H^p Multipliers on Stratified Groups (*).

LEONEDE DE MICHELE - GIANCARLO MAUCERI

Summary. – *In this paper we give a criterion for boundedness on the Hardy spaces for functions $M(\mathscr{L})$ of the sublaplacian $\mathscr{L}$ on a stratified group. The criterion requires that the function M satisfies locally a Besov condition. The proof is based on the atomic and molecular characterization of Hardy spaces.*

0. – Introduction.

Let G be a stratified Lie group. Then the sublaplacian $\mathscr{L}$ is a nonnegative essentially self adjoint operator when restricted to functions of compact support. Thus the function $M(\mathscr{L})$ can be defined by the spectral theorem according to the prescription

$$M(\mathscr{L}) = \int_0^\infty M(\lambda)\, dE(\lambda)$$

where $E(\lambda)$ is the spectral resolution of $\mathscr{L}$. If M is a bounded Borel function on $(0, \infty)$ the operator $M(\mathscr{L})$ is bounded on L^2, the space of square integrable functions on G with respect to the Haar measure.

A natural problem is to study the boundedness of the operator $M(\mathscr{L})$ on various spaces of distributions on G, in terms of smoothness properties of the function M. When G is the Heisenberg group this problem was investigated by the authors [DMM], [M1], who gave a criterion for the boundedness of $M(\mathscr{L})$ on the spaces L^p, $1 < p < \infty$. HULANICKI and STEIN [FS] proved the following Marcinkiewicz-type multiplier theorem on any stratified Lie group G.

THEOREM. – Suppose M is of class C^s on $(0, \infty)$ and

$$\sup_{\lambda > 0} |\lambda^j M^{(j)}(\lambda)| \leqq C < \infty \qquad \text{for } 0 \leqq j \leqq s\,.$$

If r is a positive integer and $s > r + (3Q/2) + 2$ (where Q is the homogeneous dimension of the group) then $M(\mathscr{L})$ is bounded on H^p for $Q/(Q + r) < p < \infty$.

(*) Entrata in Redazione il 10 dicembre 1985; versione riveduta il 3 luglio 1986.
Inditizzo degli AA.: L. DE MICHELE: Dipartimento di Matematica, Università di Milano, 20133 Milano, Italy; G. MAUCERI: Istituto di Matematica, Università di Genova, 16132 Genova, Italy.

Here H^p, $0 < p \leqq \infty$, is the Hardy space on G defined either in terms of maximal functions or in terms of an atomic decomposition [FS]. In particular for $1 < p \leqq \infty$ $H^p = L^p$.

Recently the second named author [M2] improved the multiplier theorem of Hulanicki and Stein for the range $1 < p < \infty$, replacing the smoothness condition on M by a fractional order condition of order $s > (Q/2) + 1$. Fractional versions of the multipliers theorem for H^p when $G = \boldsymbol{R}^n$ are not new [C], [CT], [TW]. In particular TAIBLESON and WEISS expressed the condition on the multiplier M as a requirement that M is locally in a Lipschitz-Besov space $\Lambda(s; 2, 2)$ and BAERNSTEIN and SAWYER [BS] proved a sharp generalization of their theorem.

The purpose of this paper is to prove a fractional multiplier theorem for operators $M(\mathscr{L})$ acting on the Hardy spaces H^p, $0 < p < \infty$, on a stratified group G. For $M\colon \boldsymbol{R} \to \boldsymbol{C}$ let

$$\Delta_h M(x) = M(x+h) - M(x)\,, \qquad \Delta_h^{k+1} M = \Delta_h(\Delta_h^k M)\,, \qquad k \geqq 0\,, \qquad \Delta_h^0 M = M\,.$$

If $1 \leqq \tau \leqq \infty$, $s > 0$, k is an integer $> s$ and $A > 0$, we say that the function M satisfies condition $C(s; \tau, \infty, A)$ if

$$\text{i)}\quad |M(\lambda)| \leqq A \quad \text{for all } \lambda > 0\,,$$

$$\text{ii)}\quad R^{\tau s-1} \sup_{|h|>0} |h|^{-\tau s} \int_{R/2}^{2R} |\Delta_h^k M(\lambda)|^\tau \, d\lambda \leqq A^\tau$$

for every $R > 0$.

Thus a function M satisfies a condition $C(s; \tau, \infty, A)$ if M is bounded on $\boldsymbol{R}_+$ and M and its dilates $M(R\lambda)$, $R > 0$, are locally in the Lipschitz-Besov space $\Lambda(s; \tau, \infty)$ uniformly with respect to R.

THEOREM 1. – Suppose M satisfies condition $C(s; \tau, \infty, A)$. If $r > 0$ (not necessarily an integer) and $s > r + (Q/2) + 1$ then the operator $M(\mathscr{L})$ is bounded on H^p for $Q/(Q + r) < p < \infty$ and is weak type $(1-1)$. Moreover there exists a con stant $C > 0$, independent of M and f, such that $\|M(\mathscr{L})f\|_{H^p} \leqq CA\|f\|_{H^p}$ for all $f \in H^p$.

An immediate consequence of Theorem 1 is the following corollary whose statement should be compared with that of [TW, Th. 4.9].

COROLLARY 2. – Let $0 < p \leqq 1$. Suppose that M satisfies a condition $C(s; \tau, \infty, A)$ for some $s > Q(p^{-1} - 2^{-1}) + 1$. Then $M(\mathscr{L})$ is bounded on H^p.

The proof of Theorem 1 is based on the atomic and molecular characterizations of H^p [FS], [TW], [He]. Namely we shall prove that the operator $M(\mathscr{L})$ can be decomposed into a sum of operators $\sum_i M_i(\mathscr{L})$ where each summand is bounded on H^p, since it maps p-atoms into p-molecules, and the series converges in the space of all bounded operators on H^p. We now describe the contents of the remaining sections of the paper.

In § 1 we briefly summarize the features of analysis on stratified groups we shall need in the sequel. In § 2 we describe the molecular characterization of the Hardy spaces on stratified groups. The last section is devoted to the proof of Theorem 1.

1. – Preliminaries on stratified groups.

The main reference for the content of this section is the monograph by FOLLAND and STEIN [FS], to which we refer the reader for all unexplained terminology and notation.

A stratified group is a connected simply connected nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ is the direct sum of vector subspaces V_j, $j = 1, \ldots, m$,s uch that $[V_i, V_j] \subset V_{i+j}$ and V_1 generates $\mathfrak{g}$ as an algebra. The algebra $\mathfrak{g}$ is equipped with a family of dilations $\{\delta_t : t > 0\}$ which are the algebra automorphisms defined by

$$\delta_t\Big(\sum_{j=1}^{m} X_j\Big) = \sum_{j=1}^{m} t^j X_j \qquad (X_j \in V_j) .$$

We shall denote also by δ_t the corresponding group dilations, writing tx instead of $\delta_t x$ whenever $x \in G$, $t > 0$. We shall denote by

$$Q = \sum_{j=1}^{m} j \dim (V_j)$$

the homogeneous dimension of G. A homogeneous norm on G is a continuous function $x \to |x|$ from G to $[0, \infty)$, which is C^∞ on $G\setminus\{0\}$ and satisfies $|x^{-1}| = |x|$, $|tx| = t|x|$ for all $x \in G$, $t > 0$; $|x| = 0$ if and only if $x = 0$ (the group identity). Moreover there exists a constant $\gamma \geqq 1$ such that $|xy| \leqq \gamma(|x| + |y|)$ for all $x, y \in G$. Actually GUIVARCH [Gu] has shown that every nilpotent Lie group with dilations has a norm with $\gamma = 1$. Henceforth we assume that G is equipped with a fixed homogeneous norm satisfying $|xy| \leqq |x| + |y|$, $x, y \in G$. The homogeneous norm defines a left invariant metric d on G by $d(x, y) = |x^{-1}y|$, $x, y \in G$. Thus for every $x \in G$ and $r > 0$ the ball of center x and radius r, $B(r, x) = \{y \in G : |x^{-1}y| < r\}$, is the left translate by x of $B(r, 0)$, which is in turn the image under δ_r of $B(1, 0)$.

We consider $\mathfrak{g}$ as the Lie algebra of all left invariant vector fields on G and fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that each X_j is an eigenvector for the dilations $\{\delta_t\}$ with eigenvalue d_j and $X_1, \ldots, X_\nu$ is a basis for V_1. We denote by $Y_1, \ldots, Y_n$ the corresponding basis for right invariant vector fields, i.e.

$$Y_j f(x) = \frac{d}{dt} f(\exp (tX_j)x)\Big|_{t=0} .$$

If $I = (i_1, \ldots, i_n) \in \boldsymbol{N}^n$ is a multiindex we set $Y^I = Y_1^{i_1} \ldots Y_n^{i_n}$. Then Y^I is a right invariant differential operator, homogeneous of degree $d(I) = \sum_{k=1}^{n} d_k i_k$ with respect to the dilations δ_t, $t > 0$.

The sublaplacian $\mathscr{L} = -\sum_{j=1}^{\nu} X_j^2$ is a second order, hypoelliptic nonnegative differential operator on G, whose restriction to smooth functions with compact support is essentially self adjoint. Let $\{E(\lambda)\colon \lambda > 0\}$ be its spectral resolution. Thus if M is a bounded Borel function on $(0, \infty)$ the operator

$$M(\mathscr{L}) = \int_0^\infty M(\lambda)\, dE(\lambda)$$

is bounded on L^2 and commutes with left translations. Thus, by Schwartz's kernel theorem, there exists a tempered distribution K on G such that $M(\mathscr{L})\varphi = \varphi * K$, for all functions φ in the Schwartz space $\mathscr{S}$. For every $t > 0$ let

$$M(t\mathscr{L}) = \int_0^\infty M(t\lambda)\, dE(\lambda)\, .$$

Then the distribution kernel of the operator $M(t\mathscr{L})$ is $K_{\sqrt{t}}$, where $K_{\sqrt{t}}$ is the distribution defined by

$$\langle K_{\sqrt{t}}, \varphi\rangle = \langle K, \varphi \cdot \delta_{\sqrt{t}}\rangle \quad \text{for } \varphi \in \mathscr{S}\, .$$

A polynomial on G is a function P on G such that $P \circ \exp$ is a polynomial on $\mathfrak{g}$ (exp is the exponential map from $\mathfrak{g}$ to G). Every polynomial on G can be written uniquely as a finite sum

$$P = \sum_I a_I \eta^I\, , \qquad a_I \in \boldsymbol{C}\, , \qquad \eta^I = \eta_1^{i_1} \ldots \eta_n^{i_n}$$

where $\eta_j = \xi_j \circ \exp$ and $\xi_1, \ldots, \xi_n$ is the basis of $\mathfrak{g}^*$ dual to the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. The monomial η^I is homogeneous of degree $d(I) = \sum_{k=1}^{n} d_k i_k$. The homogeneous degree of P is $\max\{d(I)\colon a_I \neq 0\}$. If $a \in \boldsymbol{N}$ we note by $\mathscr{P}_a$ the space of polynomials of homogeneous degree $\leqq a$. Note that $\mathscr{P}_a$ is invariant under left and right translations [FS, Prop. 1.25].

If $x \in G$, $a \in \boldsymbol{N}$ and f is a function whose derivatives $Y^I f$ are continuous functions in a neighborhood of x for $d(I) \leqq a$, the right Taylor polynomial of f at x of homogeneous degree a is the unique $P_{x,f} \in \mathscr{P}_a$ such that $Y^I P_{x,f}(0) = Y^I f(x)$ for $d(I) \leqq a$. The right Taylor polynomial of homogeneous degree a of a distribution

$T \in \mathscr{S}'$ is the unique polynomial

$$P_{\cdot,T} = \sum_{d(I) \leqq a} a_I \eta^I, \qquad a_I \in \mathscr{S}'$$

such that $Y^I P_{\cdot,T}(0) = Y^I T(\cdot)$ for $d(I) \leqq a$.

LEMMA 1.1. – Let $T \in \mathscr{S}'$ and $P_{\cdot,T} = \sum a_I \eta^I$, $a_I \in \mathscr{S}'$, be the right Taylor polynomial of T of homogeneous degree a. Then a_I is a linear combination of the $Y^J T$, $d(J) \leqq a$. Moreover for every $\varphi \in \mathscr{S}$

$$P_{x,\varphi * T}(y) = \big(\varphi * P_{\cdot,T}(y)\big)(x) = \sum_{d(I) \leqq a} \big(\varphi * a_I(x)\big)\eta^I(y),$$

$$P_{x,T*\varphi}(y) = \big(P_{\cdot,\varphi}(y) * T\big)(x) = \langle P_{x-z,\varphi}(y), T(z)\rangle .$$

PROOF. – The first statement is an easy consequence of [FS, pp. 24, 25]. The formulas for $P_{x,\varphi * T}$ and $P_{x,T*\varphi}$ follow at once from the definition of Taylor polynomial.

2. – The molecular characterization of Hardy spaces.

The molecular theory of Hardy spaces, initially developed for H^1 in the very general context of spaces of homogeneous type in [CW], was extended to all H^p spaces, $0 < p \leqq 1$, in various particular situations in [CR], [TW], [He]. In particular in [He] HEMLER developed a molecular theory for the Hardy spaces on the Heisenberg group. Since the same ideas work, with obvious modifications, in the context of stratified groups, in this section we shall limit ourselves to a brief description of the main results of the theory.

On a stratified group G the Hardy spaces H^p can be defined by means of the « heat maximal function » [FS]. Namely, let W_t, $t > 0$ be the heat kernel on G, i.e. the kernel of the operator $\exp(-t\mathscr{L})$, $t > 0$. Then $\{W_t\colon t > 0\}$ is a commutative approximate identity on G and, for every $0 < t \leqq \infty$, H^p is the space of all distributions $f \in \mathscr{S}'$ such that $f^*(x) = \sup_{t>0} |f * W_t(x)|$ is in L^p. The quasi norm on H^p is $\|f\|_{H^p}^p = \|f^*\|_p^p$. If $p > 1$, $H^p = L^p$. When $0 < p \leqq 1$ the elements of H^p can be decomposed into elementary building blocks: the (p, q, s)-atoms. Suppose $0 < p \leqq 1 \leqq q \leqq \infty$, s is an integer. We call the ordered triple (p, q, s) admissible if $p < q$ and $s \geqq [Q(p^{-1} - 1)]$ (the integer part of $Q(p^{-1} - 1)$). Suppose that (p, q, s) is admissible. A (p, q, s)-atom centered at $x_0 \in G$ is a compactly supported L^q function f such that there is a ball B of center x_0 whose closure contains supp (f) and $\|f\|_q \leqq |B|^{(1/q)-(1/p)}$, and f is orthogonal to all polynomials in $\mathscr{P}_s$.

THEOREM [FS]. – Let (p, q, s) be an admissible triple. Then any f in H^p can be represented as a linear combination of (p, q, s)-atoms

$$f = \sum \lambda_i f_i\,, \qquad \lambda_i \in \boldsymbol{C}$$

where the f_i are (p, q, s) atoms and the sum converges in H^p. Moreover $\|f\|_{H^p} \sim \sim \inf\left\{(\sum |\lambda_i|^p)^{1/p} \colon \sum \lambda_i f_i \text{ is a decomposition of } f \text{ into } (p, q, s)\text{-atoms}\right\}$.

The theory of molecules is an extension of atomic H^p theory with important applications to the study of convolution operators. If (p, q, s) is an admissible triple and $\varepsilon > \max\{s/Q, p^{-1} - 1\}$ we set

$$a = 1 - p^{-1} + \varepsilon\,, \qquad b = 1 - q^{-1} + \varepsilon\,.$$

Then *a* (p, q, s, ε)-molecule centered at x_0 is a L^q function M satisfying:

i) $M|x_0^{-1}x|^{Qb} \in L^q$,

ii) $\|M\|_q^{a/b}\,\|M|x_0^{-1}x|^{Qb}\|_q^{1-(a/b)} = \mathfrak{N}(M) < \infty$,

iii) $\int M(x)P(x)\,dx = 0$ for every $P \in \mathscr{P}_s$.

$\mathfrak{N}(M)$ is called the molecular norm of M. An easy adaptation of the arguments of [TW], [He] shows that every (p, q, s)-atom f is a (p, q, s, ε)-molecule for any $\varepsilon > \max(s/Q, p^{-1} - 1)$ and $\mathfrak{N}(f) \leqq C$, where C is a constant independent of the atom. Moreover any (p, q, s, ε)-molecule M is in H^p and $\|M\|_{H^p} \leqq C'\mathfrak{N}(M)$, where C' is independent of the molecule M.

As a consequence of the molecular characterization of H^p, to show that a linear map T is bounded on H^p it is sufficient to show that whenever f is a p-atom then Tf is a p-molecule and $\mathfrak{N}(Tf) \leqq C$ for some constant C independent of f. We shall exploit this approach to obtain estimates of certain operators on H^p spaces that will be useful in the proof of the multiplier theorem.

THEOREM 2.1. – Let k be a tempered distribution on G such that:

(2.1) $\quad k \in L^2_{\mathrm{loc}}$ on $G \setminus \{0\}$,

(2.2) $\quad \|f * k\|_2 \leqq A\|f\|_2$, for every $f \in \mathscr{S}$,

(2.3) $\quad$ there exist $a \in N$; $r > \max\left(a, [Q(p^{-1} - 1)]\right)$ and a measurable $\mathscr{P}_a$-valued function $x \to P_x$ almost everywhere defined on G such that:

$$\sup_{R>0} R^{-Q-2r} \int_{|y|<R} dy \int_{|x|>2R} |k(yx) - P_x(y)|^2\, |x|^{Q+2r}\, dx \leqq A^2\,.$$

Then the operator $Kf = f * k$ is bounded on H^p for $p > Q/(Q + r)$ and there exists a constant $C = C(Q, r)$ such that $\|Kf\|_{H^p} \leqq CA\|f\|_{H^p}$, for all $f \in H^p$.

PROOF. – Let s be an integer such that $r > s \geqq \max\big(a, [Q(p^{-1} - 1)]\big)$. Then the triple $(p, 2, s)$ and the quadruple $(p, 2, s, r/Q)$ are admissible. We shall prove that if f is a $(p, 2, s)$-atom then Kf is a $(p, 2, s, r/Q)$-molecule whose molecular norm satisfies $\mathfrak{N}(Kf) \leqq CA$. Assume that supp $(f) \subset \{x\colon |x| \leqq R\}$ and let $a = 1 - p^{-1} + (r/Q)$, $b = \frac{1}{2} + (r/Q)$. Then by (2.2):

$$\|f * k\|_2^{a/b} \leqq A^{a/b} R^{Q(a-b)a/b} .$$

On the other hand

$$\| \, |x|^{(Q/2)+r} f * k\|_2 \leqq \Big(\int_{|x|\leqq 2R} |f * k(x)|^2 |x|^{Q+2r} dx\Big)^{\frac{1}{2}} + \Big(\int_{|x|>2R} |f * k(x)|^2 |x|^{Q+2r} dx\Big)^{\frac{1}{2}} = I_1 + I_2 .$$

To estimate I_1 we use (2.2):

$$I_1 \leqq (2R)^{Q/2+r} \|f * k\|_2 \leqq C(Q, r) A R^{Qa} .$$

To estimate I_2 we use Schwarz-Hölder, (2.3) and the fact that f is orthogonal to $\mathscr{P}_a$ for $a \leqq s$:

$$I_2 = \Big[\int_{|x|>2R} \Big|\int_{|y|<R} f(y)[k(y^{-1}x) - P_x(y^{-1})]\, dy\Big|^2 |x|^{Q+2r} dx\Big]^{\frac{1}{2}} \leqq AR^{Qa} .$$

Thus $\mathfrak{N}(Kf) \leqq C(Q, r)A$. Since Kf is obviously orthogonal to $\mathscr{P}_a$, the theorem is proved.

3. – Proof of the multiplier theorem.

We shall begin by establishing some weighted norm inequalities for the distribution kernels of the operators $M(\mathscr{L})$, when M is a function with compact support in $\boldsymbol{R}^+$ in a Besov space on $\boldsymbol{R}$. We recall that for $s > 0$ the Besov space $\Lambda(s; 1, \infty)$ on $\boldsymbol{R}$ is the space of all function $f \in L^1(\boldsymbol{R})$ such that the norm

$$\|f\|_1 + \sup_{|h|>0} |h|^{-s} \|\Delta_h^k f\|_1 \tag{3.1}$$

is finite for some, and hence for all, integers $k > s$. For every $s > 0$ let $k(s)$ be the smallest integer strictly greater than s and denote by $|f|_{s;1,\infty}$ the norm defined by (3.1) when $k = k(s)$.

LEMMA 3.1. – Suppose $\alpha > 0$, $1 \leqq p \leqq 2$ and M is a function in $\Lambda(s; 1, \infty)$ such that $\operatorname{supp}(M) \subset [\frac{1}{2}, 2]$. Let H be the distribution kernel of the operator $M(\mathscr{L})$. Then H is in C^∞. If $s > \alpha p^{-1} + (Q/2)(2p^{-1} - 1) + 1$ then, for every multiindex I, $Y^I H \in L^p$ and

$$\int |x|^\alpha |Y^I H(x)|^p \, dx \leqq C(\alpha, p, s, Q, I) |M|^p_{s; 1, \infty} . \tag{3.2}$$

PROOF. – Assume first that the estimate holds for $I = 0$. Let $\omega(x) = (1 + |x|)^\alpha$. Then by [FS, Lemma 1.10] ω satisfies $\omega(x) = \omega(x^{-1})$, $\omega(xy) \leqq \omega(x)\omega(y)$ for $x, y \in G$. Thus if $g \in L^p(\omega(x)\,dx)$, $f \in L^1(\omega^{1/p}(x)\,dx)$ then $f * g \in L^p(\omega(x)\,dx)$ and

$$\|f * g\|_{L^p(\omega)} \leqq \|g\|_{L^p(\omega)} \|f\|_{L^1(\omega^{1/p})} .$$

Let $M_1(\lambda) = \exp(\lambda) M(\lambda)$. Then $M_1 \in \Lambda(s; 1, \infty)$ and $|M_1|_{s;1,\infty} \sim |M|_{s;1,\infty}$. Moreover $H = W_1 * H_1$, where H_1 is the kernel of $M_1(\mathscr{L})$ and W_1 is the heat kernel. Hence H is C^∞ and $Y^I H = (Y^I W_1) * H_1$. Then, since $W_1 \in \mathscr{S}$ and (3.2) holds for $I = 0$ by hypothesis:

$$\|Y^I H\|_{L^p(\omega)} \leqq \|H_1\|_{L^p(\omega)} \|Y^I W_1\|_{L^1(\omega^{1/p})} \leqq C(I, \alpha, p) \|H_1\|_{L^p(\omega)} \leqq C(I, \alpha, p, Q, s) |M|_{s;1,\infty} .$$

It remains only to prove the estimate (3.2) for $I = 0$. Let $F(t) = M(-\ln t)$, for $0 < t < 1$, and $F(t) = 0$ otherwise. Then $\operatorname{supp}(F) \subset [e^{-2}, e^{-\frac{1}{2}}]$ and the $\Lambda(s; 1, \infty)$ norms of F and M are comparable. As in [FS, Lemma 6.35] we expand F in a Fourier series on $(-\pi, \pi)$: $F(t) = \sum_{m \in \mathbf{Z}} a_m e^{imt}$. Then $F(0) = \sum_{m \in \mathbf{Z}} a_m = 0$ and $|a_m| \leqq$ $\leqq c |F|_{s;1,\infty} (1 + |m|)^{-s}$. But

$$M(\lambda) = F(e^{-\lambda}) = \sum_{m \in \mathbf{Z}} a_m [\exp(ime^{-\lambda}) - 1]$$

and hence

$$H(x) = \sum_{m \neq 0} a_m E_m(x)$$

where

$$E_m(x) = \sum_{k=1}^{\infty} \frac{(im)^k}{k!} W_k(x) , \qquad x \in G \tag{3.3}$$

is the distribution kernel of the operator $\exp(ime^{-\mathscr{L}}) - 1$, and

$$|a_m| \leqq C |M|_{s;1,\infty} (1 + |m|)^{-s} .$$

Hence we only need to prove the estimate

$$\left(\int |x|^\alpha |E_m(x)|^p \, dx\right)^{1/p} \leqq C(\alpha. p, Q) |m|^{\alpha p^{-1} + (Q/2)(2p^{-1} - 1)} [\ln^{Q/4}(|m|) + 1] \tag{3.4}$$

for all $\alpha \geqq 0$, $1 \leqq p \leqq 2$ and $m \in \mathbf{Z} \setminus \{0\}$. By a theorem of HULANICKI [H1], [H2] there exist two constants $k > 0$, $\theta > 1$ such that

$$\int e^{|x|} W_t(x)\, dx \leqq k\theta^t, \qquad t > 0 .$$

The function $W_t = W_{t/2} * W_{t/2}^*$ is positive definite.

Thus $\|W_t\|_\infty = W_t(0) = t^{-Q/2} W_1(0)$ and, by Hölder's inequality

$$\int e^{|x|} |W_t(x)|^p\, dx \leqq \|W_t\|_\infty^{p-1} \int e^{|x|} W_t(x)\, dx \leqq k_1 \theta^t t^{-(p-1)Q/2} .$$

Hence, by (3.3) and Minkowski's inequality

$$\left(\int e^{|x|} |E_m(x)|^p\, dx\right)^{1/p} \leqq k_2 \exp\left(|m|\theta^{1/p}\right) . \tag{3.5}$$

On the other hand we have

$$\|E_m\|_2 \leqq C[\ln^{Q/4}(|m|) + 1] \qquad \text{for } m \neq 0 . \tag{3.6}$$

Indeed let $\boldsymbol{A}$ be the closed subalgebra of L^1 generated by the functions W_t, $t > 0$. By [HJ] $\boldsymbol{A}$ is a commutative Banach $*$-subalgebra of L^1 whose maximal ideal space is homeomorphic to $[0, \infty)$. Moreover, if we denote by $\hat{f}$ the Gelfand transform of $f \in \boldsymbol{A}$, then $\hat{W}_t(\lambda) = e^{-\lambda t}$, $t > 0$, $\lambda \geqq 0$, and for all $f \in \boldsymbol{A} \cap L^2$.

$$\|f\|_2^2 = C(G) \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^{Q/2-1}\, d\lambda . \tag{3.7}$$

Then by (3.3) and (3.7) $E_m \in \boldsymbol{A} \cap L^2$ and

$$\|E_m\|_2^2 = C(G) \int_0^\infty |\exp(ime^{-\lambda}) - 1|^2 \lambda^{Q/2-1}\, d\lambda \leqq C[\ln^{Q/2}(|m|) + 1] .$$

Now we choose $A = p|m|\theta^{1/p}$ and estimate separately the integrals of $|x|^\alpha |E_m(x)|^p$ over the regions $|x| \leqq A$ and $|x| > A$. By Holder's inequality and (3.6)

$$\int_{|x| \leqq A} \leqq \left(\int_{|x| \leqq A} |x|^{2\alpha/(2-p)}\, dx\right)^{1-(p/2)} \|E_m\|_2^p \leqq C(\alpha, p, Q)\, A^{\alpha + Q(1-(p/2))} ln^{pQ/4}(|m|) .$$

On the other hand by (3.5)

$$\int_{|x| > A} \leqq \sup_{|x| \geqq A} \left(|x|^\alpha e^{-|x|}\right) \int e^{|x|} |E_m(x)|^p\, dx \leqq k_2^p A^\alpha e^{-A} \exp\left(p|m|\theta^{1/p}\right) = k_2^p A^\alpha$$

for $|m|$ large enough. Since the estimate (3.4) is easy for small $|m|$ the lemma is proved.

LEMMA 3.2. – Let $M \in \Lambda(s; 1, \infty)$ be a function such that $\operatorname{supp}(M) \subset [\frac{1}{2}, 2]$ and denote by H the distribution kernel of $M(\mathscr{L})$. Let $P_{x,H}$ be the right Taylor polynomial of H at x of homogeneous degree $a \in N$. Then if $0 < r < a + 1$ and $s > r + (Q/2) + 1$ there exist constants σ, $C_1 > 0$ such that

$$R^{-Q-2r} \int_G \int_{|y|<R} |H(yx) - P_{x,H}(y)|^2 \, dy |x|^{Q+2r} \, dx \leqq C_1 |M|_{s;1,\infty} R^\sigma \tag{3.8}$$

for all $0 < R \leqq 1$. If $r \geqq a$, $r > 0$ and $s > r + (Q/2) + 1$ there exist two constants δ, $C_2 > 0$ such that

$$R^{-Q-2r} \int_{|x|>2R} \int_{|y|<R} |H(yx) - P_{x,H}(y)|^2 \, dy |x|^{Q+2r} \, dx \leqq C_2 |M|_{s;1,\infty} R^{-\delta} \tag{3.9}$$

for all $R \geqq 1$.

PROOF. – Let $M_1(\lambda) = \exp(\lambda) M(\lambda)$. Then $H = W_1 * H_1$ and $|F_1|_{s;1,\infty} \sim |F|_{s;1,\infty}$. Let $P_x = P_{x,W_1}$ be the right Taylor polynomial of W_1 at x of homogeneous degree a. Thus by Lemma 1.1 $P_{x,H}(y) = (P(y) * H_1)(x)$ and

$$\int_G \int_{|y|<R} |H(yx) - P_{x,H}(y)|^2 \, dy |x|^{Q+2r} \, dx = \int_G \int_{|y|<R} \left| \int_G A_y(xz^{-1}) H_1(z) \, dz \right|^2 dy |x|^{Q+2r} \, dx$$

where $A_y(u) = W_1(yu) - P_u(y)$. Since, for $\alpha \geqq 0$, $|x|^\alpha \leqq C(\alpha)(|xz^{-1}|^\alpha + |z|^\alpha)$, the latter integral is bounded by

$$C(\alpha) \left\{ \int_{|y|<R} (\|B_y * H_1\|_2^2 + \|A_y * C\|_2^2) \, dy \right\}$$

where

$$B_y(u) = |A_y(u)| |u|^{(Q/2)+r}, \qquad C(u) = |H_1(u)| \, |u|^{(Q/2)+r}.$$

Now, by the right invariant Taylor inequality for stratified groups [FS, (1.44)] we have for $|y| \leqq R$:

$$|W_1(yx) - P_x(y)| \leqq C(a) |y|^{a+1} \varphi(x)$$

where $\varphi(x) = \sup\{|Y^I W_1(zx)|: |z| \leqq b^{a+1}R, \, d(I) = a + 1\}$ and $b > 0$ is a constant which depends only on the group G. Hence φ is a fast decreasing function and

$$\|A_y\|_1 = \int |W_1(yx) - P_x(y)| \, dx \leqq C(a) |y|^{a+1} \int \varphi(x) \, dx \leqq C_1 |y|^{a+1}.$$

Moreover, since $s > r + (Q/2) + 1$, by Lemma 3.1:

$$\|C\|_2 \leqq C_2 |M_1|_{s;1,\infty} \leqq C_3 |M|_{s;1,\infty}.$$

On the other hand

$$\|B_y\|_1 = \int |A_y(x)|\,|x|^{(Q/2)+r}\,dx \leq C(a)|y|^{a+1}\left(\int \varphi(x)|x|^{(Q/2)+r}\,dx\right) \leq C_4|y|^{a+1}.$$

Thus

$$\|A_y * C\|_2^2 + \|B_y * H_1\|_2^2 \leq \|A_y\|_1^2\,\|C\|_2^2 + \|B_y\|_1^2\,\|H_1\|_2^2 \leq C_5|M|^2_{s;1,\infty}\,|y|^{2(a+1)}.$$

Hence

$$\int_G \int_{|y|<R} |H(yx) - P_{x,H}(y)|^2\,dy|x|^{Q+2r}\,dx \leq C_6|M|^2_{s;1,\infty}\int_{|y|<R} |y|^{2(a+1)}\,dy \leq C_7|M|^2_{s;1,\infty}\,R^{Q+2r+\sigma}$$

where $\sigma = 2(a+1-r) > 0$. This proves (3.8). To prove (3.9) write

$$P_{x,H}(y) = \sum_{d(I)\leq a} a_I(x)\,\eta^I(y)$$

where, by Lemma 1.1, $a_I(x)$ is a linear combination of $Y^J H(x)$, $d(J) \leq a$. Since

$$|H(yx) - P_{x,H}(y)|^2 \leq C_8\big(|H(yx)|^2 + \sum_{d(I)\leq a} |a_I(x)|^2\,|\eta^I(y)|^2\big)$$

we have to estimate

$$T = R^{-Q-2r}\int_{|x|>2R}\int_{|y|<R} |H(yx)|^2\,dy|x|^{Q+2r}\,dx$$

and several terms of the form

$$T(I) = R^{-Q-2r}\int_{|x|>2R}\int_{|y|<R} |a_I(x)|^2\,|\eta^I(y)|^2\,dy|x|^{Q+2r}\,dx$$

for $d(I) \leq a$. To estimate T let $z = yx$. Then, since $|z| \geq |x| - |y| \geq R$, changing variables we have

$$T \leq R^{-Q-2r}\int_{|z|>R}\int_{|y|<R} |H(z)|^2\,|y^{-1}z|^{Q+2r}\,dy\,dz \leq C_9 R^{-Q-2r}R^Q\int_{|z|>R} |H(z)|^2\,|z|^{Q+2r}\,dz\,.$$

because $|y^{-1}z| \leq |y| + |z| \leq R + |z| \leq 2|z|$. Hence by Lemma 3.1

$$T \leq C_{10}R^{-2r}\,|M|^2_{s;1,\infty}\,. \tag{3.10}$$

Next we estimate the terms $T(I)$. Since the function η^I are homogeneous of degree $d(I)$, we have

$$\int_{|y|<R} |\eta^I(y)|^2\,dy \leq C_{11}R^{Q+2d(I)}\max_{|y|\leq 1} |\eta^I(y)|^2 \leq C_{12}R^{Q+2a}\,. \tag{3.11}$$

On the other hand, if $\delta > 0$ is such that $s > r + (Q/2) + 1 + (\delta/2)$, then, since a_I is a linear combination of $Y^J H$, $d(J) \leqq a$, by Lemma 3.1 we have

$$\int_{|x|>2R} |a_I(x)|^2 |x|^{Q+2r} dx \leqq R^{-\delta} \int_G |a_I(x)|^2 |x|^{Q+2r+\delta} dx \leqq C_{13} |M|^2_{s;1,\infty} R^{-\delta} . \tag{3.12}$$

Thus by (3.11), (3.12)

$$T(I) \leqq C_{14} |M|^2_{s;1,\infty} R^{2(a-r)-\delta} \leqq C_{14} |M|^2_{s;1,\infty} R^{-\delta} \tag{3.13}$$

provided $R \geqq 1$, $d(I) \leqq a$. Hence (3.9) follows from (3.10) and (3.13) and the lemma is proved.

Now we are ready to finish the proof of Theorem 1.1. We shall prove that if M satisfies condition $C(s; \tau, \infty, A)$ and $s > r + (Q/2) + 1$, $r > 0$, then $M(\mathscr{L})$ is bounded on H^p for $Q/(Q + r) < p \leqq 1$. The result for $1 < p < \infty$ and the weak type $(1-1)$ estimate will follow by interpolation and duality [FS, Th. 3.34, 3.37]. Clearly we may assume that r is not an integer.

Let K be the kernel of the operator $M(\mathscr{L})$. By Theorem 2.1 we only need to show that $K \in L^2_{\mathrm{loc}}(G \setminus \{0\})$ and there exist $a \in \boldsymbol{N}$, $a < r$ and a $\mathscr{P}_a$-valued function $x \to P_x$ almost everywhere defined on G such that

$$\sup_{R>0} R^{-Q-2r} \int_{|y|<R} dy \int_{|x|>2R} |K(yx) - P_x(y)|^2 |x|^{Q+2r} dx \leqq C_1 A^2 . \tag{3.14}$$

Let $\varphi \in C_c^\infty(\boldsymbol{R})$ be a function such that $\operatorname{supp}(\varphi) \subset [\frac{1}{2}, 2]$ and $\sum_{j\in \boldsymbol{Z}} \varphi(2^{-j}\lambda) = 1$ for $\lambda \neq 0$. Since M satisfies condition $C(s; \tau, \infty, A)$ it is easy to see that the functions $M_j(\lambda) = M(2^j\lambda)\varphi(\lambda)$ are in $\Lambda(s; 1, \infty)$ uniformly with respect to j and there exists a constant $C_2 > 0$ such that

$$|M_j|_{s;1,\infty} \leqq C_2 A \qquad j \in \boldsymbol{Z} . \tag{3.15}$$

Let Φ_j, H_j, K_j be the kernels of the operators $\varphi(2^{-j}\mathscr{L})$, $M_j(\mathscr{L})$, $M_j(2^{-j}\mathscr{L})$, $j \in \boldsymbol{Z}$, respectively. Then H_j, K_j are in C^∞, while Φ_j is in $\mathscr{S}$ by [M, Prop. 2.7]. Moreover $K_j(x) = K * \Phi_j(x) = 2^{jQ/2} H_j(2^{j/2}x)$ and $K = \sum_{j\in\boldsymbol{Z}} K_j$ in $\mathscr{S}'$. Let a be the integer part of r. Then, since r is not an integer, $r - 1 < a < r$. We shall prove that for every multiindex I, such that $d(I) \leqq a$, $Y^I K \in L^2_{\mathrm{loc}}(G \setminus \{0\})$. Indeed, by the homogeneity of the differential operator Y^I, we have:

$$\int_{|x|>R} |Y^I K_j(x)|^2 dx = 2^{j((Q/2)+d(I))} \int_{|y|>2^{j/2}R} |Y^I H_j(y)|^2 dy .$$

Now, applying Lemma 3.1, first for $\alpha = 0$, then for $\alpha = Q + 2r$, and using (3.15) we obtain

$$\int_{|y|>2^{j/2}R} |Y^I H_j(y)|^2\, dy \leqq \int_G |Y^I H_j(y)|^2\, dy \leqq C_3 A^2\,, \tag{3.16}$$

$$\int_{|y|>2^{j/2}R} |Y^I H_j(y)|^2\, dy \leqq (2^{j/2}R)^{-(Q+2r)} \int_G |y|^\alpha\, |Y^I H_j(y)|^2\, dy \leqq C_4 (2^{j/2}R)^{-(Q+2r)} A^2\,. \tag{3.17}$$

Thus by Minkowski's inequality:

$$\Bigg(\int_{|x|>R} |Y^I K(x)|^2\, dx\Bigg)^{\frac12} \leqq \sum_{j\in \boldsymbol{Z}} \Bigg(\int_{|x|>R} |Y^I K_j(x)|^2\, dx\Bigg)^{\frac12} \leqq C_5 A (1 + R)^{-(Q+2r)/2}$$

where we used (3.16) to estimate the terms with $j<0$ and (3.17) to estimate those with $j \geqq 0$. Thus $Y^I K \in L^2_{\mathrm{loc}}(G\setminus\{0\})$ for $d(I) \leqq a$ and the right Taylor polynomial $P_{x,K}$ of K at x of homogeneous degree a is well defined for almost every x. Let $P_{x,j}$ be the right Taylor polynomial of $K_j = \Phi_j * K$ at x of homogeneous degree a. By Lemma 1.1 $P_{x,j}(y) = [\Phi_j * P_{\cdot,K}(y)](x)$. Thus, to prove (3.14) with $P_x = P_{x,K}$ we only need to show that the series

$$\sum_{j\in\boldsymbol{Z}} \sup_{R>0} R^{-(Q+2r)/2} \Bigg[\int_{|y|<R} dy \int_{|x|>2R} |K_j(yx) - P_{x,j}(y)|^2\, |x|^{Q+2r}\, dx\Bigg]^{\frac12} \tag{3.18}$$

converges and its sum is bounded by $\sqrt{C_1}A$. Let $Q_{x,j}$ be the right Taylor polynomial at x of homogeneous degree a of H_j. In view of Lemma 3.2 and (3.15) we have

$$R^{-Q-2r} \int_{|x|>2R} \int_{|y|<R} |H_j(yx) - Q_{x,j}(y)|^2\, dy\, |x|^{Q+2r}\, dx \leqq \begin{cases} C_6 A^2 R^\sigma & \text{for } 0 < R \leqq 1\,, \\ C_6 A^2 R^{-\delta} & \text{for } 1 \leqq R\,, \end{cases} \tag{3.19}$$

where σ, $\delta > 0$. Since $P_{x,j}(y) = 2^{jQ/2} Q_{2^{j/2}x,j}(2^{j/2}y)$, (3.19) yields

$$\begin{aligned} &R^{-Q-2r} \int_{|x|>2R} \int_{|y|<R} |K_j(yx) - P_{x,j}(y)|^2\, dy\, |x|^{Q+2r}\, dx = \\ &= (2^{j/2}R)^{-Q-2r} \int_{|z|>2(2^{j/2}R)} \int_{|v|<2^{j/2}R} |H_j(vz) - Q_{z,j}(v)|^2\, dv\, |z|^{Q+2r}\, dz \leqq C_7 A^2 \min\big((2^{j/2}R)^\sigma, (2^{j/2}R)^{-\delta}\big)\,. \end{aligned}$$

This shows that the series (3.18) converges and its sum is bounded by $\sqrt{C_1}A$ where C_1 is a constant independent of R. So the proof is complete.

REFERENCES

[BS] A. BAERNSTEIN - E. T. SAWYER, *Embedding and multiplier theorems for* $H^p(\boldsymbol{R}^n)$, Memoirs of the Amer. Math. Soc., n. 318 (1985), pp. 1-80.

[C] R. R. COIFMAN, *Characterization of Fourier transforms of Hardy spaces*, Proc. Nat. Acad. Sci. U.S.A., **71** (1974), pp. 4133-4134.

[CR] R. R. COIFMAN - R. ROCHBERG, *Representation theorems for holomorphic and harmonic functions in* L^p, Astérisque, **77** (1980), pp. 11-66.

[CT] A. P. CALDERÓN - A. TORCHINSKY, *Parabolic maximal functions associated with a distribution* II, Advances in Math., **24** (1977), pp. 101-171.

[CW] R. R. COIFMAN - G. WEISS, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83** (1977), pp. 569-645.

[DMM] L. DE MICHELE - G. MAUCERI, L^p*-multipliers on the Heisenberg group*, Michigan Math. J., **26** (1979), pp. 361-373.

[FS] G. B. FOLLAND - E. M. STEIN, *Hardy spaces on homogeneous groups*, Mathematical Notes 28, Princeton University Press, Princeton, N.J. (1982).

[Gu] Y. GUIVARCH, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France, **101** (1973), pp. 333-379.

[He] H. L. HEMLER, *The molecular theory of* $H^{p,q,s}(\boldsymbol{H}^n)$, Ph. D. Dissertation, Washington University, St. Louis (1980).

[H1] A. HULANICKI, *Subalgebra of* $L^1(G)$ *associated with the Laplacian on a Lie group*, Coll. Math., **31** (1974), pp. 259-278.

[H2] A. HULANICKI, *Commutative subalgebra of* $L^1(G)$ *associated with a subelliptic operator on a Lie group* G, Bull. Amer. Math. Soc., **81** (1975), pp. 121-124.

[HJ] A. HULANICKI - J. W. JENKINS, *Almost everywhere summability on nilmanifolds*, Trans. Amer. Math. Soc., **278** (1983), pp. 703-715.

[M1] G. MAUCERI, *Zonal multipliers on the Heisenberg group*, Pacific J. of Math., **95** (1981), pp. 143-159.

[M2] G. MAUCERI, *Maximal operators and Riesz means on stratified groups*, to appear in Symp. Math.

[Mi] A. MIYACHI, *On some Fourier multipliers for* $H^p(\boldsymbol{R}^n)$, J. Fac. Sci. Univ. Tokyo, **27** (1980), pp. 157-179.

[TW] M. H. TAIBLESON - G. WEISS, *The molecular characterization of certain Hardy spaces*, Astérique, **77** (1980), pp. 67-149.