2)^{(p-2)/2}|D\phi(x)|^2]\,dx\,, \tag{1.1}$$

for any $\xi\in\mathbb{R}^{nN}$ and $\phi\in C_0^1(\Omega;\mathbb{R}^N)$.

This condition was introduced in case $p\geq 2$ in a paper by Evans (see [7]). He proved that if F satisfies (1.1) and

$$|D^2F(\xi)| \leq L(1+|\xi|^2)^{(p-2)/2} \tag{1.2}$$

(*) Entrata in Redazione il 3 febbraio 1997.

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then a minimizer of $I(u)$ is $C^{1,\alpha}$ on an open subset $\Omega_0 \subset \Omega$ such that $\text{meas}(\Omega - \Omega_0) = 0$.

This result was later generalized in [3] where condition (1.2) is dropped (see also [12]). At the time these papers were written no examples of genuine quasiconvex functions with subquadratic growth at infinity were known. However recently V. Šverak (see [16]) gave an example of a quasiconvex (and not convex neither polyconvex) function depending on 2×2 matrices and having polynomial growth with exponent $1 < p < 2$.

In this paper we extend Evans' result to the case where F satisfies (1.1) and p is any exponent between 1 and 2.

We notice that a first regularity result in this direction was obtained in [5] under the more restrictive assumption $2n/(n+2) < p < 2$.

The proof of the regularity of u is based, as usual, on a blow-up argument aimed to establish a decay estimate for the *excess* function

$$E(x_0, R) = -\!\!\!\!\!\!\int_{B_R(x_0)} |V(Du(x)) - V((Du)_{x_0, R})|^2 \, dx ,$$

where

$$V(\xi) = (1 + |\xi|^2)^{(p-2)/4} \xi .$$

Comparing to the case $p \geq 2$ and the case studied in [5], we have to face a few technical difficulties.

A point where one needs the assumption $p > 2n/(n+2)$ is in proving the following Sobolev-Poincaré type inequality

$$\left(-\!\!\!\!\!\!\int_{B_R} \left| V\left(\frac{u - u_R}{R} \right) \right|^{2n/(n-2)} \right)^{(n-2)/2n} \leq c \left(-\!\!\!\!\!\!\int_{B_R} |V(Du)|^2 dx \right)^{1/2} \tag{1.3}$$

provided $u \in W^{1,p}(\Omega; R^N)$.

It is not clear to us if (1.3) holds when $p \in (1, 2)$. Indeed we can prove that an inequality of this kind is still true if one adds a constant $c = c(n, p)$ on the right hand side of (1.3), but unfortunately this extra term would give serious troubles when one blows up the solution.

However, we have been able (see Theorem 2.4) to prove that if $p \in (1, 2)$ an inequality of the type (1.3) holds if one increases the radius of the ball on the right hand side.

Another technical point to overcome is to prove that if $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a weak solution of a linear elliptic system with constant coefficients, satisfying the strong Legendre-Hadamard condition, then u is locally $W^{1,2}$ hence C^∞: a fact that does not seem to be known in the literature.

2. – Preliminary results.

In the following Ω will denote a bounded open set of $\mathbb{R}^n$, $B_R(x_0)$ the ball $\{x \in \mathbb{R}^n : |x - x_0| < R\}$, and if h is an integrable function we define

$$h_{x_0, R} := \fint_{B_R(x_0)} h(x)\, dx = \frac{1}{\omega_n R^n} \int_{B_R(x_0)} h(x)\, dx\,,$$

where ω_n is the Lebesgue measure of the n-dimensional unit ball. When no confusion may arise we write simply h_R in place of $h_{x_0, R}$ or B_R in place of $B_R(x_0)$. Throughout the paper p will be a number between 1 and 2 and for $\xi \in \mathbb{R}^k$ we shall denote

$$V(\xi) = (1 + |\xi|^2)^{(p-2)/4} \xi\,. \tag{2.1}$$

The following statement contains some useful properties of the function V.

LEMMA 2.1. – *Let* $1 < p < 2$, *and* $V: \mathbb{R}^k \to \mathbb{R}^k$ *the function defined by* (2.1), *then for any* $\xi, \eta \in \mathbb{R}^k$, $t > 0$

(i) $2^{(p-2)/4} \min\{|\xi|, |\xi|^{p/2}\} \leq |V(\xi)| \leq \min\{|\xi|, |\xi|^{p/2}\}$,

(ii) $|V(t\xi)| \leq \max\{t, t^{p/2}\} |V(\xi)|$,

(iii) $|V(\xi + \eta)| \leq c(p)[|V(\xi)| + |V(\eta)|]$,

(iv) $\dfrac{p}{2} |\xi - \eta| \leq \dfrac{|V(\xi) - V(\eta)|}{(1 + |\xi|^2 + |\eta|^2)^{(p-2)/4}} \leq c(k, p) |\xi - \eta|$,

(v) $|V(\xi) - V(\eta)| \leq c(k, p) |V(\xi - \eta)|$,

(vi) $|V(\xi - \eta)| \leq c(p, M) |V(\xi) - V(\eta)|$ *if* $|\eta| \leq M$ *and* $\xi \in \mathbb{R}^k$.

PROOF. – Properties (i)-(ii) are easy to check.

To prove (iii) let us assume $|\eta| \leq |\xi|$. If $|\xi| \leq 1$ by (i) we get

$$|V(\xi + \eta)| \leq |\xi + \eta| \leq 2|\xi| \leq c(p) |V(\xi)|\,,$$

and if $|\xi| \geq 1$ by (i) again we get

$$|V(\xi + \eta)| \leq |\xi + \eta|^{p/2} \leq c(p) |\xi|^{p/2} \leq c(p) |V(\xi)|\,.$$

Inequality (iv) is proved in Lemma 2.2 in [4], while (v) can be immediately derived from (iv).

To prove (vi) notice that if $|\eta| \leq M$

$$|\xi - \eta|^2 \geq |\xi|^2 + |\eta|^2 - 2|\xi|\,|\eta| \geq \frac{3}{4} |\xi|^2 - 3|\eta|^2 \geq \frac{3}{4} |\xi|^2 - 3M^2\,.$$

From this inequality and from the inequality on the left in (iv) we then get

$$
\begin{aligned}
&|V(\xi-\eta)| = (1+|\xi-\eta|^2)^{(p-2)/4}|\xi-\eta| \leqslant \\
&\qquad \leqslant (1+4M^2)^{(2-p)/4}(1+4M^2+|\xi-\eta|^2)^{(p-2)/4}|\xi-\eta| \leqslant \\
&\qquad \leqslant (1+4M^2)^{(2-p)/4}\left(1+M^2+\frac{3}{4}|\xi|^2\right)^{(p-2)/4}|\xi-\eta| \leqslant \\
&\qquad \leqslant c(p,M)(1+|\xi|^2+|\eta|^2)^{(p-2)/4}|\xi-\eta| \leqslant c(p,M)\,|V(\xi)-V(\eta)|\,. \qquad \blacksquare
\end{aligned}
$$

Let $h\colon \mathbb{R}^n \to \mathbb{R}^k$ be a locally integrable function. The Hardy-Littlewood maximal function $M(h)$ is defined for any $x \in \mathbb{R}^n$ as

$$M(h)(x) = \sup_{r>0} \fint_{B_R(x)} |h(y)|\,dy\,.$$

It is well known that M is a continuous operator from L^q to L^q, if $q>1$. The following result, which is a slightly modified version of Proposition 1.2 in [10], shows that the continuity properties of the maximal operator M hold in more general situations.

PROPOSITION 2.2. – *Let* $A\colon [0,+\infty[\to [0,+\infty[$ *be a continuous function such that*

(i) $A(2t) \leqslant KA(t)$ *for any* $t>0$,

(ii) *there exists* $r>1$ *such that* $t \to A(t)/t^r$ *is increasing.*

Then there exists a constant $c \equiv c(n,K,r)$ *such that if* f *is a nonnegative, measurable function in* $\mathbb{R}^n$, *then*

$$\int_{\mathbb{R}^n} A(M(f))\,dx \leqslant c \int_{\mathbb{R}^n} A(f)\,dx\,.$$

We now apply this proposition to a particular case, which will be useful in the sequel.

PROPOSITION 2.3. – *Let* $1<p<2$ *and* $\alpha>2/p$, *then there exists* $c \equiv c(\alpha,p,n)$ *such that if* $h\colon \mathbb{R}^n \to \mathbb{R}^k$ *is measurable, then*

$$\int_{\mathbb{R}^n} |V(M(h))|^\alpha\,dx \leqslant c \int_{\mathbb{R}^n} |V(h)|^\alpha\,dx\,.$$

PROOF. – Set for $t>0$

$$A(t) = [(1+t^2)^{(p-2)/4}t]^\alpha$$

and notice that, by (ii) of Lemma 2.1, we have for any $t > 0$

$$A(2t) \leq 2^{\alpha} A(t)$$

and that, if $r = \alpha p/2 > 1$, the function

$$\frac{A(t)}{t^r} = \left[\frac{t}{(1+t^2)^{1/2}}\right]^{(\alpha(2-p))/2}$$

is increasing in $[0, +\infty[$. The result then follows from Proposition 2.2. ∎

We are now in position to prove the following Sobolev-Poincaré type inequality.

THEOREM 2.4. – *If* $1 < p < 2$, *there exist* $2/p < \alpha < 2$ *and* $\sigma > 0$ *such that if* $u \in W^{1,p}(B_{3R}(x_0), \mathbb{R}^N)$, *then*

$$\left(\fint_{B_R(x_0)} \left| V\left(\frac{u - u_{x_0,R}}{R}\right)\right|^{2(1+\sigma)} dx\right)^{1/(2(1+\sigma))} \leq c \left(\fint_{B_{3R}(x_0)} |V(Du)|^{\alpha} dx\right)^{1/\alpha}, \tag{2.2}$$

where $c \equiv c(n, p, N)$ *is independent on* R *and* u.

PROOF. – Setting $\tilde{u}(y) = (1/R)[u(x_0 + Ry) - u_{x_0,R}]$, we may always assume $x_0 = 0$, $R = 1$ and $u_{0,1} = 0$. Then for any $x \in B_1$ we have

$$|u(x)| \leq c(n, N) \int_{B_1} \frac{|Du(y)|}{|x-y|^{n-1}} dy =$$

$$= c(n, N)\left[\int_{B_1 \cap B_\varepsilon(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy + \int_{B_1 \setminus B_\varepsilon(x)} \frac{|Du(y)|}{|x-y|^{n-1}} dy\right],$$

with $0 < \varepsilon \leq 1$ to be chosen. Denoting by $\overline{Du}$ the zero extension of Du outside B_3, we have

$$|u(x)| \leq c\left[\sum_{i=0}^{\infty}\left(\frac{2^i}{\varepsilon}\right)^{n-1} \int_{\{\varepsilon 2^{-i-1} \leq |x-y| < \varepsilon 2^{-i}\}} |Du(y)|\, dy + \frac{1}{\varepsilon^{n-1}} \int_{B_2(x)} |Du(y)|\, dy\right] \leq$$

$$\leq c\left[\sum_{i=0}^{\infty} \frac{\varepsilon}{2^i} \fint_{B_{\varepsilon/2^i}(x)} |Du(y)|\, dy + \frac{1}{\varepsilon^{n-1}} \int_{B_2(x)} |Du(y)|\, dy\right] \leq$$

$$\leq c\left[\varepsilon (M(\overline{Du}))(x) + \frac{1}{\varepsilon^{n-1}} \fint_{B_2(x)} |Du(y)|\, dy\right].$$

Noticing that the function $t \to (1+t^2)^{(p-2)/2} t^2$ is increasing in $[0, +\infty[$ and using

(ii) and (iii) of Lemma 2.1, from the inequality above we deduce, since $0 < \varepsilon \leq 1$,

$$
\text{(2.3)} \qquad |V(u(x))|^2 \leq c(n, p, N)\left[|V(\varepsilon M(\overline{Du})(x))|^2 + \left| V\left(\frac{1}{\varepsilon^{n-1}} \fint_{B_2(x)} |Du(y)|\, dy \right)\right|^2 \right] \leq
$$

$$
\leq c\left[\frac{\varepsilon^p}{p} |V(M(\overline{Du})(x))|^2 + \frac{1}{2(n-1)\varepsilon^{2(n-1)}} \left| V\left(\fint_{B_2(x)} |Du(y)|\, dy \right)\right|^2 \right].
$$

The quantity in square brackets attains its minimum when

$$
\varepsilon = \left[\frac{\left| V\left(\fint_{B_2(x)} |Du(y)|\, dy \right)\right|}{|V(M(\overline{Du})(x))|} \right]^{2/(2(n-1)+p)}.
$$

Since

$$
\fint_{B_2(x)} |Du(y)|\, dy \leq (M(\overline{Du}))(x),
$$

the value of ε given above is less than or equal to 1. Inserting this value in (2.3) we obtain easily

$$
\text{(2.4)} \qquad |V(u(x))|^2 \leq c\, |V(M(\overline{Du})(x))|^{(4(n-1))/(2(n-1)+p)} \left| V\left(\fint_{B_2(x)} |Du(y)| \right)\right|^{2p/(2(n-1)+p)}.
$$

Let us choose now α such that

$$
\max\left\{ \frac{2}{p}, \frac{4(n-1)}{2(n-1)+p} \right\} < \alpha < 2 .
$$

Raising both sides of (2.4) to $\alpha((2(n-1)+p)/4(n-1))$, integrating on B_1, and using (ii) of Lemma 2.1 and Proposition 2.3 we have

$$
\int_{B_1} |V(u(x))|^{\alpha[2(n-1)+p]/2(n-1)}\, dx \leq
$$

$$
\leq c \left| V\left(\fint_{B_3} |Du(y)|\, dy \right)\right|^{\alpha p/2(n-1)} \int_{B_1} |V(M(\overline{Du})(x))|^{\alpha}\, dx \leq
$$

$$
\leq c \left[\left| V\left(\fint_{B_3} |Du(y)|\, dy \right)\right|^{\alpha} \right]^{p/2(n-1)} \int_{B_3} |V(Du(x))|^{\alpha}\, dx .
$$

Finally, notice that from (i) of Lemma 2.1 it is clear that there exists c_α such that

$$c_\alpha^{-1} g_\alpha(|\xi|) \leqslant |V(\xi)|^\alpha \leqslant c_\alpha g_\alpha(|\xi|), \qquad \text{for any } \xi,$$

where

$$g_\alpha(t) := \begin{cases} t^\alpha, & \text{if } 0 \leqslant t \leqslant 1, \\ \dfrac{2}{p}\, t^{\alpha p/2} + 1 - \dfrac{2}{p}, & \text{if } t \geqslant 1. \end{cases}$$

Since $\alpha > 2/p$, g_α is convex, hence we have, using Jensen's inequality

$$\int_{B_1} |V(u(x))|^{\alpha[2(n-1)+p]/2(n-1)}\, dx \leqslant$$

$$\leqslant c \left[g_\alpha \left(\fint_{B_3} |Du(x)|\, dx \right) \right]^{p/2(n-1)} \int_{B_3} |V(Du(x))|^\alpha\, dx \leqslant$$

$$\leqslant c \left(\fint_{B_3} g_\alpha(|Du(x)|)\, dx \right)^{p/2(n-1)} \int_{B_3} |V(Du(x))|^\alpha\, dx \leqslant c \left(\int_{B_3} |V(Du(x))|^\alpha\, dx \right)^{1+(p/2(n-1))}.$$

Setting $1 + \sigma = \alpha[2(n-1)+p]/4(n-1)$ and noticing that from the definition of α it is clear that $\sigma > 0$, from the inequality above we then get

$$\int_{B_1} |V(u(x))|^{2(1+\sigma)}\, dx \leqslant c \left(\int_{B_3} |V(Du(y))|^\alpha\, dy \right)^{2(1+\sigma)/\alpha},$$

which proves the result. ∎

REMARK 2.5. – The Sobolev-Poincaré inequality we have just proved uses some ideas from [14]. This inequality is an essential tool in order to get the regularity result Theorem 3.2. It is also one point where our case most differs from the case $p \geqslant 2$, when $|V(\xi)|$ is equivalent to $|\xi|^2 + |\xi|^p$. Under the extra assumption $p > 2n/(n+2)$ one can show that the inequality proved in Theorem 2.4 holds with the same ball (see [5]). It is not clear to us if one can still get in our situation the same sort of estimate without having to pass from the ball B_R to B_{3R}.

The following simple lemma is proved in [11] (see Lemma 3.1, Chap. 5).

LEMMA 2.6. – *Let* $f: [r/2, r] \to [0, +\infty[$ *be a bounded function such that for all* $r/2 < t < s < r$

$$f(t) \leqslant \theta f(s) + \frac{A}{(s-t)^\alpha},$$

where A, α, θ *are nonnegative constants such that* $\theta < 1$. *Then there exists* $c \equiv c(\theta)$

such that

$$f\left(\frac{r}{2}\right) \leqslant c(\theta)\,\frac{A}{r^{\alpha}}\,.$$

This result can be easily extended, using condition (ii) in place of homogenity.

LEMMA 2.7. – *Let* $f:[r/2, r] \to [0, +\infty[$ *be a bounded function such that for all* $r/2 < t < s < r$

$$f(t) \leqslant \theta f(s) + A \int_{B_r} \left| V\left(\frac{h(x)}{s-t}\right)\right|^2 dx\,,$$

where $h \in L^p(B_r)$, $A > 0$, *and* $0 < \theta < 1$. *Then there exists* $c \equiv c(\theta)$ *such that*

$$f\left(\frac{r}{2}\right) \leqslant c(\theta)\, A \int_{B_r} \left| V\left(\frac{h(x)}{r}\right)\right|^2 dx\,.$$

We are now in position to prove the following higher integrability result (see [3], for the case $p \geqslant 2$).

LEMMA 2.8. – *Let* $g\colon \mathbb{R}^{nN} \to \mathbb{R}$ *be a continuous function such that*

$$|g(\xi)| \leqslant L\,|V(\lambda\xi)|^2\,,$$

$$\int_{\Omega} g(D\phi(x))\,dx \geqslant \nu \int_{\Omega} |V(\lambda D\phi(x))|^2\,dx$$

for any $\phi \in C_0^1(\Omega, \mathbb{R}^N)$ *and suitable constants* L, ν, $\lambda > 0$. *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *satisfy*

$$\int_{\Omega} g(Du(x))\,dx \leqslant \int_{\Omega} g(Du(x) + D\phi(x))\,dx$$

for all $\phi \in C_0^1(\Omega, \mathbb{R}^N)$, *where* $1 < p < 2$. *Then there exist* c, δ, *depending only on* p, n, N, L, ν *but not on* λ *and* u *such that for any* $B_R(x_0) \subset \Omega$

$$\fint_{B_{R/2}} |V(\lambda Du)|^{2(1+\delta)}\,dx \leqslant c\left(\fint_{B_R} |V(\lambda Du)|^2\,dx\right)^{1+\delta}. \tag{2.5}$$

PROOF. – Fix B_r such that $B_{3r} \subset \Omega$, $r/2 < t < s < r$, and take a cut-off function $\zeta \in C_0^1(B_s)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ on B_t and $|D\zeta| \leqslant 2/(s-t)$. If we set

$$\phi_1 = [u - u_r]\,\zeta\,, \qquad \phi_2 = [u - u_r](1-\zeta)\,,$$

then $D\phi_1 + D\phi_2 = Du$, and we have, by the minimality of u

$$\nu \int_{B_s} |V(\lambda D\phi_1)|^2 dx \leq \int_{B_s} g(D\phi_1)\, dx = \int_{B_s} g(Du - D\phi_2)\, dx =$$

$$= \int_{B_s} g(Du)\, dx + \int_{B_s} [g(Du - D\phi_2) - g(Du)]\, dx \leq$$

$$\leq \int_{B_s \setminus B_t} g(D\phi_2)\, dx + \int_{B_s \setminus B_t} [g(Du - D\phi_2) - g(Du)]\, dx \leq$$

$$\leq L \int_{B_s \setminus B_t} [|V(\lambda D\phi_2)|^2 + |V(\lambda(Du - D\phi_2))|^2 + |V(\lambda Du)|^2]\, dx .$$

From this inequality, using (iii) of Lemma 2.1 we then get

$$\int_{B_t} |V(\lambda Du)|^2\, dx \leq \int_{B_s} |V(\lambda D\phi_1)|^2\, dx \leq \tilde{c} \int_{B_s \setminus B_t} |V(\lambda Du)|^2\, dx + \tilde{c} \int_{B_s \setminus B_t} \left| V\left(\lambda \frac{u - u_r}{s - t}\right) \right|^2 dx .$$

Adding to both sides of the previous inequality the quantity

$$\tilde{c} \int_{B_t} |V(\lambda Du)|^2\, dx ,$$

we obtain that for any $r/2 < t < s < r$:

$$\int_{B_t} |V(\lambda Du)|^2\, dx \leq \frac{\tilde{c}}{1 + \tilde{c}} \int_{B_s} |V(\lambda Du)|^2\, dx + \frac{\tilde{c}}{1 + \tilde{c}} \int_{B_r} \left| V\left(\lambda \frac{u - u_r}{s - t}\right) \right|^2 dx .$$

Now, Lemma 2.7 implies that

$$\int_{B_{r/2}} |V(\lambda Du)|^2\, dx \leq c \int_{B_r} \left| V\left(\lambda \frac{u - u_r}{r}\right) \right|^2 dx$$

and so, by (2.2) we get

$$⨍_{B_{r/2}} |V(\lambda Du)|^2 \leq c \left(⨍_{B_r} \left| V\left(\lambda \frac{u - u_r}{r}\right) \right|^{2(1+\sigma)} \right)^{1/(1+\sigma)} \leq c \left(⨍_{B_{3r}} |V(\lambda Du)|^\alpha \right)^{2/\alpha},$$

with $2/p < \alpha < 2$. From this inequality the result follows immediately just applying the Gehring's Lemma version due to Giaquinta and Modica (see [11] Theorem 1.1, Chap. 5). ■

The following lemma is a slightly modified version of the approximation result proved in [2].

LEMMA 2.9. – *Let* $u \in W^{1,q}(\mathbb{R}^n, \mathbb{R}^N)$, *with* $q \geq 1$. *For every* $K > 0$, *if we set*

$$H_K = \{x \in \mathbb{R}^n : M(Du) \leq K\},$$

then there exists a lipschitz function $w : \mathbb{R}^n \to \mathbb{R}^N$ *such that*

$$\|Dw\|_\infty \leq cK, \qquad w = v \quad on\ H_K,$$

$$\operatorname{meas}(\mathbb{R}^n \setminus H_K) \leq \frac{c\|Du\|_q^q}{K^q},$$

where c depends only on n, N, q.

Next result is a simple consequence of the a priori estimates for solutions of linear elliptic systems with constant coefficients.

PROPOSITION 2.10. – *Let* $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ *such that*

$$\int_\Omega A_{\alpha\beta}^{ij} D_\alpha u^i D_\beta \phi^j \, dx = 0 \tag{2.6}$$

for any $\phi \in C_0^1(\Omega, \mathbb{R}^N)$, *where* $(A_{\alpha\beta}^{ij})$ *is a constant matrix satisfying the strong Legendre-Hadamard condition:*

$$A_{\alpha\beta}^{ij} \lambda^i \lambda^j \mu_\alpha \mu_\beta > \nu |\lambda|^2 |\mu|^2, \quad for\ any\ \lambda \in \mathbb{R}^N, \quad \mu \in \mathbb{R}^n.$$

Then u is C^∞ *and for any* $B_R(x_0) \subset \Omega$ *and* $0 < \varrho < R$

$$\sup_{B_{R/2}} |Du| \leq \frac{c}{R^n} \int_{B_R} |Du| \, dx, \tag{2.7}$$

where c depends only on n, N, p, ν *and* $\max |A_{\alpha\beta}^{ij}|$.

PROOF. – *Step* 1. Let $v \in W_{\mathrm{loc}}^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of (2.6). It is well known that v is C^∞ in Ω. We want to show that if $B_R(x_0) \subset\subset \Omega$ then

$$\sup_{B_{R/2}} |Dv| \leq \frac{c}{R^n} \int_{B_R} |Dv| \, dx. \tag{2.8}$$

By a rescaling argument it is clear that to prove this inequality we may assume $x_0 = 0$ and $R = 1$. From Caccioppoli inequality for solutions of linear elliptic systems we get that for any $1/2 < t < s < 1$

$$\int_{B_t} |Dv|^2 \, dx \leq \frac{c}{(s-t)^2} \int_{B_s} |v|^2 \, dx.$$

Since also higher order derivatives of v are solutions of (2.6), iterating the inequality above on suitably nested balls we get that for any $h = 1, 2, \dots$

$$\int_{B_t} |D^h v|^2 \, dx \le \frac{c(h)}{(s-t)^{2h}} \int_{B_s} |v|^2 \, dx .$$

Therefore, since $1/2 < t < 1$, we may estimate

$$\sup_{B_t} |v| \, dx \le c(n) \|v\|_{W^{n,2}(B_t)} \le \frac{c}{(s-t)^n} \left(\int_{B_s} |v|^2 \, dx \right)^{1/2} \le \frac{c}{(s-t)^n} \left(\int_{B_s} |v| \, dx \right)^{1/2} \left(\sup_{B_s} |v| \right)^{1/2} .$$

Using Young inequality we obtain

$$\sup_{B_t} |v| \le \frac{1}{2} \sup_{B_s} |v| + \frac{c}{(s-t)^{2n}} \int_{B_1} |v| \, dx .$$

Finally, from Lemma 2.6 we conclude that

$$\sup_{B_{1/2}} |v| \le c \int_{B_1} |v| \, dx .$$

Applying this estimate to the derivatives $D_i v$ of v we then get (2.8).

Step 2. Let $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ be a solution of (2.6). If $\varrho(x)$ is a simmetric mollifier we set for any $x \in \Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$

$$u_\varepsilon(x) = \int_{|z|<1} \varrho(z) \, u(x + \varepsilon z) \, dz .$$

Then u_ε is smooth and one readily checks that for any $\phi \in C_0^1(\Omega_\varepsilon, \mathbb{R}^N)$

$$\int_{\Omega_\varepsilon} A_{\alpha\beta}^{ij} D_\alpha u_\varepsilon^i D_\beta \phi^j \, dx = 0 .$$

From (2.8) it follows that if $B_R \subset \Omega_\varepsilon$

$$\sup_{B_{R/2}} |Du_\varepsilon| \le \frac{c}{R^n} \int_{B_R} |Du_\varepsilon| \, dx .$$

Since $Du_\varepsilon \to Du$ locally in L^1 it follows that u satisfies (2.7), and so it is smooth. ■

We conclude this section recalling a selection lemma due to Eisen (see [6]).

LEMMA 2.11. – *Let G be a measurable subset of $\mathbb{R}^k$, with meas$(G) < +\infty$. Assume (M_h) is a sequence of measurable subets of G such that, for some $\varepsilon > 0$, the following* *estimate holds*:

$$\text{meas}(M_h) \geq \varepsilon \quad \textit{for all } h \in \mathbb{N}.$$

Then a subsequence (M_{h_k}) *can be selected such that* $\cap_k M_{h_k} \neq \emptyset$.

3. – Proof of the main result.

In this section we will prove the partial regularity of minimizers of the functional

$$I(v) := \int_\Omega F(Dv(x))\, dx,$$

where $v \in W^{1,p}(\Omega, \mathbb{R}^N)$, $1 < p < 2$, and $F: \mathbb{R}^{nN} \to \mathbb{R}$ is a C^2 function satisfying the following assumptions

$$\text{(H}_1\text{)} \qquad |F(\xi)| \leq L(1 + |\xi|^p),$$

$$\text{(H}_2\text{)} \qquad \int_\Omega F(\xi + D\phi(x))\, dx \geq \int_\Omega [F(\xi) + \nu(1 + |\xi|^2 + |D\phi(x)|^2)^{(p-2)/2} |D\phi(x)|^2]\, dx$$

for any $\xi \in \mathbb{R}^{nN}$ and any $\phi \in C_0^1(\Omega, \mathbb{R}^N)$, with $\nu > 0$.

REMARK 3.1. – Condition (H$_2$), introduced in [7] in the case $p \geq 2$, is called *uniform strict quasiconvexity* and implies that for any $\xi \in \mathbb{R}^{nN}$, $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}^n$

$$\frac{\partial^2 F}{\partial \xi^i_\alpha \partial \xi^j_\beta}(\xi)\, \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq c\nu(1 + |\xi|^2)^{(p-2)/2} |\lambda|^2 |\mu|^2,$$

with c independent on ξ, λ, μ.

Notice that we do not assume any control on second derivatives. However, if a function F is *quasiconvex*, i.e. verifies (H$_2$) with $\nu = 0$, and has the growth control (H$_1$), then it is well known (see [15]) that

$$\text{(3.1)} \qquad |DF(\xi)| \leq c(n, N, p)\, L(1 + |\xi|^{p-1}).$$

We also recall that a function $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a *minimizer* of $I(v)$ if for any function $\phi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$

$$I(u) \leq I(u + \phi).$$

We can now state the main result of this section.

THEOREM 3.2. – *Let* F *be a* C^2 *function satisfying* (H$_1$) *and* (H$_2$) *and* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a minimizer of functional* $I(v)$. *Then there exists an open subset* Ω_0 *of* Ω *such that* $\text{meas}(\Omega \setminus \Omega_0) = 0$ *and* u *is* $C^{1,\gamma}(\Omega_0, \mathbb{R}^N)$ *for any* $\gamma < 1$.

A standard technique in order to prove such kind of results is to look at the decay in small balls around a point x_0 of the so called *excess* of the gradient of the solution u. Roughly speaking the excess $E(x_0, R)$ measures how far is the gradient from being constant in the ball $B_R(x_0)$. In our case it is convenient to define

$$E(x_0, R) = \fint_{B_R(x_0)} |V(Du(x)) - V((Du)_{x_0, R})|^2 \, dx \,,$$

where V is the function given by (2.1). Notice that if $p \geq 2$ this quantity is equivalent to

$$\fint_{B_R(x_0)} |Du(x) - (Du)_{x_0, R}|^2 + \fint_{B_R(x_0)} |Du(x) - (Du)_{x_0, R}|^p \, dx \,,$$

but it is not so in our case.

We shall prove our decay estimate, Proposition 3.4 below, by a more or less standard argument consisting in blowing up the solution in small balls and reducing the problem to the study of convergence in the unit ball of solutions of suitably rescaled functionals. To this aim we need the following simple technical result which is a modified version of Lemma 2.4 in [4] and is proved exactly in the same way.

LEMMA 3.3. – *Let $f \colon \mathbb{R}^k \to \mathbb{R}$ be a function of class C^2 satisfying for any $\xi \in \mathbb{R}^k$*

$$|Df(\xi)| \leq L(1 + |\xi|^2)^{(p-1)/2} \,,$$

with $1 < p < 2$. Then for any $M > 0$ there exists a constant c depending only on M, p, L, such that if we set for any $\lambda > 0$ and $A \in \mathbb{R}^k$ with $|A| \leq M$

$$f_{A, \lambda}(\xi) = \lambda^{-2}[f(A + \lambda\xi) - f(A) - \lambda Df(A)\xi] \,,$$

then

$$|f_{A, \lambda}(\xi)| \leq c(p, L, M)(1 + |\lambda\xi|^2)^{(p-2)/2} |\xi|^2 \,.$$

We can now establish the decay estimate of $E(x_0, R)$. The proof we give is based on an idea contained in [8], later modified in [4] in order to deal with functionals with no control on the second derivatives (see also [5]). We will follow closely the various steps of the proof as presented in [4].

PROPOSITION 3.4 [*Decay estimate*]. – *Fix $M > 0$; there exists a constant C_M such that for every $0 < \tau < 1/4$ there is an $\varepsilon \equiv \varepsilon(\tau, M)$ such that if*

$$|(Du)_{x_0, R}| \leq M \quad \text{and } E(x_0, R) < \varepsilon$$

then

$$E(x_0, \tau R) \leq C_M \tau^2 E(x_0, R) \,.$$

PROOF. – Fix M and τ. We shall determine C_M at the end of the proof.

Step 1: *blow-up.* We argue by contradiction, assuming that there is a sequence $B_{R_h}(x_h)$ of balls contained in Ω such that

$$|(Du)_{x_h, R_h}| \leq M \,, \quad \lim_h E(x_h, R_h) = 0$$

and

$$E(x_h, \tau R_h) > C_M \tau^2 E(x_h, R_h). \tag{3.2}$$

We introduce the following notations:

$$a_h = u_{x_h, R_h}, \qquad A_h = (Du)_{x_h, R_h}, \qquad \lambda_h^2 = E(x_h, R_h);$$

and rescale the function u in each ball $B_{R_h}(x_h)$ to obtain a sequence of functions on $B_1(0)$:

$$v_h(y) = \frac{1}{\lambda_h R_h}[u(x_h + R_h y) - a_h - R_h A_h y].$$

Clearly, we have

$$Dv_h(y) = \frac{1}{\lambda_h}[Du(x_h + R_h y) - A_h], \qquad (v_h)_{0,1} = 0, \qquad (Dv_h)_{0,1} = 0.$$

Moreover from (ii) and (vi) of Lemma 2.1 we have

$$\fint_{B_1(0)} |V(Dv_h(y))|^2\, dy = \fint_{B_{R_h}(x_h)} \left| V\left(\frac{Du(x) - (Du)_{x_h, R_h}}{\lambda_h} \right) \right|^2 dx \leqslant$$

$$\leqslant \frac{\tilde{c}(M)}{\lambda_h^2} \fint_{B_{R_h}(x_h)} |V(Du(x)) - V((Du)_{x_h, R_h})|^2\, dx = \tilde{c}.$$

Hence from (i) of Lemma 2.1 we may conclude that the sequence (Dv_h) is bounded in $L^p(B_1, \mathbb{R}^{nN})$:

$$\|Dv_h\|_p \leqslant c, \qquad \text{for any } h, \tag{3.3}$$

and assume, without loss of generality, that

$$v_h \rightharpoonup v, \qquad \text{weakly in } W^{1,p}(B_1, \mathbb{R}^N)$$

and, since $|A_h| \leqslant M$,

$$A_h \to A.$$

Step 2: *v solves a linear system.* From the Euler system for u, rescaled in each $B_{R_h}(x_h)$, we deduce for every $\phi \in C_0^1(B_1, \mathbb{R}^N)$

$$\int_{B_1} \frac{\partial F}{\partial \xi_\alpha^i}(A_h + \lambda_h Dv_h)\, D_\alpha \phi^i\, dy = 0,$$

and also

$$\int_{B_1}\left[\frac{\partial F}{\partial \xi_\alpha^i}(A_h+\lambda_h Dv_h)-\frac{\partial F}{\partial \xi_\alpha^i}(A_h)\right]D_\alpha \phi^i\,dy=0\,.$$

If we split B_1 as

$$E_h^+\cup E_h^-=\{y\in B_1\colon \lambda_h\,|Dv_h(y)|\geqslant 1\}\cup\{y\in B_1\colon \lambda_h\,|Dv_h(y)|<1\}\,,$$

we get by (3.3)

$$\text{(3.4)}\qquad \operatorname{meas}(E_h^+)\leqslant\int_{B_1}\lambda_h^p\,|Dv_h|^p\,dy\leqslant c\lambda_h^p.$$

Therefore, using (3.1) and (3.3), we deduce that

$$\frac{1}{\lambda_h}\int_{E_h^+}|[DF(A_h+\lambda_h Dv_h)-DF(A_h)]\,D\phi|\,dy\leqslant$$

$$\leqslant\frac{c}{\lambda_h}\int_{E_h^+}(1+\lambda_h^{p-1}\,|Dv_h|^{p-1})\,dy\leqslant$$

$$\leqslant c\,\frac{|E_h^+|}{\lambda_h}+c\lambda_h^{p-2}\left(\int_{E_h^+}|Dv_h|^p\,dy\right)^{(p-1)/p}|E_h^+|^{1/p}\leqslant c\lambda_h^{p-1}\,,$$

which implies

$$\text{(3.5)}\qquad \lim_h\frac{1}{\lambda_h}\int_{E_h^+}|[DF(A_h+\lambda_h Dv_h)-DF(A_h)]\,D\phi|\,dy=0\,.$$

Now we observe that (3.4) implies that $\chi_{E_h^-}\to 1$ in $L^q(B_1)$ for all $q<\infty$ and that by (3.3) we have $\lambda_h Dv_h(y)\to 0$ a.e. in B_1. Then on E_h^- we may write

$$\int_{E_h^-}[DF(A_h+\lambda_h Dv_h)-DF(A_h)]\,D\phi\,dy=\int_{E_h^-}dy\int_0^1 D^2F(A_h+s\lambda_h Dv_h)\,Dv_h D\phi\,ds=$$

$$=\int_{E_h^-}dy\int_0^1[D^2F(A_h+s\lambda_h Dv_h)-D^2F(A_h)]\,Dv_h D\phi\,ds+\int_{E_h^-}D^2F(A_h)\,Dv_h D\phi\,dy\,.$$

letting $h\to\infty$ and using the uniform continuity of D^2F on bounded sets we obtain

$$\lim_h\frac{1}{\lambda_h}\int_{E_h^-}[DF(A_h+\lambda_h Dv_h)-DF(A_h)]\,D\phi\,dy=\int_{B_1}D^2F(A)\,DvD\phi\,dy\,,$$

which together with (3.5) implies

$$\int_{B_1} \frac{\partial^2 F}{\partial \xi^i_\alpha \partial \xi^j_\beta}(A)\, D_\alpha v^i D_\beta \phi^j \, dy = 0 .$$

By Remark 3.1 the coefficients of this linear system satisfy the inequality

$$c(\nu, M)\,|\lambda|^2\,|\mu|^2 \leq \frac{\partial^2 F}{\partial \xi^i_\alpha \partial \xi^j_\beta}(A)\,\lambda^i \lambda^j \mu_\alpha \mu_\beta \leq c(M)\,|\lambda|^2\,|\mu|^2 ,$$

hence from Lemma 2.10 we deduce that v is C^∞ in B_1. Moreover from the theory of linear systems (see [11], Theorem 2.1, Chap. 3) and by (2.7) and (3.3) we get that if $0 < \tau < 1/2$

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2 \, dy \leq c(M)\,\tau^2 \fint_{B_{1/2}} |Dv - (Dv)_{1/2}|^2 \, dy \leq \tag{3.6}$$

$$\leq c(M)\,\tau^2 \sup_{B_{1/2}} |Dv|^2 \leq c(M)\,\tau^2 \left(\fint_{B_1} |Dv|^p \right)^{2/p} \leq C^*(M) \tau^2 .$$

Step 3: higher integrability of v_h. If we set

$$F_h(\xi) := \lambda_h^{-2}[F(A_h + \lambda_h \xi) - F(A_h) - \lambda_h DF(A_h)\xi],$$

by (3.1) we may apply Lemma 3.3 and deduce

$$|F_h(\xi)| \leq \frac{c(M)}{\lambda_h^2} |V(\lambda_h \xi)|^2 . \tag{3.7}$$

On the other hand (H_2) implies that

$$\int_{B_1} F_h(D\phi(y))\, dy \geq \frac{\nu}{\lambda_h^2} \int_{B_1} |V(\lambda_h D\phi(y))|^2 \, dy \tag{3.8}$$

for any $\phi \in C^1_0(B_1, \mathbb{R}^N)$. Set for any $0 < r \leq 1$

$$I^h_r(w) := \int_{B_r} F_h(Dw(y))\, dy ;$$

it is then easily verified that v_h is a minimizer of I^h_r. Therefore, applying Lemma 2.8 to the functions $g_h(\xi) = \lambda_h^2 F_h(\xi)$, we get by (2.5) and (vi) of Lemma 2.1

$$\fint_{B_{1/2}} |V(\lambda_h Dv_h)|^{2(1+\delta)} \, dy \leq c \left(\fint_{B_1} |V(\lambda_h Dv_h)|^2 \, dy \right)^{1+\delta} = \tag{3.9}$$

$$= c \left(\fint_{B_{R_h}(x_h)} |V(Du(x) - A_h)|^2 \, dx \right)^{1+\delta} \leq$$

$$\leq c(M) \left(\fint_{B_{R_h}(x_h)} |V(Du(x)) - V(A_h)|^2 \, dx \right)^{1+\delta} \leq c \lambda_h^{2(1+\delta)} ,$$

and this gives immediately by (i) of Lemma 2.1 that the sequence (Dv_h) is bounded in $L^{p(1+\delta)}(B_{1/2}, \mathbb{R}^{nN})$.

Step 4: upper bound. Fix $r < 1/3$. Passing to a (not relabelled) subsequence, we may always assume that

$$\lim_h [I_r^h(v_h) - I_r^h(v)]$$

exists. We claim that

$$\lim_h [I_r^h(v_h) - I_r^h(v)] \leq 0 .$$

Choose $s < r$ and take $\zeta \in C_0^\infty(B_r)$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on B_s and $|D\zeta| \leq \leq 2/(r-s)$. If we set $\phi_h = (v - v_h)\zeta$ by (iii) and (ii) of Lemma 2.1, (3.7) and the minimality of v_h it follows

$$I_r^h(v_h) - I_r^h(v) \leq I_r^h(v_h + \phi_h) - I_r^h(v) = \int_{B_r \setminus B_s} [F_h(Dv_h + D\phi_h) - F_h(Dv)]\, dy \leq$$

$$\leq \frac{c}{\lambda_h^2} \int_{B_r \setminus B_s} [|V(\lambda_h Dv)|^2 + |V(\lambda_h(v - v_h)D\zeta + \lambda_h \zeta Dv + \lambda_h(1-\zeta)Dv_h)|^2]\, dy \leq$$

$$\leq \frac{c}{\lambda_h^2} \int_{B_r \setminus B_s} \left[|V(\lambda_h Dv)|^2 + |V(\lambda_h Dv_h)|^2 + \frac{1}{(r-s)^2} |V(\lambda_h(v - v_h))|^2 \right] dy .$$

Now from (3.9) we have

$$\int_{B_r \setminus B_s} |V(\lambda_h Dv_h)|^2\, dy \leq$$

$$\leq \left(\int_{B_r \setminus B_s} |V(\lambda_h Dv_h)|^{2(1+\delta)}\, dy \right)^{1/(1+\delta)} |B_r \setminus B_s|^{\delta/(1+\delta)} \leq c\lambda_h^2 (r-s)^{\delta/(1+\delta)} ,$$

and by (2.2), taking θ such that $1/2 = \theta + (1-\theta)/(2(1+\sigma))$, we obtain, using (iii) and (ii) of Lemma 2.1,

$$\int_{B_r \setminus B_s} |V(\lambda_h(v - v_h))|^2\, dy \leq \left(\int_{B_r \setminus B_s} |V(\lambda_h(v - v_h))|\, dy \right)^{2\theta} \cdot$$

$$\cdot \left(\int_{B_r \setminus B_s} |V(\lambda_h(v - v_h))|^{2(1+\sigma)}\, dy \right)^{(1-\theta)/(1+\sigma)} \leq c\lambda_h^{2\theta} \left(\int_{B_1} |v - v_h|\, dy \right)^{2\theta} \cdot$$

$$\cdot \left(\int_{B_{1/3}} |V(\lambda_h(v - v_h) - \lambda_h(v - v_h)_{0,1/3})|^{2(1+\sigma)}\, dy + |V(\lambda_h(v - v_h)_{0,1/3})|^{2(1+\sigma)} \right)^{(1-\theta)/(1+\sigma)} \leq$$

$$\leq c\lambda_h^{2\theta} \left(\int_{B_1} |v - v_h|\, dy \right)^{2\theta} \left[\left(\int_{B_1} |V(\lambda_h Dv_h)|^2\, dy \right)^{1-\theta} + \lambda_h^{2(1-\theta)} \right] \leq c\lambda_h^2 \left(\int_{B_1} |v - v_h|\, dy \right)^{2\theta} ,$$

where we used the estimate (see (3.9))

$$\fint_{B_1} |V(\lambda_h Dv_h)|^2 \, dy \leq c\lambda_h^2 . \tag{3.10}$$

Finally from the estimates above we conclude that

$$I_r^h(v_h) - I_r^h(v) \leq c\left[\left(\sup_{B_r} |Dv|^2\right)(r-s) + (r-s)^{\delta/(1+\delta)} + \frac{1}{(r-s)^2}\left(\int_{B_1} |v - v_h| \, dy\right)^{2\theta}\right].$$

Since $v_h \to v$ in $L^p(B_1, \mathbb{R}^N)$, letting first $h \to \infty$ and then $s \to r$ we prove the claim.

Step 5: *lower bound.* We claim that if $t < r < 1/4$

$$\limsup_h \frac{1}{\lambda_h^2} \int_{B_t} |V(\lambda_h(Dv - Dv_h))|^2 \, dy \leq c \lim_h [I_r^h(v_h) - I_r^h(v)] .$$

Let $\phi \in C_0^1(B_{1/3})$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_{1/4}$ and $|D\phi| \leq c$. Set

$$\tilde{v}_h = v_h \phi , \qquad \tilde{v} = v\phi .$$

We may always assume that the exponent δ given by the higher integrability estimate (3.9) is less than or equal to the exponent σ provided by the Sobolev-Poincaré inequality (2.2). Therefore we get by (3.9) and (3.10)

$$\int_{\mathbb{R}^n} |V(\lambda_h D\tilde{v}_h)|^{2(1+\delta)} \, dy \leq c \int_{B_{1/3}} |V(\lambda_h Dv_h)|^{2(1+\delta)} \, dy + c \int_{B_{1/3}} |V(\lambda_h v_h)|^{2(1+\delta)} \, dy \leq$$

$$\leq c \int_{B_{1/3}} |V(\lambda_h Dv_h)|^{2(1+\delta)} \, dy + c \int_{B_{1/3}} |V(\lambda_h v_h - \lambda_h (v_h)_{0,1/3})|^{2(1+\delta)} \, dy \, \cdot$$

$$\cdot c |V(\lambda_h (v_h)_{0,1/3})|^{2(1+\delta)} \leq c\lambda_h^{2(1+\delta)} + c\left(\int_{B_1} |V(\lambda_h Dv_h)|^2 \, dy\right)^{1+\delta} \leq c\lambda_h^{2(1+\delta)} .$$

From this estimate and Proposition 2.3 it then follows that

$$\lambda_h^{-1} [\|V(\lambda_h D\tilde{v}_h)\|_{L^{2(1+\delta)}(\mathbb{R}^n)} + \|V(\lambda_h M(D\tilde{v}_h))\|_{L^{2(1+\delta)}(\mathbb{R}^n)}] \leq c \tag{3.11}$$

for all h. Fix $\varepsilon > 0$, from the estimate above it is clear that there exists $\eta > 0$ such that if $G \subset \mathbb{R}^n$ is a measurable set, with $\operatorname{meas}(G) < \eta$

$$\frac{1}{\lambda_h^2}\left[\int_G |V(\lambda_h D\tilde{v}_h)|^2 \, dy + \int_G |V(\lambda_h M(D\tilde{v}_h))|^2 \, dy\right] < \varepsilon . \tag{3.12}$$

Notice that (3.11) implies also that $(\tilde{v}_h)$ is bounded in $W^{1,p(1+\delta)}(\mathbb{R}^n, \mathbb{R}^N)$, therefore by the continuity of the maximal function in L^q spaces we deduce that there exists

$K > 1$ such that, setting $S_h = \{y \in \mathbb{R}^n : M(D\tilde{v}_h)(y) > K\}$,

$$\text{meas}(S_h) < \eta \quad \text{for all } h. \tag{3.13}$$

Having chosen K, we now apply Lemma 2.9 to find a sequence of functions $w_h \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$w_h = \tilde{v}_h \ \text{ on } \mathbb{R}^n \setminus S_h, \qquad \|Dw_h\|_\infty \leq cK. \tag{3.14}$$

Therefore, passing to a (not relabelled) subsequence we may also suppose that

$$w_h \rightharpoonup w \quad \text{weakly* in } W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N).$$

Notice that by (3.12), (3.13) and the definition of S_h we have the estimate

$$\text{meas}(S_h)(1 + \lambda_h^2 K^2)^{(p-2)/2} K^2 \leq \frac{1}{\lambda_h^2} \int_{S_h} |V(\lambda_h M(D\tilde{v}_h))|^2 \, dy \leq \varepsilon$$

which gives

$$\text{meas}(S_h) \leq \varepsilon \frac{(1 + \lambda_h^2 K^2)^{(2-p)/2}}{K^2} < \frac{2\varepsilon}{K^2} \tag{3.15}$$

for h large enough. We turn now to estimate the difference

$$I_r^h(v_h) - I_r^h(v) = \tag{3.16}$$

$$= [I_r^h(\tilde{v}_h) - I_r^h(w_h)] + [I_r^h(w_h) - I_r^h(w)] + [I_r^h(w) - I_r^h(v)] = R_1^h + R_2^h + R_3^h.$$

By (3.7), (3.12), (3.13) and (3.14) we get

$$|R_1^h| \leq \int_{S_h \cap B_r} |F_h(D\tilde{v}_h) - F_h(Dw_h)| \, dy \leq \tag{3.17}$$

$$\leq \frac{c}{\lambda_h^2} \int_{S_h} [V(\lambda_h D\tilde{v}_h)|^2 + V(\lambda_h M(D\tilde{v}_h))|^2] \, dy < c\varepsilon.$$

Now choose $t < s < r$ and take a cut-off function ζ as in Step 4. Setting $\psi_h = (w_h - w)\zeta$ we split R_2^h as follows:

$$R_2^h = [I_r^h(w_h) - I_r^h(w + \psi_h)] + \tag{3.18}$$

$$+ [I_r^h(w + \psi_h) - I_r^h(w) - I_r^h(\psi_h)] + I_r^h(\psi_h) = R_4^h + R_5^h + R_6^h.$$

Again by (3.7), (3.14) and (iii), (ii) of Lemma 2.1 we have

$$|R_4^h| \leqslant \int_{B_r\setminus B_s} |F_h(Dw_h) - F_h(Dw + D\psi_h)|\, dy \leqslant$$

$$\leqslant \frac{c}{\lambda_h^2}\int_{B_r\setminus B_s}\left[|V(\lambda_h Dw_h)|^2 + |V(\lambda_h Dw)|^2 + \frac{1}{(r-s)^2}|V(\lambda_h(w_h - w))|^2\right] dy \leqslant$$

$$\leqslant c(K)(r-s) + \frac{c}{(r-s)^2}\int_{B_r\setminus B_s} |w_h - w|^2\, dy\,.$$

Since $w_h \to w$ uniformly we conclude that

$$\limsup_h |R_4^h| \leqslant c(K)(r-s)\,. \tag{3.19}$$

To bound R_5^h we observe that for any $A, B \in \mathbb{R}^{nN}$

$$F_h(A+B) - F_h(A) - F_h(B) = \int_0^1\int_0^1 D^2F_h(sA + tB)\, ds\, dt$$

and therefore

$$R_5^h = \int_{B_r} dx \int_0^1\int_0^1 D^2F(A_h + s\lambda_h Dw_h + t\lambda_h D\psi_h)\, DwD\psi_h\, ds\, dt\,.$$

Since $D^2F(A_h + s\lambda_h Dw_h + t\lambda_h D\psi_h)$ converges to $D^2F(A)$ uniformly, and $w_h \rightharpoonup w$ weakly* in $W^{1,\infty}$ we easily get

$$\lim_h R_5^h = 0\,. \tag{3.20}$$

Moreover (3.8) implies that

$$R_6^h = \int_{B_r} F_h(D\psi_h)\, dy \geqslant \frac{\nu}{\lambda_h^2}\int_{B_r} |V(\lambda_h D\psi_h)|^2\, dy \geqslant \frac{\nu}{\lambda_h^2}\int_{B_s} |V(\lambda_h(Dw_h - Dw))|^2\, dy\,.$$

Passing possibly to a subsequence we may suppose that

$$\lim_h R_2^h$$

exists too. Therefore by (3.18), (3.19), (3.20) we deduce

$$\lim_h R_2^h \geqslant \limsup_h \frac{\nu}{\lambda_h^2}\int_{B_s} |V(\lambda_h(Dw_h - Dw))|^2\, dy - c(K)(r-s)\,. \tag{3.21}$$

To deal with R_3^h we use a technique introduced in [1]. First we prove that

$$\text{meas}\,\{y \in B_r\colon v(y) \neq w(y)\} \leqslant \frac{3\varepsilon}{K^2}\,. \tag{3.22}$$

Set $S = \{y \in B_r\colon v(y) \neq w(y)\}$ and

$$\tilde{S} = S \cap \{y \in B_r\colon v(y) = \lim_h v_h(y)\}\,.$$

Then $\text{meas}\,(S) = \text{meas}\,(\tilde{S})$. We argue by contradiction. If

$$\text{meas}\,(S) > \frac{3\varepsilon}{K^2}\,,$$

then by (3.15)

$$\text{meas}\,(\tilde{S} \backslash S_h) > \frac{\varepsilon}{K^2}\,,$$

for h large enough and by Lemma 2.11 there exists $\bar{y} \in B_r$ such that

$$\bar{y} \in \tilde{S} \backslash S_h \quad \text{for infinitely many } h\,.$$

Passing to this subsequence, we have

$$v(\bar{y}) = \lim_h v_h(\bar{y}) = \lim_h w_h(\bar{y}) = w(\bar{y})\,;$$

hence $\bar{y} \notin S$, which is a contradiction. This proves (3.22). Now, since $Dv = Dw$ a.e. in $B_r \backslash S$, by (3.7) and (3.22) we get

$$|R_3^h| \leqslant \int_{B_r \cap S} |F_h(Dw) - F_h(Dv)|\,dy \leqslant \tag{3.23}$$

$$\leqslant \frac{c}{\lambda_h^2} \int_{B_r \cap S} [\,|V(\lambda_h Dv)|^2 + |V(\lambda_h Dw)|^2]\,dy \leqslant c(1+K^2)\,\text{meas}\,(S) \leqslant \frac{c(1+K^2)\,\varepsilon}{K^2} \leqslant c\varepsilon\,,$$

since $K > 1$. By this inequality, (3.16),(3.17) and (3.21) we conclude that

$$\lim_h\,[I_r^h(v_h) - I_r^h(v)] \geqslant \tag{3.24}$$

$$\geqslant \limsup_h \frac{\nu}{\lambda_h^2} \int_{B_s} |V(\lambda_h(Dw_h - Dw))|^2\,dy - c(K)(r-s) - c\varepsilon\,.$$

By (iii) of Lemma 2.1 we then have

$$\frac{1}{\lambda_h^2} \int_{B_t} |V(\lambda_h(Dv - Dv_h))|^2 dy \leqslant \frac{c}{\lambda_h^2} \int_{B_s} |V(\lambda_h(Dw - Dw_h))|^2 dy + \tag{3.25}$$

$$+ \frac{c}{\lambda_h^2} \int_{B_s \cap S_h} |V(\lambda_h(Dw_h - Dv_h))|^2\,dy + \frac{c}{\lambda_h^2} \int_{B_s \cap S_h} |V(\lambda_h(Dv - Dw))|^2\,dy\,.$$

With the same argument used to prove (3.23) we get also

$$\frac{c}{\lambda_h^2}\int_{B_s\cap S}|V(\lambda_h(Dv-Dw))|^2\,dy\leq\frac{c}{\lambda_h^2}\int_{B_r\cap S}[|V(\lambda_h Dv)|^2+|V(\lambda_h Dw)|^2]\,dy\leq c\varepsilon \tag{3.26}$$

and as in (3.17) we get

$$\frac{c}{\lambda_h^2}\int_{B_s\cap S_h}|V(\lambda_h(Dv_h-Dw_h))|^2\,dy\leq\frac{c}{\lambda_h^2}\int_{S_h}[|V(\lambda_h D\tilde{v}_h)|^2+|V(\lambda_h M(D\tilde{v}_h))|^2]\,dy\leq c\varepsilon\,.$$

Therefore, from this estimate, (3.24), (3.25), (3.26) we finally conclude that

$$\limsup_h\frac{1}{\lambda_h^2}\int_{B_t}|V(\lambda_h(Dv-Dv_h))|^2\,dy\leq c\lim_h[I_r^h(v_h)-I_r^h(v)]+c\varepsilon+c(K)(r-s)\,.$$

The proof of the claim then follows by letting first $s\to r$ and then $\varepsilon\to 0^+$.

Step 6: *conclusion of the proof.* From the two previous steps we have that for any $0<\tau<1/4$

$$\lim_h\frac{1}{\lambda_h^2}\int_{B_\tau}|V(\lambda_h(Dv-Dv_h))|^2\,dy=0\,.$$

Now from this inequality, (v), (iii) of Lemma 2.1 and (3.6) we get

$$\limsup_h\frac{E(x_h,\tau R_h)}{\lambda_h^2}=\limsup_h\frac{1}{\lambda_h^2}\fint_{B_{\tau R_h}(x_h)}|V(Du)-V((Du)_{x_h,\tau R_h})|^2\,dx\leq$$

$$\leq\limsup_h\frac{c}{\lambda_h^2}\fint_{B_{\tau R_h}(x_h)}|V(Du-(Du)_{x_h,\tau R_h})|^2\,dx=$$

$$=\limsup_h\frac{c}{\lambda_h^2}\fint_{B_\tau}|V(\lambda_h(Dv_h-(Dv_h)_\tau))|^2\,dy\leq$$

$$\leq\limsup_h\frac{c}{\lambda_h^2}\fint_{B_\tau}[|V(\lambda_h(Dv_h-Dv))|^2+$$

$$+|V(\lambda_h(Dv-(Dv)_\tau))|^2+|V(\lambda_h((Dv)_\tau-(Dv_h)_\tau))|^2]\,dy\leq$$

$$\leq[C^*(M)\tau^2+\lim_h|(Dv)_\tau-(Dv_h)_\tau|^2]\,,$$

and since $Dv_h \rightharpoonup Dv$ weakly in $L^p(B_1, \mathbb{R}^{nN})$ we deduce that

$$\lim_h \frac{E(x_h, \tau R_h)}{\lambda_h^2} \leq C^*(M)\tau^2,$$

which contradicts (3.2) if we choose $C_M = 2C^*(M)$. ■

PROOF OF THE THEOREM 3.2. – Following the argument used in section 6 of [9], from the Decay estimate just proved, one can easily obtain that for any $M > 0$ there exist $0 < \tau < 1/4$ and $\eta > 0$ such that if

$$|(Du)_{x_0, R}| \leq M \quad \text{and} \quad E(x_0, R) < \eta \tag{3.27}$$

then for any $k = 1, 2, \ldots$

$$E(x_0, \tau^k R) \leq c(M)\tau^{2k} E(x_0, R), \qquad |(Du)_{x_0, \tau^k R}| \leq 2M.$$

From this estimate one then gets that if (3.27) holds for any $0 < \varrho < R$ we have

$$|(Du)_{x_0, \varrho}| \leq c(M) \quad \text{and} \quad E(x_0, \varrho) \leq c(M)\left(\frac{\varrho}{R}\right)^2 E(x_0, R).$$

Therefore from Lemma 2.1 we get that

$$\begin{aligned} \fint_{B_\varrho(x_0)} |Du - (Du)_{x_0, \varrho}|\, dy &\leq \fint_{B_\varrho(x_0) \cap \{x:\, |Du - (Du)_{x_0, \varrho}| \leq 1\}} |Du - (Du)_{x_0, \varrho}|\, dy + \\ &+ \fint_{B_\varrho(x_0) \cap \{x:\, |Du - (Du)_{x_0, \varrho}| > 1\}} |Du - (Du)_{x_0, \varrho}|\, dy \leq \\ &\leq c \fint_{B_\varrho(x_0)} |V(Du - (Du)_{x_0, \varrho})|\, dy + c\left(\fint_{B_\varrho(x_0)} |V(Du - (Du)_{x_0, \varrho})|^2\, dy\right)^{1/p} \leq \\ &\leq c(M)\left[\fint_{B_\varrho(x_0)} |V(Du) - V((Du)_{x_0, \varrho})|\, dy + \left(\fint_{B_\varrho(x_0)} |V(Du) - V((Du)_{x_0, \varrho})|^2\, dy\right)^{1/p}\right] \leq \\ &\leq c(M)[E^{1/2}(x_0, \varrho) + E^{1/p}(x_0, \varrho)] \leq c(M, R)\, \varrho. \end{aligned} \tag{3.28}$$

From this estimate it is then clear that if we set

$$\Omega_0 = \left\{x \in \Omega : \sup_{r > 0} |(Du)_{x_0, r}| < \infty \text{ and } \lim_{r \to 0} E(x_0, r) = 0\right\}$$

Ω_0 is an open set such that meas $(\Omega - \Omega_0) = 0$ and by (3.28) $u \in C^{1, \alpha}(\Omega_0)$ for any $0 < \alpha < 1$. ■

REFERENCES

[1] E. ACERBI - N. FUSCO, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal., **86** (1984), pp. 125-145.

[2] E. ACERBI - N. FUSCO, *An approximation lemma for $W^{1,p}$ functions*, Proc. of the Symposium on Material Instabilities and Continuum Mech., J. M. BALL editor, Oxford Science Publications (1986), pp. 1-5.

[3] E. ACERBI - N. FUSCO, *A regularity theorem for minimizers of quasiconvex integrals*, Arch. Rational Mech. Anal., **99** (1987), pp. 261-281.

[4] E. ACERBI - N. FUSCO, *Regularity for minimizers of non-quadratic functionals: the case* $1 < p < 2$, Journ. Math. Anal. Appl., **140** (1989), pp. 115-135.

[5] M. CAROZZA - A. PASSARELLI DI NAPOLI, *A regularity theorem for minimizers of quasiconvex integrals: the case* $1 < p < 2$, Proc. Royal Math. Soc. Edinburgh, **126** Part 6 (1996), pp. 1181-1199.

[6] G. EISEN, *A selection lemma for sequences of measurable sets and lower semicontinuity of multiple integrals*, Manuscr. Math., **27** (1979), pp. 73-79.

[7] L. C. EVANS, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Rational Mech. Anal., **95** (1986), pp. 227-252.

[8] L. C. EVANS - R. F. GARIEPY, *Compactness and partial regularity in the calculus of variations*, Rend. Circolo Mat. Palermo, **15** (1987), pp. 101-108.

[9] N. FUSCO - J. E. HUTCHINSON, $C^{1,\alpha}$ *partial regularity of functions minimizing quasiconvex integrals*, Manus. Math., **54** (1985), pp. 121-143.

[10] N. FUSCO - C. SBORDONE, *Higher integrability of the gradient of minimizers of functionals with nonstandard growth condition*, Comm. Pure Appl. Math., **43** (1990), pp. 673-683.

[11] M. GIAQUINTA, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. Math. Studies, Princ. University Press, **105** (1983).

[12] M. GIAQUINTA, *Quasiconvexity, growth conditions and partial regularity*, Partial differential equations and calculus of variations, Lecture Notes in Math., **1357**, Springer, Berlin (1988), pp. 211–237.

[13] E. GIUSTI, *Metodi diretti nel calcolo delle variazioni*, U.M.I., Bologna, (1994).

[14] L. I. HEDBERG, , On certain convolution inequalities, Proc. A.M.S., **36** (1972), pp. 505-510.

[15] P. MARCELLINI, *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals*, Manuscr. Math., **51** (1985), pp. 1-28.

[16] V. ŠVERAK, *Quasiconvex functions with subquadratic growth*, Proc. Royal Soc. London Ser. A, **433** (1991), pp. 723-725.