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On rings whose prime radical contains all nilpotent elements of index two

By

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Dedicated to the memory of Professor Hisao Tominaga

Let R be an associative ring with identity. The set of all nilpotent elements (resp. all nilpotent elements of index two) of R is denoted by $\mathbf{N}(R)$ (resp. $\mathbf{N}_2(R)$). We use $\mathbf{P}(R)$ for the prime radical of R, i.e., the intersection of all prime ideals of R. Following Birkenmeier, Heatherly and Lee [3], a ring R is said to be *2-primal* if $\mathbf{P}(R) = \mathbf{N}(R)$. Clearly commutative rings and reduced rings (i.e., rings without non-zero nilpotent elements) are 2-primal.

Historically, some of the earliest results known to us about 2-primal rings (although not so called at the time) and prime ideals were due to Shin [12]. He showed that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime. Hirano [6] considered the 2-primal condition in the context of strongly π-regular rings. He used the term *N-ring* for what we call a 2-primal ring. The 2-primal condition was taken up independently by Sun [13], where in the setting of rings with identity he introduced a condition called *weakly symmetric*, which is equivalent to the 2-primal condition for rings. Sun [13] showed that if R is weakly symmetric, then each minimal prime ideal of R is a completely prime ideal, and that the ring of n-by-n upper triangular matrices over R inherits the weakly symmetric condition. The name *2-primal rings* originally and independently came from the context of left near rings by Birkenmeier, Heatherly and Lee in [3].

For 2-primal rings, there is a question of Birkenmeier, Heatherly and Lee [3, Problem 3, p. 373] which asks if the prime radical of a ring R contains all nilpotent elements of index two, is R a 2-primal ring?

In this note we give the answer to this question in the negative. Furthermore we consider a certain class of rings in which the question of [3] is true.

The following example shows that the answer to the question above is negative.

Example 1. Let F be a field, $F < X, Y >$ the free algebra on X, Y over F and I denote the ideal $(X^2)^2$ of $F < X, Y >$, where (X^2) is the ideal of $F < X, Y >$ generated by X^2. Consider the ring $R = F < X, Y > / I$. Then we have $\mathbf{N}(R) = xRx + Rx^2R + Fx$ and $\mathbf{N}_2(R) = Rx^2R = \mathbf{P}(R)$, where $x = X + I$ in R.

Now we will prove this fact. Since $(Rx^2R)^2 = 0$, we have $Rx^2R \subseteq \mathbf{P}(R)$. It is clear that R/Rx^2R is isomorphic to the ring $S = F < X, Y > /(X^2)$. For an element u in $F < X, Y >$, we denote the residue class $u + (X^2)$ in S by $\bar{u}$. It is easy to see that for any two non-zero elements a, b of S, $a\bar{Y}b \neq 0$. Thus S is a prime ring. Since $Rx^2R \subseteqq \mathbf{P}(R)$ and R/Rx^2R is a prime ring, we conclude that $Rx^2R = \mathbf{P}(R)$.

Next we *claim* that if a, b are two non-zero elements of S such that $ab = 0$ then $a \in S\bar{X}$ and $b \in \bar{X}S$. Let us set $\bar{X}^0 = 1$. First observe that

$$B = \{1, \bar{X}\} \cup \{\bar{X}^i \bar{Y}^{n_1} \bar{X} \cdots \bar{X} \bar{Y}^{n_k} \bar{X}^j \mid 0 \leqq i, j \leqq 1, n_1, n_2, \ldots, n_k \in \mathbb{N}, k \in \mathbb{N}\}$$

is an F-basis of S. We define the *length* of fb ($f \in F \setminus \{0\}$, $b \in B$) as follows:

$$\text{length}(f1) = 0, \ \text{length}(f\bar{X}) = 1$$

and

$$\text{length}(f\bar{X}^i \bar{Y}^{n_1} \bar{X} \cdots \bar{X} \bar{Y}^{n_k} \bar{X}^j) = i + j + k - 1 + n_1 + \cdots + n_k.$$

Now suppose that a, b are two non-zero elements of S such that $ab = 0$. We write a, b in the form $a = r\bar{X} + s\bar{Y}$ and $b = \bar{X}t + \bar{Y}u$ with $r, s, t, u \in S$. Then $ab = r\bar{X}\bar{Y}u + s\,\bar{Y}\bar{X}t + s\bar{Y}^2u = 0$. Since $a \neq 0$, we have either $r \neq 0$ or $s \neq 0$. Similarly we have either $t \neq 0$ or $u \neq 0$. Now we can easily see that the sum of terms of the highest length in

$$r\bar{X}\bar{Y}u + s\,\bar{Y}\bar{X}t + s\bar{Y}^2u$$

is zero if and only if $s = u = 0$. This proves our claim.

Now we can easily see that $\mathbf{N}(S) = \mathbf{N}_2(S) = \bar{X}S\bar{X} + F\bar{X}$, and hence $\mathbf{N}(R) = xRx + Rx^2R + Fx$.

Finally we show that $\mathbf{N}_2(R) = Rx^2R$. Let z be an element of R with $z^2 = 0$. Then, since $z \in \mathbf{N}(R)$, we can write

$$z = xax + \sum b_i x^2 c_i + fx$$

with $a, b_i, c_i \in R$ and $f \in F$. Since $z^2 = 0$ and $(Rx^2R)^2 = 0$, we have $xax^2ax + fxax^2 + xax(\sum b_i x^2 c_i) + fx^2ax + f^2x^2 + fx(\sum b_i x^2 c_i) + (\sum b_i x^2 c_i)xax + f(\sum b_i x^2 c_i)x = 0$. Thus $f^2x^2 = 0$ and so we have $f = 0$. Therefore $z = xax + \sum b_i x^2 c_i$ and thus

$$(*) \qquad xax^2ax + xax(\sum b_i x^2 c_i) + (\sum b_i x^2 c_i)xax = 0.$$

If $xax \in Rx^2R$, then $z \in Rx^2R$ and so we are done. Thus we may assume that $xax \notin Rx^2R$. So there is a non-zero term, say $\alpha y^{m_1} x y^{m_2} x \cdots x y^{m_k}$ of a, where $\alpha \in F$ and $k, m_1, m_2, \ldots, m_k$ are positive integers. But it is impossible from the equation (*). Thus $xax \in Rx^2R$. This proves $\mathbf{N}_2(R) = Rx^2R = \mathbf{P}(R)$.

Let $J(R)$ denote the Jacobson radical of a ring R. We say that idempotents *lift modulo* $J(R)$ in case every idempotent in $R/J(R)$ can be lifted to an idempotent in R. A ring R is called an *I-ring* if every non-nil right ideal of R contains a non-zero idempotent. Right or left Artinian rings, more generally, π-regular rings are I-rings.

Theorem 2. *Let R be a ring such that $R/J(R)$ is an I-ring and suppose that idempotents lift modulo $J(R)$. If $J(R)$ contains $\mathbf{N}_2(R)$, then $R/J(R)$ has no non-zero nilpotent elements.*

Proof. Suppose, on the contrary, that $R/J(R)$ contains a non-zero nilpotent element a. We may assume that $a^2 = 0$, that is, a is of index 2. Then by [9, Theorem 1, p. 237], the ideal (a) of $R/J(R)$ generated by a contains a system $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ of 2^2 matrix units. By hypothesis e_{11} can be lifted to an idempotent of R, say E. Let r be an arbitrary element of R. Then $(ErE - Er)^2 = 0$, and hence $ErE - Er \in J(R)$. Similarly we have $ErE - rE \in J(R)$. If we put $\bar{r} = r + J(R) \in R/J(R)$, then these imply that $e_{11}\bar{r} = e_{11}\bar{r}e_{11} = \bar{r}e_{11}$. Hence e_{11} is a central idempotent of $R/J(R)$. This is a contradiction, because $e_{11}e_{12} = e_{12} \neq 0 = e_{12}e_{11}$. □

The following example shows that the assumption "idempotents lift modulo $J(R)$" cannot be dropped from Theorem 2.

Example 3. Let R denote the localization of the ring $\mathbb{Z}$ of integers at the prime ideal (3). Consider the quaternions Q over R, that is, a free R-module with basis $1, i, j, k$ and multiplication satisfying $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$. Then Q is a noncommutative domain, and so $N_2(Q) = 0$. However $Q/J(Q)$ is an I-ring with non-zero nilpotent elements. In fact, $J(Q) = 3Q$ and $Q/J(Q)$ is isomorphic to the 2-by-2 full matrix ring over $\mathbb{Z}/(3)$.

Corollary 4. *Assume that R is a ring and $J(R)$ contains $N_2(R)$. If R satisfies any of the following conditions, then $R/J(R)$ has no non-zero nilpotent elements*:

(1) *R is a semiperfect ring.*
(2) *R is a right or left self-injective ring.*
(3) *R is an I-ring.*

Proof. (1) By its definition (see [8, p. 73]), note that a semiperfect ring satisfies the hypotheses of Theorem 2.

(2) By [8, Proposition 4.4.1, p. 102], $R/J(R)$ is a von Neumann regular ring and idempotents lift modulo $J(R)$. Now the assertion follows from Theorem 2.

(3) The Jacobson radical $J(R)$ of the I-ring R is a nil ideal and the ring $R/J(R)$ also is an I-ring. Thus it follows immediately from Theorem 2. □

Next we deal with a ring R which is a right order in a right Artinian ring. A condition for a ring R to be a right order in a right Artinian ring can be found in [5, p. 172 after Exercise 10H].

The following lemma is almost evident and its proof may be omitted.

Lemma 5. *The following statements are equivalent for a ring R:*
(1) $\mathrm{N}_2(R) \subseteqq \mathbf{P}(R)$.
(2) *For any a, b in R with $ab = 0$, it holds $bRa \subseteqq \mathbf{P}(R)$.*

Theorem 6. *Let R be a right order in a right Artinian ring Q. If the prime radical $\mathbf{P}(R)$ of R contains $\mathrm{N}_2(R)$, then $\mathbf{P}(R) = \mathrm{N}(R)$, i.e., R is 2-primal.*

Proof. By [5, Exercise 10H], $\mathbf{P}(Q) = \mathbf{P}(R)Q$ and $R/\mathbf{P}(R)$ is a right order in the semisimple Artinian ring $Q/\mathbf{P}(Q)$. Hence it suffices to prove that $Q/\mathbf{P}(Q)$ has no non-zero nilpotent elements. Suppose, on the contrary, that $Q/\mathbf{P}(Q)$ has a non-zero nilpotent element. Then by [9, Theorem 2.1] $Q/\mathbf{P}(Q)$ contains a system of 2^2 matrix units $\mathscr{E}_{11}, \mathscr{E}_{12}, \mathscr{E}_{21}, \mathscr{E}_{22}$. Since $\mathbf{P}(Q)$ is a nil ideal, $\mathscr{E}_{11}$ and $\mathscr{E}_{22}$ can be lifted to orthogonal idempotents E_{11} and E_{22} of Q, respectively. Since the image of $E_{11}QE_{22}$ in $Q/\mathbf{P}(Q)$ contains $\mathscr{E}_{12}$, we have $E_{11}QE_{22} \nsubseteq \mathbf{P}(Q)$. Since $(E_{11}QE_{22})^2 = 0$, we obtain an element $a \in Q$ such that $a^2 = 0$ and $a \notin \mathbf{P}(Q)$ $(= J(Q))$. Then again by [9, Theorem 2.1] Q contains a system of 2^2 matrix units, say $e_{11}, e_{12}, e_{21}, e_{22}$. Let $e_{12} = ad^{-1}$ where $a, d \in R$ with d regular. Then we can write $d^{-1}a = bc^{-1}$ where $b, c \in R$ with c regular. Then $0 = (e_{12})^2 = ad^{-1}ad^{-1} = abc^{-1}d^{-1}$, and hence $ab = 0$. Therefore $bRa \subseteq \mathbf{P}(R)$ by Lemma 5.

Now let $\{P_k \mid k \in K\}$ be the set of all prime ideals of R. Then $\mathbf{P}(R) = \bigcap_{k \in K} P_k$. Since $bRa \subseteq P_k$, we have either $a \in P_k$ or $b \in P_k$. Now let $I_a = \bigcap \{P_k \mid a \in P_k\}$ and $I_b = \bigcap \{P_k \mid b \in P_k\}$. Then $I_a \cap I_b = \mathbf{P}(R)$. By [5, Theorem 9.20 (a)], I_aQ and I_bQ are ideals of Q. Clearly we have $\mathbf{P}(Q) = \mathbf{P}(R)Q \subseteq I_aQ \cap I_bQ$. To prove the converse inclusion, let $z \in I_aQ \cap I_bQ$. Then by [5, Lemma 5.1 (c)], we can write $z = gf^{-1} = hf^{-1}$ where $g \in I_a$, $h \in I_b$ and f is a regular element of R. Then $g = (gf^{-1})f = (hf^{-1})f = h \in I_a \cap I_b$, and hence $z \in (I_a \cap I_b)Q = \mathbf{P}(R)Q = \mathbf{P}(Q)$. This proves $I_aQ \cap I_bQ = \mathbf{P}(Q)$. Since $a \in I_a$, $e_{12} = ad^{-1} \in I_aQ$. Since $b \in I_b$ and I_b is an ideal of R, we get $e_{12} = ad^{-1} = dbc^{-1}d^{-1} \in I_bQ$. Hence we obtain $e_{12} \in I_aQ \cap I_bQ = \mathbf{P}(Q)$, and so $e_{11} = e_{12}e_{21} \in \mathbf{P}(Q)$. Since $\mathbf{P}(Q)$ is a nil ideal, this is a contradiction. □

Finally we consider a certain class of PI-rings in which the question in [3, Problem 3, p. 373] is true. We denote the center of a ring by $\mathbf{Z}(R)$.

Theorem 7. *Let R be a PI-ring with $\mathbf{Z}(R/\mathbf{P}(R)) = (\mathbf{Z}(R) + \mathbf{P}(R))/\mathbf{P}(R)$. If $\mathbf{P}(R)$ contains $\mathbf{N}_2(R)$, then R is 2-primal.*

Proof. Let $\bar{R}$ denote $R/\mathbf{P}(R)$ and $Q(\bar{R})$ denote the maximal right quotient ring of $\bar{R}$. Then $Q(\bar{R})$ is a self-injective von Neumann regular ring and satisfies a polynomial identity by [10, Theorem 2]. Then by [1, Theorem 3.1], $Q(\bar{R})$ is a direct product of full matrix rings over strongly regular rings. Hence, to prove that $\bar{R}$ has no non-zero nilpotent elements, it suffices to show that $Q(\bar{R})$ has no non-central idempotents.

Suppose, on the contrary, that $Q(\bar{R})$ contains a non-central idempotent e. By the definition of $Q(\bar{R})$, there exists an essential right ideal I of $\bar{R}$ such that $eI \subseteq \bar{R}$. Let C denote the center of $\bar{R}$. By [1, Lemma 2.2], $(I \cap C)\bar{R}$ is an essential right ideal of $\bar{R}$ contained in I. Let $K = \{k \in \mathbf{Z}(R) \mid k + \mathbf{P}(R) \in I \cap C\}$. Take an element $k \in K$ and set $\bar{k} = k + \mathbf{P}(R)$. Then $e\bar{k} \in \bar{R}$. Let F be an element of R such that $F + \mathbf{P}(R) = e\bar{k}$. Then $Fk - F^2 \in \mathbf{P}(R)$. Hence, there exists a positive integer m such that $(Fk - F^2)^m = 0$.

Now we can write $k^m - (k - F)^m = FG$ with some element G in the subring generated by k and F. Then, $0 = (Fk - F^2)^m = F^m(k - F)^m = F^m(k^m - FG) = F^m k^m - F^{m+1}G$. From the equation $F^m k^m = F^{m+1}G$, we obtain $F^m k^{m^2} = F^{2m}G^m = F^m(FG)^m$. By Lemma 5, $F^m(k^{m^2} - (FG)^m) = 0$ implies that $(k^{m^2} - (FG)^m)RF^m \subseteq \mathbf{P}(R)$. Since $FG + \mathbf{P}(R) = \bar{k}^m - (\bar{k} - e\bar{k})^m = e\bar{k}^m$, this implies $(1 - e)\bar{R}e\bar{k}^{m^2+m} = 0$ in $Q(\bar{R})$. Since $Q(\bar{R})$ is semiprime and since $\bar{k}$ is in the center of $Q(\bar{R})$, we obtain $(1-e)\bar{R}e\bar{k} = 0$. Since $\mathbf{Z}(R/\mathbf{P}(R)) = (\mathbf{Z}(R) + \mathbf{P}(R))/\mathbf{P}(R)$, we have $I \cap C \subseteq \{\bar{k} \mid k \in K\}$. Thus we get $(1-e)\bar{R}e\{(I \cap C)\bar{R}\} = 0$. Since $(I \cap C)\bar{R}$ is an essential right ideal of $\bar{R}$, this implies $(1 - e)\bar{R}e = 0$. Similarly we can obtain $e\bar{R}(1 - e) = 0$. From these, we deduce that e centralizes $\bar{R}$.

Let q be an arbitrary element of $Q(\bar{R})$ and let A be an essential right ideal of $\bar{R}$ with $qA \subseteq R$. Then, for any $a \in A$, $(eq-qe)a = e(qa)-(qa)e = 0$, and hence $(eq-qe)A = 0$. Therefore $eq - qe = 0$ for all $q \in Q(\bar{R})$. This is a contradiction. □

Let R^{op} denote the opposite ring of R. By [2, Proposition 2.3], if R is quasi-projective as a left module over the ring $R \otimes_{\mathbf{Z}(R)} R^{op}$, then R satisfies the condition $\mathbf{Z}(R/\mathbf{P}(R)) = (\mathbf{Z}(R) + \mathbf{P}(R))/\mathbf{P}(R)$. In particular, we have the following

Corollary 8. *Let R be an Azumaya algebra. If* $\mathbf{P}(R)$ *contains* $\mathbf{N}_2(R)$, *then R is 2-primal.*

Proof. By [11, Proposition 13.7.7], R is a PI-ring. Also, by [4, Proposition 1.11], we have $\mathbf{Z}(R/\mathbf{P}(R)) = (\mathbf{Z}(R) + \mathbf{P}(R))/\mathbf{P}(R)$. Hence we can apply Theorem 7. □

An algebra R over a commutative ring C is called a *module-finite C-algebra* if R is a finitely generated C-module. A module M over a commutative ring C is said to be *torsion-free* provided $xc = 0$, with $x \in M$, $c \in C$, implies either $x = 0$ or c is a zero divisor of C.

Proposition 9. *Let R be a module-finite algebra over a commutative Noetherian ring C such that* $R/\mathbf{P}(R)$ *is torsion-free over* $\bar{C} = (C + \mathbf{P}(R))/\mathbf{P}(R)$. *If* $\mathbf{P}(R)$ *contains* $\mathbf{N}_2(R)$, *then R is 2-primal.*

Proof. Let $\bar{R}$ denote the ring $R/\mathbf{P}(R)$ and let S denote the set of regular elements of $\bar{C}$. Since $\bar{R}$ is torsion-free over $\bar{C}$, all elements of S are regular in $\bar{R}$. By hypotheses, the ring $S^{-1}\bar{R}$ of fractions of $\bar{R}$ with respect to S is a semisimple Artinian ring. In a similar way as in the proof of Theorem 7, we can show that $S^{-1}\bar{R}$ has no non-central idempotents. Therefore $\bar{R}$ has no non-zero nilpotent elements. □

We conclude this paper with the following two questions.

Question 1. Let R be a module-finite algebra over its center. If $\mathbf{P}(R)$ contains $\mathbf{N}_2(R)$, is R 2-primal?

If the answer of Question 1 is affirmative, we ask

Question 2. Let R be a PI-ring. If $\mathbf{P}(R)$ contains $\mathbf{N}_2(R)$, is R 2-primal?

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