Riccati Equations with Unbounded Coefficients (*).

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Summary. – *We give some results on the operator Riccati equation arising in linear quadratic optimal control problems. Our theory covers boundary control as well as pointwise control problems.*

1. – Introduction.

In this paper we consider a controlled semigroup model which covers parabolic equations with boundary and pointwise control. We study directly the Riccati equation associated with it. Then using the dynamic programming arguments we solve quadratic control problems.

The semi-group approach has already been employed by other authors [1], [4], [10], [14], [16]. CURTAIN and PRITCHARD [4] covers pointwise control in one dimension and the mixed boundary control. The Dirichlet boundary control was first studied by BALAKRISHNAN [1] and much work has been added by LASIECKA and TRIGGIANI [14]-[17]. However the Riccati equation is not directly studied there. Recently FLANDOLI [10] has shown the existence and uniqueness of a solution to a general Riccati equation which covers the Dirichlet case. Similar quadratic control problems have also been solved by SORINE [20], [21] using the variational approach. A unified treatment of quadratic control problems for partial differential equations and retarded functional differential equations is also reported in [19], but the Dirichlet boundary control is not included.

Following [10] we study the Riccati equation directly. But our proof is elementary and we obtain existence and uniqueness and some improvements. Then using dynamic programming we solve quadratic problems. We also characterize the domain of the infinitesimal generator of a closed system with constant feedback (see [9], [17], [23] for similar results). Our model covers Dirichlet control as well as pointwise control in $\boldsymbol{R}^d$, $d\leqslant 3$. We may take unbounded cost operators such as first derivatives and point evaluations in some cases. This is illustrated by examples.

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2. – The semigroup model and the Riccati equations.

Let H and U be real Hilbert spaces with scalar products $\langle\, ,\rangle$ and norms $|\ |$. The set of linear bounded operators mapping $U(H)$ into H is denoted by $L(U, H)$ ($L(H)$ respectively). Consider the controlled system:

$$y' = Ay + (-A)^\theta Bu\,, \quad y(0) = y_0 \tag{2.1}$$

where A is the infinitesimal generator of an analytic semigroup of negative type $S(t)$ on H, $B \in L(U, H)$, $u \in L^2(0, T; U)$, $0 < T < \infty$, $\theta \in [0, 1[$ and $(-A)^\theta$ are the fractional powers of $-A$ (see for instance [18]). We denote by $D((-A)^{-\theta})$ the dual space of $D((-A^*)^\theta)$.

Since $|(-A)^\theta S(t)| \leqslant C/t^\theta$ for $t \in]0, T]$, we define the mild solution of (2.1) by

$$y(t) = S(t)y_0 + \int_0^t (-A)^\theta S(t-r)Bu(r)\, dr\,. \tag{2.2}$$

In fact $y \in C([0, T]; H) \cap L^2(0, T; D((-A))^{1-\theta})$ if $\theta < \frac{1}{2}$ and $y \in L^2(0, T; H) \cap \cap L^2(0, T; D((-A))^{1-\theta-\varepsilon})$ if $\theta \geqslant \frac{1}{2}$ for $\varepsilon > 0$ small.

We now associate with (2.2) the following quadratic functional:

$$J(u) = (P_0 y(T), y(T)) + \int_0^T [(My(t), y(t)) + |u(t)|^2]\, dt\,, \tag{2.3}$$

where $P_0 \in L^+(D((-A)^\mu); D((-A)^{-\mu}))$, $M \in L^+(D((-A)^\nu); D((-A)^{-\nu}))$, $\mu, \nu \geqslant 0$, $(\cdot, \cdot)$ denotes duality and $L^+(D((-A)^\mu); D((-A)^{-\mu}))$ denotes the set of self-adjoint non-negative operators in H which are bounded as application of $D((-A)^\mu)$ into $D((-A)^{-\mu})$. Assume for the moment $\mu = \nu = 0$ and P_0, $M \in L^+(H)$. If $\theta < \frac{1}{2}$, $J(u)$ is finite for any $u \in L^2(0, T; U)$. If $\theta \geqslant \frac{1}{2}$, the first term in (2.3) in general does not make sense. But $J(u)$ is finite for any $u \in C([0, T]; U)$. Thus for any $0 \leqslant \theta < 1$ the minimization of $J(u)$ over all $u \in L^2(0, T; U)$ makes sense. The Riccati equation for this problem is formally given by

$$P' = A^*P + PA + M - P(-A)^\theta BB^*(-A^*)^\theta P\,, \quad P(0) = P_0 \tag{2.4}$$

and its mild version by

$$P(t)y = S^*(t)P_0 S(t)y + \int_0^t S^*(t-r)[M - P(r)(-A)^\theta BB^*(-A^*)^\theta P(r)]S(t-r)y\, dr\,. \tag{2.5}$$

We shall consider three cases separately and establish a unique global solution by showing that local solutions are uniformly bounded over all intervals of existence.

Let $C_s([0, T]; L(H))$ be the Banach space of strongly continuous operators P on H with norm $\|P\|_T = \operatorname{Sup}_{0\leqslant t\leqslant 1} |P(t)|$.

Let K be a real Hilbert space and let $L^\infty_\alpha(0, T; L(H, K))$ be the Banach space of strongly measurable operators in $L(H, K)$ with norm $\|P\|^\infty_{\alpha,T} = \operatorname{ess.\,sup}_{0\leqslant t\leqslant 1} t^\alpha|P(t)|$.

REMARK 2.1. – We have assumed that $S(t)$ is of negative type, but results in this paper can be easily extended to the case of general analytic semigroups.

2.1. *The case* $0\leqslant\theta + \nu < \frac{1}{2}$.

We set formally

$$Q(t) = (-A^*)^\theta P(t)(-A)^\theta\,, \tag{2.6}$$

then (2.5) yields

$$Q(t)y = S^*(t)\hat{P}_0 S(t)y + \int_0^t S^*(t-r)[\hat{M} - Q(r)BB^*Q(r)]S(t-r)y\,dr \tag{2.7}$$

where $\hat{P}_0 = (-A^*)^\theta P_0(-A)^\theta$ and $\hat{M} = (-A^*)^\theta M(-A)^\theta$.

Now we study (2.7) under general conditions.

PROPOSITION 2.1. – Suppose $\hat{P}_0 \in L^+\big(D((-A)^{\alpha/2}), D((-A)^{-\alpha/2})\big)$, $M \in L^+\big(D((-A)^\beta), D((-A)^{-\beta}\big)$ for some $0\leqslant\alpha, \beta < \frac{1}{2}$.

Then there exists a unique solution Q to (2.7) in $L^\infty_\alpha(0, T; L^+(H)) \cap C_s(]0, T]; L_+(H))$. Moreover if $\hat{P}_0 \in L^+(H)$, then $Q \in C_s([0, T]; L^+(H))$.

PROOF. – Note that the right hand side of (2.7) (as a function of Q) maps $L^\infty_\alpha(0, T_0; L_s(H))$ into itself for any $T_0 > 0$, as in the classical case of bounded coefficients (see for example [5], [6]). Here $L_s(H)$ is the set of self-adjoint operators in $L(H)$. If T_0 is small enough, the standard contraction mapping theorem assures the existence and uniqueness of a solution Q in $L^\infty_\alpha(0, T_0; L_s(H))$ [2].

Then from (2.7) we also have $Q \in C_s(]0, T_0]; L_s(H))$.

Using the evolution operator generated by $A - (-A)^\theta BB^*Q(t)$ we can also show (if necessary via approximations as in [2]) that $Q(t) \in L^+(H)$, for $0 < t \leqslant T_0$. To show the global existence it is sufficient to prove $\|Q\|^\infty_{\alpha,T_0} \leqslant c$ for some constant independent of T_0 [2], then from (2.7) we have an a priori estimate:

$$\|Q\|^\infty_{\alpha,T_0} \leqslant \|S^*(t)\hat{P}_0 S(t)\|^\infty_{\alpha,T_0} + \left\|\int_0^t S^*(t-r)\,\hat{M}S(t-r)\,dr\right\|^\infty_{\alpha,T_0} \leqslant c \tag{2.8}$$

independent of T_0.

If $\hat{P}_0 \in L^+(H)$, we can directly establish a unique solution in $C_s([0, T]; L^+(H))$ as in the classical case of bounded coefficients, see for example [1], [5].

Coming back to the original equation (2.5) we have:

COROLLARY 2.1. – Let $0 \leqslant \theta + \mu \leqslant (\theta + \nu)/2 < \frac{1}{4}$. Then there exists a unique solution P to (2.5) in $L^\infty_{2\mu}(0, T; L^+(H)) \cap L^\infty_{2(\theta+\mu)}\big(0, T; L^+(D(-A)^{-\theta}), D((-A^*)^\theta)\big) \cap$ $\cap L^\infty_{\theta+2\mu}\big(0, T; L(H; D((-A^*)^\theta))\big)$. Moreover if $P_0 \in L\big(D((-A)^{-\theta}), D((-A^*)^\theta)\big)$, $P \in$ $\in C_s\big([0, T]; L^+(H) \cap L^+\big(D((-A)^{-\theta}), D((-A^*)^\theta)\big) \cap L\big(H, D((-A^*)^\theta)\big)\big)$.

2.2. *The case* $\theta + \nu = \frac{1}{2}$.

In this case we assume that A is self-adjoint. Then we have ([8])

$$\int_0^T |(-A)^{\frac{1}{2}} S(t) y|^2 \, dt \leqslant \tfrac{1}{2} |y|^2 , \qquad y \in H .$$

Hence we can take $\beta = \frac{1}{2}$ in Proposition 2.1. In fact we have

PROPOSITION 2.2. – Suppose $\hat{P}_0 \in L^+(D(-A)^{\alpha/2})$, $\big(D((-A)^{-\alpha/2})\big)$, $0 \leqslant \alpha < \frac{1}{2}$ and $M \in$ $\in L^+\big(D((-A)^{\frac{1}{2}}), D((-A)^{-\frac{1}{2}})\big)$. Then there exists a unique solution to (2.7) in $L^\infty_\alpha(0,$ $T; L^+(H)) \cap C_s(]0, T]; L^+(H))$. Moreover, if $\hat{P}_0 \in L^+(H)$, then $Q \in C_s([0, T]; L^+(H))$.

COROLLARY 2.2. – Assume $0 \leqslant \theta + \mu < (\theta + \nu)/2 = \frac{1}{4}$. Then the conclusion of Corollary 2.1 still holds.

REMARK 2.2. – We can easily generalize Proposition 2.2 to the case where A is replaced by $A + A_1$, with A self-adjoint and $A_1 \in \mathcal{L}\big(D((-A)^{\frac{1}{2}}, H)\big)$ see [7], [8]. In this case (2.5) is replaced by

$$\text{(2.9)} \qquad P(t)y = S(t)P_0S(t)y + \int_0^t S(t-r)[M + A_1^* P(r) + P(r)A_1 - \\ - P(r)BB^*P(r)]S(t-r)y \, dr ,$$

and the control system (2.1) by

$$\text{(2.10)} \qquad y' = (A + A_1)y + (-A)^\theta Bu , \qquad y(0) = y_0 .$$

REMARK 2.3. – The special case: $\theta = 0$, $\nu = \frac{1}{2}$ has an interesting application, see Example 5.1.

2.3. *The general case.*

If $2\nu \geqslant 1$ we cannot use the transformation (2.6), since the integral

$$\int_0^t S^*(t-r)\hat{M}S(t-r)y \, dr$$

does not make sense. So we shall directly study (2.5):

$$P(t)y = S^*(t)P_0S(t)y + \int_0^t S^*(t-r)[M - P(r)(-A)^\theta BB^*(-A^*)^\theta P(r)]S(t-r)y\,dr\,.$$

Our main assumption is the following:

$$(2.11)\qquad \begin{cases} \text{i) } 0 \leqslant \theta + 2\nu < 1; \\ \text{ii) } 0 \leqslant \theta + 2\mu < \frac{1}{2} \text{ if } \theta < \frac{1}{2} \text{ and} \\ \qquad P_0 \in L_+(H) \cap L\big(H; D((-A^*)^{\bar\mu})\big) \quad \text{for } \bar\mu/2 > \theta - \frac{1}{2} \text{ if } \theta \geqslant \frac{1}{2}\,. \end{cases}$$

The following is our main result of this section and generalizes [10], giving also a simpler proof.

PROPOSITION 2.3. – Assume (2.11). Then (2.5) has a unique solution in $L^\infty_\beta\big(0, T; L\big(H; D((-A^*)^\theta)\big)\big) \cap C_s([0, T]; L_+(H)) \cap C_s\big(]0, T]; L\big(H; D((-A^*)^\theta)\big)\big)$ (¹). Moreover, if $P_0 \in L^+(H) \cap L\big(H; D((-A^*)^\theta)\big)$, then $P \in C\Big([0, T]; L\big(H; D((-A^*)^\theta)\big)\Big)$.

PROOF – Following [10] we set $(-A^*)^\theta P(t) = R(t)$, then:

$$(2.12)\qquad R(t)y = S^*(t)(-A^*)^\theta P_0S(t)y + \int_0^t (-A^*)^\theta S^*(t-r)[M - R(r)BB^*R(r)]S(t-r)y\,dr\,.$$

We shall consider only the case $\theta \geqslant \frac{1}{2}$, the other case being similar. Then the right hand side of (2.12) maps $L^\infty_{\theta-\bar\mu}(0, T; L(H))$ into itself. Again it is well known that there exists a unique local solution to (2.12) in $L^\infty_{\theta-\bar\mu}(0, T_0; L(H))$ for sufficiently small T_0 Then $P(t) = (-A^*)^{-\theta}R(t)$ is a unique solution to (2.5). Moreover, $P(t)$ satisfies on $[0, T_0]$.

$$(2.13)\qquad P(t)y = S^*(t)P_0U(t, 0)y + \int_0^t S^*(t-r)MU(t, r)y\,dr$$

where

$$(2.14)\qquad U(t, s)y = S(t-s)y - \int_s^t S(r-s)(-A)^\theta BB^*(-A^*)^\theta P(r)U(t, r)y\,dr\,, \qquad 0 \leqslant s \leqslant t \leqslant T_0\,.$$

(¹) Where $\beta = \theta + 2\mu$ if $\theta < \frac{1}{2}$ and $\beta = \theta - \bar\mu$ if $\theta \geqslant \frac{1}{2}$.

From (2.13) we obtain

$$(2.15)\quad (-A^*)^\theta P(t)y = (-A^*)^\theta S^*(t)P_0 U(t,0)y + \int_0^t (-A^*)^\theta S(t-r)MU(t,r)y\,dr$$

To obtain a global solution $R(t)$ on $[0,T]$ it is sufficient to show that

$$\|(-A^*)^\theta P(t)\|^\infty_{\theta-\bar{\mu},T_0} \leqslant c$$

independent of T_0. This will follow via (2.15) if we can show that

$$\|U(\cdot,\cdot)\|_{T_0} = \sup_{0\leqslant s\leqslant t\leqslant T_0} |U(t,s)| \leqslant c \quad \text{independent of } T_0 .$$

For this purpose we take $0 < T_1 \leqslant T_0$ and consider

$$(2.16)\quad (-A^*)^\theta P(r)U(t,r)y = S^*(r)(-A^*)^\theta P_0 U(t,0)y + \\ + \int_0^r (-A^*)^\theta S^*(r-s)MU(t,s)\,ds ,$$

where we have used the semigroup property

$$U(r,s)U(t,r) = U(t,s) \quad \text{for} \quad 0\leqslant s\leqslant r\leqslant t\leqslant T_0 .$$

Note that from (2.5) we have $|P(t)|\leqslant b_0$ for some $b_0 = b_0(T) > 0$. Let $b = \max(b_0, |P_0|)$. Now from (2.16) we have

$$(2.17)\quad |(-A^*)^\theta P(r)U(t,r)| \leqslant [C_1/r^{\theta-\bar{\mu}} + C_2 r^{1-\theta-2\bar{\mu}}]\cdot\|U(\cdot,\cdot)\|_{T_1} \quad \text{for } r\in\,]0,T_1] ,$$

where c_1 and c_2 are independent of T_1 (or T_0).

Then from (2.14) we obtain

$$\|U(\cdot,\cdot)\|_{T_1} \leqslant C_3 + C_4\int_s^t \frac{dr}{(r-s)^\theta}[C_1 b r^{\bar{\mu}-\theta} + C_2 r^{1-\theta-2\bar{\mu}}]\|U(\cdot,\cdot)\|_{T_1} ,$$

where C_3, C_4 are constants independent of T_1. Then for each $0<\eta<1$ we can find T_1 sufficiently small such that

$$\|U(\cdot,\cdot)\|_{T_1} \leqslant C_3 + \eta\|U(\cdot,\cdot)\|_{T_1} .$$

Now we fix $0<\eta<1$ and $T_1 = T_1(\eta)$. Then $\|U(\cdot,\cdot)\|_{T_1} \leqslant C_3/(1-\eta)$.

Now considering equations similar to (2.13), (2.14) and (2.16) on $[T_1, 2T_1]$ (initial time 0 is replaced by T_1) and so on, we can easily obtain after a finite number of setps

$$\|U(\cdot,\cdot)\|_{T_1} \leqslant c(\eta)\,, \quad \text{independent of } T_0\,.$$

Thus we have shown that (2.12) has a unique global solution to (2.5) as asserted in Proposition 2.3.

REMARK 2.4. – The result in FLANDOLI [10] is our special case $\nu = 0$, $\bar{\mu} = \theta$.

REMARK 2.5. – It is not clear whether or not we can take $\bar{\mu} = 0$ in Proposition 2.3, while LASIECKA and TRIGGIANI [16] obtain a solution to (2.4) when $\bar{\mu} = \nu = 0$, $\theta = \frac{3}{4} + \varepsilon$, $\varepsilon > 0$ small.

3. – The closed loop system.

In this section we consider (2.1) with feedback controls

$$u(\cdot) = K(T - \cdot)y\,, \tag{3.1}$$

where $K \in L^\infty_\alpha(0, T; L(H, U))$ for some $0 \leqslant \alpha < 1$. In this case (2.3) becomes an integral equation

$$y(t) = S(t)y_0 + \int_0^t (-A)^\theta S(t-r)BK(T-r)y(r)\,dr\,. \tag{3.2}$$

Since $K(T - \cdot)$ is bounded on $]0, T-\varepsilon]$ for any $\varepsilon > 0$, there exists a unique solution y to (3.2) in $C([0, T-\varepsilon]; H) \cap L^\infty_\beta(0, T-\varepsilon; D((-A)^\beta)) \cap C]0, T-\varepsilon]; D((-A^\beta)))$ for any $0 \leqslant \beta < 1 - \theta$. It is interesting to know the behavior of $y(\cdot)$ as $\varepsilon \to 0$. The following Proposition is easily proved.

PROPOSITION 3.1. – Let $K \in L^\infty_\alpha(0, T; L(H, U))$ for some $0 \leqslant \alpha < 1$. Suppose $\theta + \alpha < 1$, then there exists a unique solution y to (3.2) in $C([0, T]; H)$. If $\alpha + \beta + \theta < 1$, then $y \in L^\infty_\beta(0, T; D((-A)^\beta)) \cap C(]0, T]; D((-A)^\beta))$.

Let μ, ν, θ in (2.2), (2.3) (resp. $\nu \geqslant 0$, $\theta \geqslant 0$, $\theta + 2\nu < 1$) be given. Then $K \in L^\infty_\alpha(0, T; L(H, U))$ is admissible if $\theta + \nu + \alpha < 1$ and $\theta + \mu + \alpha < 1$ hold.

If, in particular, $\theta = 0$ and A is self-adjoint, then for $\mu < \nu = \frac{1}{2}$, $K \in L^\infty_\alpha(0, T; L(H; U))$ is admissible for any $\alpha < \frac{1}{2}$.

Thus the feedback control given by the solution of the Riccati equation

$$u = -B^*(-A^*)^\theta P(T - \cdot)\,y \tag{3.3}$$

is admissible and is in fact optimal as we shall see in Section 4.

If $V(t, s)$ is the evolution operator relative to $A - (-A)^\theta BB^*(-A^*)^\theta P(T-t)$ then $U(t, s) = V(T-s, T-t)$ where U is the operator in (2.14).

Remark 3.1. – Similar conclusions hold when A is replaced by $A + A_1$ with A self-adjoint and A_1 bounded from $D((-A)^{\frac{1}{2}})$ into H. In this case we take (2.9), (2.10) instead of (2.1), (2.5).

Now we consider the constant feedback controls:

$$u = Ky \,, \qquad K \in \mathcal{L}(H, U) \,.$$

Consider

$$y(t) = S(t)y_0 + (-A)^\theta \int_0^t S(t-r)BKy(r)\, dr \,. \tag{3.4}$$

By Proposition 3.1 there exists a unique solution y to (3.4) in $C([0, T]; H)$ for any $0 < T < \infty$. So we can define a strongly continuous semigroup, say $S_K(t)$. We now characterize the domain of its generator A_K.

Proposition 3.2. – $S_K(t)$ is an analytic semigroup and its generator is given by

$$\begin{aligned} A_K y &= (-A)^\theta[-(-A)^{1-\theta}y + BK_y] \\ D(A_K) &= \{y \in D((-A)^{1-\theta}), -(-A)^{1-\theta}y + Ky \in D((-A)^\theta)\} \,. \end{aligned} \tag{3.5}$$

Moreover, if $S(t)$ is compact, then $S_K(t)$ is also compact.

Proof. – Taking the Laplace transform of (3.4) we obtain

$$R(\lambda, A_K)y_0 = R(\lambda, A)y_0 + (-A)^\theta R(\lambda, A)BKR(\lambda, A_K)y_0 \,.$$

Thus

$$R(\lambda, A_K) = [I - (-A)^\theta R(\lambda, A)BK]^{-1}R(\lambda, A) \tag{3.6}$$

for sufficiently large λ. Note that multiplication by $\exp[-at]$ to any semigroup does not change the domain of its generator. So we may assume $0 \in \varrho(A) \cap \varrho(A_K)$, resolvent sets. Hence setting $\lambda = 0$ in (3.6) we obtain

$$A_K^{-1} = [I + (-A)^{-(1-\theta)}BK]^{-1}A^{-1}$$

from which (3.5) follows. The analyticity and compactness also follow from (3.6).

REMARK 3.2. – As we shall see in Section 5, there are some cases where $B = (-A)^{1-\theta}B_0$, $B_0 \in L(U, D((-A)^{1-\theta}))$. Then

$$A_K y = A[I - B_0 K]y$$
$$D(A_K) = \{y \in H;\ (I - B_0 K)y \in D(A)\}\ .$$

In general let $S(t)$ be any strongly continuous semigroup and let K be any linear unbounded operator on H. Suppose

$$y(t) = S(t)y_0 + A\int_0^t S(t-r)Ky(r)\,dr$$

has a unique solution and defines a strongly continuous semigroup $S_K(t)$. Then

$$A_K = A(I - K)$$
$$D(A_K) = \{y \in D(K);\ y - Ky \in D(A)\}\ .$$

Sufficient conditions for $A(I-K)$ to generate a strongly continuous semigroup are studied in [3], [9], [17], [23].

4. – Quadratic control.

Consider the quadratic control problem (2.1), (2.3). Since we have established a unique solution to the Riccati equation (3.5), dynamic programming gives us the optimal control and the minimum cost. In fact we have

PROPOSITION 4.1. – Suppose one of the following conditions holds:

(4.1)
(i) $0 \leqslant \theta + \nu < \frac{1}{2}$, $\theta + \mu \leqslant (\theta + \nu)/2$;
(ii) $\theta + \nu = \frac{1}{2}$, $\mu < \frac{1}{2}$, where A is self-adjoint;
(iii) $0 \leqslant \theta + 2\nu < 1$ and $0 \leqslant \theta + 2\mu < \frac{1}{2}$ if $\theta < \frac{1}{2}$ whereas $P_0 \in L^+(H) \cap L(H; D((-A^*)^{\bar{\mu}}))$ for $\bar{\mu}/2 > \theta - \frac{1}{2}$ if $\theta \geqslant \frac{1}{2}$.

Then the optimal control is given by the feedback law

$$u_*(\cdot) = -B^*(-A^*)^\theta P(T - \cdot)y(\cdot) \tag{4.2}$$

and the minimum cost by

$$J(u_*) = \langle P(T)y_0, y_0\rangle \tag{4.3}$$

where P is the unique solution of (2.5) given either in Corollaries 2.1, 2.2 or in Proposition 2.3.

PROOF. – As in [2] we can use bounded approximations to (2.1) and then pass to the limit.

REMARK 4.1. – In (4.1) (ii) we may replace A by $A + A_1$ as in Remark 2.2.

REMARK 4.2. – The algebraic Riccati equation is also considered by FLANDOLI [11] under stabilizability and detectability conditions [22].

5. – Examples.

In this section we give some examples which are covered by our abstract model. In the following $\mathcal{O}$ is an open bounded domain in $\boldsymbol{R}^d$ with smooth boundary $\partial\mathcal{O}$. We will consider Laplace operator Δ in $\mathcal{O}$, but everything in the sequel remains true for a general second order elliptic operator.

EXAMPLE 5.1. – Unbounded cost operators. Consider

$$\left\{\begin{aligned} &\frac{\partial y(t,x)}{\partial t} = \Delta y(t,x) + u(t,x)\,, && x \in \mathcal{O}\\ &y(t,x) = 0\,, && x \in \partial\mathcal{O}\\ &y(0,x) = y_0(x) \end{aligned}\right. \tag{5.1}$$

where $y_0 \in L^2(\mathcal{O})$ and $u \in L^2([0,T]\times\mathcal{O})$. In this case we take $H = L^2(\mathcal{O})$, $A = \Delta$ with $D(A) = H^2(\mathcal{O}) \cap H_1^0(\mathcal{O})$, $\theta = 0$, $B = I$ and $U = L^2(\mathcal{O})$. We take the following cost

$$J(u) = \int_0^T dt \int_{\mathcal{O}} \left[m \left|\frac{\partial y(t\;x)}{\partial x}\right|^2 + |u(t,x)|^2 \right] dx + \\ + p_0 \int_{\mathcal{O}} |y(T,x)|^2\, dx\,, \qquad m \text{ and } p_0 \text{ being positive numbers}\,. \tag{5.2}$$

In this case $M = -mA$ and the Riccati equation is given by

$$P' = AP + PA - mA - P^2\,, \qquad P(0) = P_0 I\,.$$

Here we can apply Proposition 2.2 with $\alpha = 0$ and Proposition 4.1 with $\mu = 0$, thus it has a unique solution in $C_s([0, T]; L^+(H))$.

If $d = 1$, then we may replace the last term in (5.2) by $|y(T, \xi)|^2$, $\xi \in \mathcal{O}$.

If $d \leqslant 3$ we may replace the integrand $|\partial y(t, x)/\partial x|^2_{L^2(\mathcal{O})}$ by $|y(t, \xi)|^2$, $\xi \in \mathcal{O}$, since C defined by $Cy = y(\xi)$ is a bounded linear functional on $D((-A)^\theta)$, $\theta > d/4$.

EXAMPLE 5.2. – The mixed boundary control problem [10, IEEE], [20]. Consider

$$
\begin{cases}
\dfrac{\partial y(t, x)}{\partial t} = \Delta y(t, x)\,, & x \in \mathcal{O}\,, \\[2ex]
\dfrac{\partial y(t, x)}{\partial n} + a(x) y(t, x) = u(t, x)\,, & x \in \partial\mathcal{O}\,, \\[2ex]
y(0, x) = y_0(x)\,,
\end{cases}
\tag{5.3}
$$

where $\partial/\partial n$ denotes the outward normal derivative and $a(x) > 0$ is a continuous function on $\partial\mathcal{O}$. We take $H = L^2(\mathcal{O})$, $U = L^2(\partial\mathcal{O})$. Let $D(A)$ be the closure in $H^2(\mathcal{O})$ of the subspace $\{y \in C^1(\mathcal{O}),\ \partial y/\partial n + ay = 0\}$. A is the restriction of Δ to $D(A)$. Let B_0 be the map: $U \to H$ defined by $y = B_0 u$ where y is the solution of

$$\Delta y = 0\,, \qquad \frac{\partial y}{\partial n} + ay = u\,.$$

Then $B_0 \in L\big(U, D((-A)^{1-\theta})\big)$ for any $\theta > \frac{1}{4}$ [15], [17].

We take $\theta = \frac{1}{4} + \varepsilon$, $\varepsilon > 0$ small and set $B = (-A)^{1-\theta} B_0$. Now $y(t)$ defined by (2.3) is the semigroup model of (5.3) studied in [17]. For each $u \in L^2(0, T; U)$ we have $y \in C([0, T]; H) \cap L^2(0, T; D(-A)^{\frac{1}{4}+\varepsilon})$. As a cost functional we may take

$$J(u) = \int_0^T [m|y(t, x)|^2_{L^2(\partial\mathcal{O})} + |u(t, x)|^2_{L^2(\partial\mathcal{O})}]\, dt\,. \tag{5.4}$$

In this case $M = mC^*C$ with $Cy = y|_\Gamma$, the trace operator. Since $C \in L\big(D((-A)^\alpha), L^2(\partial\mathcal{O})\big)$ for any $\alpha > \frac{1}{4}$, we have $M \in L\big(D((-A)^\nu), D((-A)^{-\nu})\big)$, $\nu = \frac{1}{4} + \varepsilon$. So we can apply Proposition 2.3 and Proposition 4.1 (4.1 (iii)) with $\theta = \nu = \frac{1}{4} + \varepsilon$, $\mu = 0$.

The Riccati equation is written as

$$P' = A^*P + PA + M - PAB_0B_0^*A^*P\,, \qquad P(0) = 0$$

and has a unique solution in $C_s([0, T]); L^+(H) \cap L\big(H, D((-A^*)^{\frac{1}{4}+\varepsilon})\big)$. For (5.3), (5.4) we may of course take $\theta = \frac{1}{2} - 2\varepsilon$, $\varepsilon > 0$ small. Then the Riccati operator lies in $C_s([0, T]); L\big(H; D((-A)^{\frac{1}{2}-2\varepsilon})\big)$, giving more regularity.

EXAMPLE 5.3. – The Dirichlet boundary control [16]. Consider

$$
\left\{
\begin{aligned}
&\frac{\partial y(t,x)}{\partial t} = \Delta y(t,x)\,, && x \in \mathcal{O}\,,\\
&y(t,x) = u(t,x)\,, && x \in \partial\mathcal{O}\,,\\
&y(0,x) = y_0(x)\,,
\end{aligned}
\right. \tag{5.5}
$$

where H and A are as in Example 5.1 and $U = L^2(\partial\mathcal{O})$.

Here we take for $B_0\colon U \to H$, the Dirichlet map defined by $y = B_0 u\colon \Delta y = 0$, $y(x) = u(x)$, $x \in \partial\mathcal{O}$. It is known [15], [16] that $B_0 \in L\big(U, D((-A)^{1-\theta})\big)$ for any $\theta > \frac{3}{4}$. Thus we choose $\theta = \frac{3}{4} + \varepsilon$, $\varepsilon > 0$ small and set $B = (-A)^{1-\theta}B_0$ (see [16]).

We may take the cost functional

$$
J(u) = \int_0^T [\,|y(t,x)|^2_{L^2(\mathcal{O})} + |u(t,x)|^2_{C^2(\partial\mathcal{O})}]\,dt\,. \tag{5.6}
$$

Then $M = I$ and the Riccati equation for this problem has the same form as in Example 5.2 and has a unique solution in $C([0,T]); L^+(H) \cap L\big(H, D((-A^*)^\theta)\big)$ for any $\frac{3}{4} < \theta < 1$.

EXAMPLE 5.4. – The pointwise control. Consider

$$
\left\{
\begin{aligned}
&\frac{\partial y(t,x)}{\partial t} = \Delta y(t,x) + \delta(x-\xi)u(t)\,, && x, \xi \in \mathcal{O}\\
&y(t,x) = 0\,, && x \in \partial\mathcal{O}\\
&y(0,x) = y_0(x)\,,
\end{aligned}
\right. \tag{5.7}
$$

where δ is the delta function ([13]). Here we take H and A as in Example 5.1 and $U = R'$.

Since $(-A)^{-\theta}\delta(x-\xi) \in H$ for $\theta > d/4$, we assume $d \leqslant 3$. We take $\theta = d/4 + \varepsilon$ and set $B = (-A)^{-\theta}\delta(x-\xi)$. Then the following cost is well-defined:

$$
J(u) = p_0|y(T,x)|^2_{L_2(\mathcal{O})} + \int_0^T [m|y(t,x)|^2_{L_2(\mathcal{O})} + |u(t)|^2]\,dt\,, \quad p_0,\ m > 0\,. \tag{5.8}
$$

If $d = 1$, we may replace the integrand $|y(t,x)|^2_{L_2(\mathcal{O})}$ by $|y(t,\eta)|^2$, $\eta \in \mathcal{O}$.

REMARK 5.1. – For a given partial differential equation, we have a choice of θ in some interval contained in $[0,1[$. If we take θ small, we may allow for large unboundedness in cost operators. On the other hand if the cost functional is fixed, then choosing θ large we get more regularity for the Riccati operator.

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