Critical Points with Lack of Compactness and Singular Dynamical Systems (*) (**).

ANTONIO AMBROSETTI - VITTORIO COTI ZELATI (***)

Sunto. – *Si prova l'esistenza di punti critici di funzionali che non verificano la condizione* (PS). *I teoremi astratti vengono applicati per trovare soluzioni periodiche di sistemi dinamici con potenziali sia limitati sia con singolarità.*

0. – Introduction.

This paper has two main purposes: first, to prove the existence of T-periodic solutions of n-dimensional dynamical systems like

$$-\ddot{y} = \mathrm{grad}_y V(t, y), \tag{0.1}$$

with potential V which (is T-periodic in t and) has singularities; second, to study the critical points of functionals whose Euler equations are (0.1). More precisely, the kind of functionals we are interested in are that ones of the form

$$f(u) = \tfrac{1}{2}(Au, u) + g(u), \tag{0.2}$$

where A is a linear selfadjoint operator acting on a Hilbert space E, $(\cdot,\cdot)$ denotes the scalar product in E and g is a nonlinear C^1 map.

The main specific features of f are:

(*a*) A has a kernel X with $n \equiv \dim X < \infty$ and, roughly, both $g(x)$ and $\mathrm{grad}\, g(x) \to 0$ as $x \in X$ and $\|x\|_E \to \infty$;

(*b*) g (and hence f) are possibly defined on an open subset Λ of E.

(*) Entrata in Redazione il 9 novembre 1986.

(**) This work has been done during a visit at the Math. Dept. of the University of Chicago. The Authors whish to thank for the kind hospitality.

(***) Supported by Ministero Pubblica Istruzione, 40%, Gruppo Nazionale « Calcolo delle Variazioni ».

Indirizzo degli A.A.: A. AMBROSETTI: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56100 Pisa; V. COLI ZELATI: International School for Advanced Studies, Strada Costiera 11, 34014 Miramare, Trieste.

As a consequence of (a), f does not satisfy the compactness condition introduced by Palais and Smale (see § 1). Actually here we show that $c = 0$ is the only level where PS fails to hold.

The idea to overcome such a lack of compactness is the following: under some mild assumptions involving, roughly, the behaviour of $g|_X$ at infinity, we are able to prove that the level set $\{f \leqslant \varepsilon\}$ is topologically equivalent to an n-dimensional sphere. This enables us to find critical points for f on Λ, via Morse type arguments. If Λ is a proper subset of E, a control on the behaviour of f on $\partial\Lambda$ is needed; on the othe hand, our approach permits to take advantage of the possibly rich topological structure of Λ: for example, if Λ has infinitely many non trivial homology groups, then we show that f has infinitely many critical points on Λ.

The abstract setting is contained in Part I, consisting of sections 1, 2 and 3: in § 1 we list the preliminaries, while § 2 and § 3 contain the theorems on the existence of critical points of f under two different kinds of assumptions at infinity for $g|_X$.

The results of Part I are applied in Part II to find periodic solutions of (0.1). More precisely, we assume $V\colon \boldsymbol{R}\times\Omega \to \boldsymbol{R}$ (Ω open subset of $\boldsymbol{R}^n$) is T-periodic in t and is such that:

$$V(t, y) \to 0 \quad \text{and} \quad \operatorname{grad}_y V(t, y) \to 0 \quad \text{as } |y| \to \infty\,, \tag{0.3}$$

and look for T-periodic solutions of (0.1) as critical points of

$$f(u) = \tfrac{1}{2}\int_0^T |u'|^2\, dt - \int_0^T V(t, u)\, dt \tag{0.4}$$

on $\Lambda = \{u \in H^{1,2}(S^1, \boldsymbol{R}^n)\colon u(t) \in \Omega\}$. The functional f in (0.4) is of the form (0.2) and exhibits (a) and (b), so that the abstract results apply. After dealing in § 5 with bounded potentials (i.e. $\Omega = \boldsymbol{R}^n$ and $\Lambda = H^{1,2}$), we consider in § 6, 7 the situa-we are mainly interested in: when Ω has a compact boundary $\partial\Omega$ and $V(t, y) \to -\infty$ as $y \to \partial\Omega$. In such a case Λ has a richer topological structure and the stronger critical points theorems of Part I can be employed provided a further condition on the behaviour of V near $\partial\Omega$ is assumed; namely that $\operatorname{grad}_y V(t, y)$ is a « strong force » in the sense of GORDON [16]. The kind of results we can prove are illustrated by the following example: if $\Omega = \boldsymbol{R}^n\setminus\{0\}$ $(n \geqslant 2)$, and $V(t, y)$ behaves like $-|x|^{-\alpha}$ with $\alpha \geqslant 2$ near $x = 0$ and like $\pm|x|^{-\beta}$ with $\beta > 0$ as $|x| \to \infty$, then (0.1) has infinitely many T-periodic solutions.

According to the Abstract Setting, we point out that in the applications to (0.1) only asymptotic conditions on V are required.

In the last section (§ 7) we shortly discuss extensions to cover autonomous systems, i.e. to the case when V is does not depend on t.

Papers somewhat related to ours are [6; 9; 10; 12; 13; 16; 17; 18; ...]. We refer to Remarks 3.6, 5.5 and 6.8 for comparison with those papers. Here we would like to spend few words to indicate the differences with [12] and [16].

The idea to overcome the lack of PS evaluating directly the topology of $\{f\leqslant\varepsilon\}$ has been first used in [12], even if in a particular case. Actually, Part I here furnishes a general abstract tool which can be used to study the specific problem of [12].

In [16], the definition of « strong force » has been first introduced. The main difference with the present paper is that Gordon assumes $\partial\Omega$ is complicated enough, in such a way that the corresponding Λ splits in components where the functional f is coercive. In particular no lack of PS arises in [16]. For example, in the case listed before (i.e. $\Omega = \boldsymbol{R}^n\setminus\{0\}$), Gordon's result applies only if $n = 2$ (see also [9, 10, 18]).

Some results and the main ideas of this paper have been presented at the meeting « Recent developements in Hamiltonian systems », held at L'Aquila, Italy, June 1986 [2]. At that meeting we learned that GRECO [19] had meantime proved the existence of one periodic solution for (0.1) with singular potentials of the type we study in Theorems 6.3 and 7.1.

PART I: ABSTRACT SETTING

1. – Notations and preliminaries.

In this section we will state some basic tools and results in critical point and Morse theory. Such results are essentially known even if not in the specific way we shall need in the following.

Let E be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We will deal in the following with functionals f wich are possibly defined on an open subset Λ of E. We denote by $\partial\Lambda$ the (possibly empty) boundary of Λ. We also set $B_R = \{u \in E\colon \|u\| \leqslant R\}$.

If $f \in C^1(\Lambda; \boldsymbol{R})$, we set $f'(u) = \operatorname{grad} f(u)$, $Z(f) = \{u \in \Lambda\colon f'(u) = 0\}$, and, for $c \in \boldsymbol{R}$, $Z_c(f) = \{u \in Z(f)\colon f(u) = c\}$, $\{f < c\} = \{u \in \Lambda\colon f(u) < c\}$, $\{f\leqslant c\} = \{u \in \Lambda\colon f(u) \leqslant c\}$, $\{a \leqslant f \leqslant b\} = \{u \in \Lambda\colon a \leqslant f(u) \leqslant b\}$, etc. We will also abbreviate $f^\varepsilon = \{f \leqslant \varepsilon\}$.

If S is a subset of Λ, we say that f *satisfies the* PS (*Palais-Smale*) *condition on* S if for every sequence $\{u_h\}$ in S such that $f(u_h)$ is bounded and $f'(u_h) \to 0$ there exists a converging subsequence $u_{h_i} \to u \in \Lambda$.

We will often refer to the *steepest descent flow* associated to a functional f. This is, essentially, the flow defineted by the differential equation $x' = -f'(x)$. We will not enter here in the details of the construction of such a flow; we only recall its properties which we will need in the paper.

PROPOSITION 1.1. – Let Λ be an open subset of a Hilbert space E, and let $f \in C^1(\Lambda; \boldsymbol{R})$ be bounded from below on Λ and such that $f(u) \to +\infty$ as $u \to \partial\Lambda$.

Then there exists $\eta \in C(\boldsymbol{R}^+ \times \Lambda; \Lambda)$ (the steepest descent flow) such that the function $f(\eta(\cdot, u))$ is not increasing.

PROOF. – See [21]. $\therefore$

We will say that a subset Y of a Hilbert space E is *positively invariant* under the steepest descent flow or, simply, positively invariant if $\eta(t, y) \in Y$, $\forall y \in Y$, $\forall t \geqslant 0$.

PROPOSITION 1.2. – Let Λ be an open subset of a Hilbert space E, and let $f \in C^1(\Lambda; \boldsymbol{R})$ be bounded from below on Λ and such that $f(u) \to +\infty$ as $u \to \partial\Lambda$. Let Y be a closed subset of Λ such that Y is positively invariant and PS holds for f in Y. Then it exists $u \in Y$ such that $f(u) = \min\{f(u): u \in Y\}$.

PROOF. – The proof is the usual one (see [21], [20]). We only point out that the deformation arguments can be carried over in our setting because: (i) Y is positively invariant; (ii) since Y is closed and PS holds in Y, then $\forall a, b \in \boldsymbol{R}$, with $Z(f) \cap \cap \{a \leqslant f \leqslant b\} \cap Y = \emptyset$, $\exists \delta > 0$ such that $\|f'(u)\| \geqslant \delta$, $\forall u \in \{a \leqslant f \leqslant b\} \cap Y$. $\therefore$

Essentially using the same arguments, one can prove:

PROPOSITION 1.3. – Let Λ be an open subset of a Hilbert space E, and let $f \in C^1(\Lambda; \boldsymbol{R})$ be such that $f(u) \to +\infty$ as $u \to \partial\Lambda$. If PS holds in $\{a \leqslant f \leqslant b\}$ (where $-\infty < a \leqslant b \leqslant +\infty$), and $Z(f) \cap \{a \leqslant f \leqslant b\} = \emptyset$, then f^a is a deformation retract of f^b. Moreover, if Y is a closed and positively invariant subset of Λ such that PS holds in $Y \cap \{a \leqslant f \leqslant b\}$ and $Z(f) \cap \{a \leqslant f \leqslant b\} \cap Y = \emptyset$, then $f^a \cap Y$ is a deformation retract of $f^b \cap Y$.

PROOF. – See [20, Lemma 3.3. - *a*), *b*)] and the proof of Proposition 1.2. $\therefore$

With $H_*(A)$, $(H_*(A, B), A \supset B)$ we will indicate the Singular Homology groups of the topological space A (of the couple of topological spaces (A, B)). From proposition 1.3. it follows that, if PS holds in $\{a \leqslant f \leqslant b\}$ (where $-\infty < a \leqslant b \leqslant +\infty$), and $Z(f) \cap \{a \leqslant f \leqslant b\} = \emptyset$, then

$$H_*(f^a) = H_*(f^b) . \tag{1.1}$$

Moreover:

PROPOSITION 1.4. – Let $f \in C^2(\Lambda; \boldsymbol{R})$ be such that $f(u) \to +\infty$ as $u \to \partial\Lambda$ and let Y be a closed, positively invariant subset of Λ. Suppose: *a*) f satisfies PS in $\{a \leqslant f \leqslant b\} \cap Y$, with $-\infty < a < b \leqslant +\infty$; *b*) $Z(f) \cap \{a \leqslant f \leqslant b\} \cap Y$ is compact (if $b < +\infty$ this follows from *a*)); *c*) $Z(f) \cap \partial(\{a \leqslant f \leqslant b\} \cap Y) = \emptyset$; *d*) f is Fredholm of index 0 in $Z(f) \cap \{a \leqslant f \leqslant b\} \cap Y$. Then $\exists q' \in \boldsymbol{N}$ such that

$$H_q(f^b \cap Y) \simeq H_q(f^a \cap Y)\,, \qquad \forall q \geqslant q'\,. \tag{1.2}$$

PROOF. – Since PS holds in $\{a \leqslant f \leqslant b\} \cap Y$, and $Z(f) \cap \{a \leqslant f \leqslant b\} \cap Y$ is compact we can use [20, thm. 2.2] to deduce the existence of g for which: (i) a), b) and c) hold; (ii) g is close to f in the C^1 norm and differs from f only in a small neighborhood of the critical points; (iii) g has only a finite number of nondegenerate critical points in $\{a \leqslant g \leqslant b\} \cap Y$. From (ii) it follows that Y is positively invariant for g, too. From this fact, and well known results of Morse theory (see, for example [7, thm. B]) it follows (taking into account also the proof of Proposition 1.2) that (1.2) holds for g. Since from (ii) one deduces that $f^b \cap Y = g^b \cap Y$ and $f^a \cap Y = g^a \cap Y$, the Proposition follows. $\therefore$

From this Proposition we deduce

PROPOSITION 1.5. – Let A be an open set of a Hilbert space E, and let $f \in C^2(E; \boldsymbol{R})$ be Fredholm of index 0 and such that $f(u) \to +\infty$ as $u \to \partial A$. Moreover suppose f satisfies PS in the set $\{f \geqslant \varepsilon\}$, $\forall \varepsilon > 0$.

(i) if $\exists \varepsilon^*$: $H_q(f^{\varepsilon^*}) \neq H_q(A)$, then $Z(f) \cap \{f \geqslant \varepsilon^*\} \neq \emptyset$;

(ii) if:

$$H_q(A) \neq 0 \qquad \text{for infinitely many } q \in \boldsymbol{N}\,, \tag{A}$$

while $\exists \varepsilon^* > 0$, $\exists q_1 \in \boldsymbol{N}$ such that $H_q(f^\varepsilon) = 0$, $\forall q \geqslant q_1$, $\forall \varepsilon \in \,]0, \varepsilon^*]$, then f has infinitely many critical points.

PROOF. – (i) Suppose $Z(f) \cap \{f \geqslant \varepsilon^*\} = \emptyset$. Then from (1.1) it follows $H_q(A) \equiv \equiv H_q(f^{\varepsilon^*})$, $\forall q \in \boldsymbol{N}$, a contradiction.

(ii) Suppose, by contradiction, that $Z(f)$ is finite and take $\varepsilon \in \,]0, \varepsilon^*]$ such that $Z_\varepsilon(f) = \emptyset$. From the assumptions and using Proposition 1.4 with $a = \varepsilon$ and $b = +\infty$ one finds $q_2 > 0$:

$$H_q(A) \simeq H_q(f^\varepsilon)\,, \qquad \forall q \geqslant q_2\,,$$

a contradiction. $\therefore$

REMARK 1.6. – Actually a stronger result can be obtained: suppose the assumptions of Proposition 1.5 - (ii) hold. Further, suppose $Z_\varepsilon(f) = \emptyset$ for some $\varepsilon \in \,]0, \varepsilon^*]$. Then f has infinitely many critical points u_h such that $f(u_h) \to +\infty$.

In fact if $\sup \{f(u)\colon u \in Z(f)\} \equiv \sigma < +\infty$, then, applying Proposition 1.4 with $b = \sigma + 1$ and $a = \varepsilon$, we reach a contradiction as before. $\therefore$

2. – Existence of critical points: a first case.

2.*a*. *The* PS *condition*.

We will deal here with functionals $f \in C^1(\Lambda; \boldsymbol{R})$ of the form

$$f(u) = \tfrac{1}{2}(Au, u) + g(u) ,$$

where $A\colon E \to E$ and $g\colon \Lambda \to \boldsymbol{R}$ satisfy the assumptions listed below. First of all:

*A*1. A is a linear bounded selfadjoint operator in E with finite dimensional kernel:

$$X = \operatorname{Ker} A , \qquad \dim X = n < \infty .$$

$\forall x \in X$, the norm $\|x\|$ will be simply denoted (according to the notation of Part II), by $|x|$. Denoted by W the orthogonal complement to X, one has

$$E = X \oplus W .$$

If P indicates the orthogonal projection onto X, we will set

$$x_u = Pu , \qquad w_u = u - x_u .$$

If no confusion arises, the subscript u will be omitted. We will also suppose:

*A*2. $$\exists \alpha > 0\colon (Aw, w) \geqslant \alpha \|w\|^2 , \qquad \forall w \in W .$$

On g we will assume:

*g*1. $g \in C^1(\Lambda; \boldsymbol{R})$ and $\exists m \geqslant 0\colon g(u) \geqslant -m$, $\forall u \in \Lambda$;

*g*2. $u_n \in \Lambda$, u_n converges weakly to $u \in \partial\Lambda$ implies $g(u_n) \to +\infty$;

*g*3. let $c^* = (2/\alpha)(m+1)$. Corresponding to c^* there exists $r^* > 0$ and $g_1, g_2 \in C(\boldsymbol{R}^n, \boldsymbol{R})$ such that:

(2.1) $$g_i(x) > 0 , \qquad \forall x \in \boldsymbol{R}^n,\ i = 1, 2,\ |x| \geqslant r^* ;$$

(2.2) $$g_i(x) \to 0 \qquad \text{as } |x| \to \infty,\ i = 1, 2 ;$$

and

(2.3) $$g_1(Pu) \leqslant g(u) \leqslant g_2(Pu) , \qquad \forall u \in \Lambda \quad \text{with } |Pu| \geqslant r^* \text{ and } \|w_u\| \leqslant c^* .$$

REMARK 2.1. – We esplicitely note that from (*g*1) and (*g*3) it follows that $\{u = Pu + w_u \in E\colon |Pu| > r^*,\ \|w_u\| \leqslant c^*\} \cap \partial\Lambda = \emptyset$. $\therefore$

First of all we derive from the preceding assumptions some consequences.

LEMMA 2.2. – (i) If $u \in f^{\varepsilon}$, $0 < \varepsilon \leqslant 1$, then $\|w_u\| \leqslant c^*$. (ii) There exists $\varepsilon^* > 0$ such that if $u \in f^{\varepsilon}$, $0 < \varepsilon \leqslant \varepsilon^*$, then $|Pu| \neq r^*$.

PROOF. – (i) Let $u \in f^{\varepsilon}$, with $0 < \varepsilon \leqslant 1$. Using $(A2)$ and $(g1)$ it follows:

$$\tfrac{1}{2}\alpha\|w\|^2 - m \leqslant \tfrac{1}{2}(Aw, w) + g(u) = f(u) \leqslant \varepsilon .$$

Hence

$$\|w\|^2 \leqslant (2/\alpha)(m + \varepsilon) \leqslant c^* .$$

(ii) By contradiction, let $u_n \in \Lambda$ be such that $f(u_n) \leqslant 1/n$ and $|Pu_n| = r^*$. Letting $w_n = w_{u_n}$, from (the preceding) (i) one has $\|w_n\| \leqslant c^*$. Then (2.3) of $(g3)$ implies:

$$g(u_n) \geqslant g_1(Pu_n) .$$

Therefore it follows:

$$f(u_n) = \tfrac{1}{2}(Aw_n, w_n) + g(u_n) \geqslant g(u_n) \geqslant g_1(Pu_n) . \tag{2.4}$$

Since $|Pu_n| = r^*$, then $g_1(Pu_n) \geqslant \delta \equiv \min\{g_1(x) : |x| = r^*\}$. By (2.1) $\delta > 0$ and hence we deduce from (2.4):

$$f(u_n) \geqslant \delta > 0 ,$$

a contradiction which proves (ii). ∴

COROLLARY 2.3. – For all $\varepsilon \leqslant \varepsilon^*$ one has

$$f^{\varepsilon} = \Gamma_1^{\varepsilon} \cup \Gamma_2^{\varepsilon}$$

where

$$\Gamma_1^{\varepsilon} = \{u \in f^{\varepsilon} : |Pu_n| < r^*\} ,$$

$$\Gamma_2^{\varepsilon} = \{u \in f^{\varepsilon} : |Pu_n| > r^*\} .$$

In particular $\bar{\Gamma}_1^{\varepsilon} \cap \bar{\Gamma}_2^{\varepsilon} = \emptyset$. Moreover the sets Γ_1^{ε}, Γ_2^{ε}, are positively invariant under the steepest descent flow of f.

PROOF. – The first statement is a direct consequence of Lemma 2.2. The positive invariance of Γ_i^{ε} $(i = 1, 2)$ follows from the fact that $\eta(t, u) \in f^{\varepsilon}$, $\forall t \geqslant 0$, $\forall u \in f^{\varepsilon}$ and by continuity, from (ii) of Lemma 2.2. ∴

REMARKS 2.4. – (i) Γ_1^ε could possibly be empty, while $\Gamma_2^\varepsilon \neq \emptyset$ for $\varepsilon > 0$. In fact the set $\{x \in X : |x| > r\}$ is contained in Γ_2^ε provided $r > 0$ is large enough, because $g(x) \to 0$ as $x \in X$, $|x| \to \infty$ (immediate consequence of (2.2) and (2.3)).

(ii) $f(u) > 0$, $\forall u \in \Gamma_2^\varepsilon$. This follows from ($g3$) and Lemma 2.2 - (ii). $\therefore$

LEMMA 2.5. – Let $u_n \in \Lambda$ be such that $f(u_n) \leqslant 1/n$ and $|Pu_n| > r^*$. Then, setting $w_n = w_{u_n}$, one has

(i) $\|w_n\| \to 0$;

(ii) $|Pu_n| \to \infty$.

PROOF. – From Lemma 2.2 - (i) one has $\|w_n\| \leqslant c^*$. Since $|Pu_n| > r^*$ by assumption, we can use ($g3$) to infer that $g(u_n) \geqslant g_1(Pu_n) > 0$. Hence, using also ($A2$),

$$\tfrac{1}{2}\alpha\|w_n\|^2 \leqslant \tfrac{1}{2}(Aw_n, w_n) + g(u_n) = f(u_n) \leqslant 1/n$$

and (i) follows.

As for (ii), we remark that (2.4) still holds here. If $|Pu_n| \leqslant$ const., it would follow that $Pu_n \to x \in X$, for a subsequence, and $g_1(Pu_n) \to g_1(x)$. Since $g_1(x) > 0$ by (2.1), (2.4) implies $f(u_n) \geqslant \delta > 0$ for n large, a contradiction. $\therefore$

We are now in position to investigate the PS condition. An assumption on g' is in order:

$g4$. (i) $u_n \in \Lambda$, u_n converges weakly to $u \in \Lambda$ implies $g'(u_n) \to g'(u)$ and

(ii) $g'(u_n) \to 0$ for all $u_n = w_n + x_n$ such that $\|w_n\| \leqslant$ const. and $|x_n| \to \infty$.

Let us point out that ($g4$) implies that

$$f' \text{ is Fredholm of index } 0\,. \tag{2.5}$$

In fact $f'(u) = Au + g'(u) = Au + Pu - Pu + g'(u)$ and $Au + Pu$ is a linear homeomorphism, while $g'(u) - Pu$ is compact.

Under the above assumptions, the PS condition fails to hold at the level $c = 0$. In fact, if $x_n \in X$ and $|x_n| \to \infty$ then ($x_n \in \Lambda$, see Remark 2.1, and) $f(x_n) = g(x_n) \to 0$, while, as a consequence of ($g4$), one has $g'(x_n) \to 0$. The following Lemma says that the preceding one is essentially the only situation in which PS fails to hold.

LEMMA 2.6. – (i) For all $\varepsilon > 0$ PS holds in the set $\{f \geqslant \varepsilon\}$; (ii) PS holds in every set where $|Pu| \leqslant$ const.

PROOF. – (i) Let $w_n \in \Lambda$ be such that

$$\varepsilon \leqslant f(u_n) = \tfrac{1}{2}(Aw_n, w_n) + g(u_n) \leqslant c_1 \qquad (w_n = w_{u_n}) \tag{2.6}$$

$$f'(u_n) = Aw_n + g'(u_n) \to 0 \,. \tag{2.7}$$

From the righ-hand side of (2.6) and ($g1$) we deduce that $\|w_n\| \leqslant c_2$ and hence w_n converges weakly, up to a subsequence, to w. We claim that $|Pu_n| \leqslant c_3$. In fact, otherwise, from ($g4$) we have that

$$g'(u_n) \to 0 \,. \tag{2.8}$$

Multiplying (2.7) by w_n, it follows:

$$(Aw_n, w_n) = \big(f'(u_n), w_n\big) - \big(g'(u_n), w_n\big)$$

and hence

$$\alpha\|w_n\|^2 \leqslant \|f'(u_n)\|\,\|w_n\| + \|g'(u_n)\|\,\|w_n\| \,.$$

Since both $f'(u_n)$ and $g'(u_n)$ tend to 0 (see (2.7) and (2.8)), then $w_n \to 0$. We can now use the right-hand of (2.3) to find

$$f(u_n) \leqslant \tfrac{1}{2}(Aw_n, w_n) + g_2(Pu_n) \,. \tag{2.9}$$

Since $w_n \to 0$ and $|Pu_n| \to \infty$, we have from (2.2)

$$\tfrac{1}{2}(Aw_n, w_n) + g_2(Pu_n) \to 0 \,.$$

From (2.9) it finally follows that $f(u_n) \to 0$, in contradiction with the left hand side of (2.6), so that the claim is proved.

Now, if $|Pu_n| \leqslant c_3$, one has that $Pu_n \to \xi$ (up to a subsequence) and then $u_n = w_n + Pu_n$ converges weakly to $w + \xi = u$. Since $g(u_n) \leqslant c_1$, then by ($g2$) we infer that $u \in \Lambda$. Then from ($g4$) it follows that $g'(u_n) \to g'(u)$, and

$$Aw_n = f'(u_n) - g'(u_n) \to g'(u) \,.$$

Then $w_n \to w$ and $u_n \to u$. This completes the proof of (i).

(ii) As in the proof of (i), $f(u_n) \leqslant c_1$ implies $\|w_n\| \leqslant c_2$. If, in addition, $|Pu_n| \leqslant$ const, the PS follows as above. ∴

2.b. The topology of f^ε.

We will always assume ($A1, 2$) and ($g1, 2, 3, 4$). We recall that, by Corollary 2.3 one has

$$f^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \,, \qquad \forall \varepsilon \leqslant \varepsilon^*$$

with $\Gamma_1^\varepsilon \cap \Gamma_2^\varepsilon = \emptyset$. The purpose of this subsection is to study the topology of Γ_2^ε. Let

$$\Pi(t, u) \equiv tw_u + Pu .$$

LEMMA 2.7. For all $0 < \varepsilon \leqslant \varepsilon^*$, there exists $\varepsilon' \leqslant \varepsilon$ such that

$$\Pi(t, u) \in \Gamma_2^\varepsilon , \qquad \forall t \in [0, 1], \ \forall u \in \Gamma_2^{\varepsilon'} .$$

PROOF. – First we remark that $|P(\Pi(t, u))| = |Pu| > r^*$, $\forall t \in [0, 1]$, $\forall u \in \Gamma_2^{\varepsilon'}$. Then, arguing by contradiction, we let $t_n \in [0, 1]$, $u_n \in \Gamma_2^{1/n}$ be such that

$$(2.10) \qquad 0 < \varepsilon < f(\Pi(t_n, w_n + x_n)) = \tfrac{1}{2} t_n^2 (Aw_n, w_n) + g(t_n w_n + x_n)$$

with $x_n = Pu_n$ and $w_n = u_n - x_n$. From the definition of $\Gamma_2^{1/n}$ we have

$$f(u_n) \leqslant 1/n \quad \text{and} \quad |x_n| > r^* .$$

Using Lemma 2.5 we get

$$(2.11) \qquad w_n \to 0 \quad \text{and} \quad |x_n| \to +\infty .$$

Since $t_n \in [0, 1]$ and $P(t_n w_n + x_n) = x_n$, then (2.11) permits to use ($g3$) to estimate

$$g(t_n w_n + x_n) \leqslant g_2(x_n) .$$

Since from (2.11) it follows that $(Aw_n, w_n) \to 0$ as well as $g_2(x_n) \to 0$, this gives a contradiction with (2.10). ∴

Set $S^{n-1} = \{x \in X : |x| = 1\}$. We can now state:

LEMMA 2.8. – If $Z(f) \cap \Gamma_2^\varepsilon = \emptyset$, $0 < \varepsilon \leqslant \varepsilon^*$, then S^{n-1} is a deformation retract of Γ_2^ε.

PROOF. – Since $Z(f) \cap \Gamma_2^\varepsilon = \emptyset$ and by the positive invariance of Γ_2^ε under the steepest descent flow η (see Corollary 2.3), one has (see § 1, Proposition 1.3) that, $\forall \varepsilon' \leqslant \varepsilon$, $\Gamma_2^{\varepsilon'}$ is a deformation retract of Γ_2^ε. From the fact that $g(Pu) \to 0$ as $|Pu| \to +\infty$ we infer that $\exists r' > 0$ such that $X - B_{r'}$ is a subset of Γ_2^ε. Let $\varepsilon' \leqslant \varepsilon$ be such that Lemma 2.7 holds. By Lemma 2.5 - (ii) it is possible to take $\varepsilon' \leqslant \varepsilon$ in such a way that (Lemma 2.7 continues to hold and)

$$|Pu| > r' , \qquad \forall u \in \Gamma_2^{\varepsilon'} .$$

Lastly, fix $\varrho \geqslant r'$ in such a way that $\Gamma_2^{\varepsilon'} \supset \partial B_\varrho \cap X$ and let θ be the radial projection

$$\theta(t, x) \equiv t\varrho \frac{x}{|x|} + (1 - t)x , \qquad x \in X, \ x \neq 0, \ t \in [0, 1] .$$

By the preceding remarks it is readily verified that $\theta(t, Pu) \in \Gamma_2^\varepsilon$, $\forall t \in [0, 1]$, $\forall u \in \Gamma_2^{\varepsilon'}$. Hence $\partial B_\varrho \cap X$ turns out to be a deformation retract of Γ_2^ε through the homotopy obtained combining η, Π and θ, and the lemma follows. $\therefore$

From the preceding lemma we infer:

COROLLARY 2.9. – Under the hypothesis of Lemma 2.8 one has

$$H_*(\Gamma_2^\varepsilon) = H_*(S^{n-1}). \quad \therefore \tag{2.12}$$

2.c. *Existence results.*

Our first result is an immediate consequence of Lemma 2.8 and will be applied when, essentially, $\Lambda = E$.

THEOREM 2.10. – Suppose ($\Lambda 1, 2$), ($g1, 2, 3, 4$) and

$$H_*(\Lambda) \neq H_*(S^{n-1}) . \tag{2.13}$$

Then f has at least a critical point in Λ.

PROOF. – Since PS holds in $\{f \geqslant \varepsilon\}$, $\forall \varepsilon > 0$ (Lemma 2.6 - (i)), then, if $Z(f) = \emptyset$, we can use the steepest descent flow to obtain (see § 1, Proposition 1.3)

$$H_*(\Lambda) = H_*(f^\varepsilon) , \qquad \forall \varepsilon > 0 .$$

Using Corollary 2.3 we have (see [15, Proposition 4.12])

$$H_*(\Lambda) = H_*(\Gamma_1^\varepsilon) \oplus H_*(\Gamma_2^\varepsilon) , \qquad \forall 0 < \varepsilon \leqslant \varepsilon^* . \tag{2.14}$$

If $\Gamma_1^\varepsilon = \emptyset$, we can use Corollary 2.9 to find

$$H_*(\Lambda) = H_*(\Gamma_2^\varepsilon) = H_*(S^{n-1}) ,$$

which contradicts (2.13).

If $\Gamma_1^\varepsilon \neq \emptyset$, a critical point of f on Λ will be found as the minimum of f on Γ_1^ε. Such a minimum exists because: (*a*) f is bounded from below on Λ (hence on Γ_1^ε), since, as a consequence of ($g1$):

$$f(u) = \tfrac{1}{2}(Au, u) + g(u) \geqslant -m , \qquad \forall u \in \Lambda;$$

(*b*) PS holds on Γ_1^ε (see Lemma 2.6 - (ii)); (*c*) Γ_1^ε is positively invariant under the steepest descent flow η (see Corollary 2.3). Then Proposition 1.2 applies. $\therefore$

COROLLARY 2.11. – Suppose $(A1, 2)$, $(g1, 3, 4)$ and let $\Lambda = E$. Then $Z(f) \neq \emptyset$. If, in addition, $\Gamma_1^\varepsilon \neq \emptyset$ for some $\varepsilon \leqslant \varepsilon^*$, then $\# Z(f) \geqslant 2$.

PROOF. – $Z(f) \neq \emptyset$ since for $\Lambda = E$ (2.13) holds and Theorem 2.10 applies. Let $\Gamma_1^\varepsilon \neq \emptyset$ for some $\varepsilon \in]0, \varepsilon^*]$. According to the proof of Theorem 2.10, f has a minimum $u_1 \in \Gamma_1^\varepsilon$. A second critical point can be found using a mountain pass type argument taking paths connecting the minimum and a point in Γ_2^ε. Every such a path has to cross the surface $|Pu| = r^*$ where f takes values $\geqslant \varepsilon^*$, see Lemma 2.2 - (ii). Then the mountain pass level $c \geqslant \varepsilon^*$ is a critical value since PS holds in $\{f \geqslant \varepsilon\}$, $\forall \varepsilon > 0$. We leave the details to the reader. $\therefore$

REMARK 2.12. – In the case in which Corollary 2.11 applies one can be slightly more precise. Namely, if $f \in C^2(E, \boldsymbol{R})$ and is a Morse functional one can find at least 3 critical points provided $n \geqslant 2$. In fact the mountain pass critical point has Morse index $= 1$ [1] and to find a third critical point it suffices to argue by contradiction, using the Morse inequalities, as in Theorem 2.10. We do not carry over the details. $\therefore$

Our main existence theorem of this section is:

THEOREM 2.13. – Suppose $(A1, 2)$, $(g1, 2, 3, 4)$ and (Λ) hold. Suppose, also, that $g \in C^2(\Lambda; \boldsymbol{R})$. Then f has infinitely many critical points on Λ.

PROOF. – First of all, Corollary 2.3 yields

$$H_q(f^\varepsilon) = H_q(\Gamma_1^\varepsilon) \oplus H_q(\Gamma_2^\varepsilon)\,, \quad \forall \varepsilon \in]0, \varepsilon^*],\ \forall q \in \boldsymbol{N}\,. \tag{2.15}$$

Next suppose, by contradiction, that $Z(f)$ is finite. Then $\exists \varepsilon \in]0, \varepsilon^*]$ such that

$$Z(f) \cap \Gamma_2^\varepsilon = \emptyset \tag{2.16}$$

$$Z_\varepsilon(f) \cap \Gamma_1^\varepsilon = \emptyset \tag{2.17}$$

(2.16) allows us to use (2.11) to find

$$H_q(\Gamma_2^\varepsilon) = \{0\}\,, \quad \forall q \neq 0, n-1\,. \tag{2.18}$$

Next let $Y = \Gamma_1^\varepsilon$, $b = \varepsilon$ and $a < -m$. From Lemma 2.6 - (ii) and (2.5) assumptions $a)$, $b)$ and $d)$ of Proposition 1.4 hold. Moreover $\partial\Gamma_1^\varepsilon = \{f = \varepsilon\} \cap \Gamma_1^\varepsilon$ and (2.17) yield $c)$. Then from Proposition 1.4 we deduce the existence of $q_2 \in \boldsymbol{N}$ such that

$$H_q(\Gamma_1^\varepsilon) = \{0\}\,, \quad \forall q \geqslant q_2\,. \tag{2.19}$$

Thus (2.15), (2.18) and (2.19) imply:

$$H_q(f^\varepsilon) = \{0\}\,, \quad \forall q \geqslant \max(q_2, n)\,.$$

We can now apply Proposition 1.5 - (ii) and find a contradiction. $\therefore$

According to remark 1.6 we can prove here a stronger result.

THEOREM 2.14. – Suppose ($A1$, 2), ($g1$, 2, 3, 4) and (Λ) hold. Suppose, also, that $g \in C^2(\Lambda; \boldsymbol{R})$ and that

$$(2.20) \quad \exists R, \delta > 0 \quad \text{such that} \quad (g'(u), Pu) < 0, \quad \forall u \text{ such that } \|w_u\| \leqslant \delta, \ |Pu| \geqslant R .$$

Then f has infinitely many critical points $u_k \in \Lambda$ such that $f(u_k) \to +\infty$.

PROOF. – We first remark that, under the assumption (2.20), $Z(f) \cap \Gamma_2^{\varepsilon^*}$ is compact. In fact let $u_n \in Z(f) \cap \Gamma_2^{\varepsilon^*}$. We know that $f(u_n) > 0$, $\forall n$. Actually we claim that $f(u_n) \geqslant \mu > 0$, $\forall n$. Otherwise, up to a subsequence, $f(u_n) \to 0$, and we deduce from Lemma 2.5 that $\exists N > 0$ such that $\forall n \geqslant N$, $|Pu_n| > R$ and $\|w_n\| \leqslant R$. For such n's we therefore have

$$(g'(u_n), Pu_n) < 0 , \quad \forall n \geqslant N .$$

But $u_n \in Z(f)$ implies

$$\begin{aligned} 0 &= (f'(u_n), Pu_n) \\ &= (Aw_n, Pu_n) + (g'(u_n), Pu_n) \\ &= (g'(u_n), Pu_n) < 0 , \end{aligned}$$

contradiction which proves the claim. Now the precompactness of $\{u_n\}$ follows from the PS. Set $\varepsilon = \inf\{f(u): u \in Z(f) \cap \Gamma_2^{\varepsilon^*}\} > 0$, and suppose, by contradiction, that there exists $b < +\infty$ such that $Z(f)$ is contained in f^b. Then $Z(f)$ is compact and, as in proposition 1.4 we can find $g \in C^2(\Lambda; \boldsymbol{R})$ which has only finitely many nondegenerate critical points in $g^{b+1} = f^{b+1}$. Reasoning as in Proposition 1.4 we deduce the existence of $q_1, q_2 \in \boldsymbol{N}$ such that

$$\begin{aligned} H_q(\Lambda) &\simeq H_q(f^{b+1}) , \quad && \forall q \in \boldsymbol{N} \\ &\simeq H_q(g^{b+1}) , && \forall q \in \boldsymbol{N} \\ &\simeq H_q(g^{\varepsilon/2}) , && \forall q \geqslant q_1 . \end{aligned}$$

It is easy to see that Corollary 2.3 and Lemma 2.6 holds for g as well. Hence

$$\begin{aligned} H_q(g^{\varepsilon/2}) &\simeq H_q(\{u \in \Lambda: g(u) \leqslant \varepsilon/2, \ |Pu| \leqslant r^*\}) \oplus H_q(\Gamma_2^{\varepsilon/2}) , \quad && \forall q \geqslant q_1 \\ &\simeq H_q(\Gamma_2^{\varepsilon/2}) , && \forall q \geqslant q_2 \\ &\simeq H_q(S^{n-1}) , && \forall q \geqslant q_2 . \end{aligned}$$

This is a contradiction which proves the Theorem. ∴

REMARK 2.15. – Using Lusternik-Schnirelman category it is possible to prove a result similar to Theorem 2.14, namely:

Suppose ($A1, 2$), ($g1, 2, 3, 4$) hold and let $\mathrm{cat}_\Lambda \Lambda = +\infty$. Then f has infinitely many critical points in Λ. Moreover there is a sequence of critical points $\{u_n\}$ such that $f(u_n) \to +\infty$.

The proof is based on the fact that under such assumptions $\mathrm{cat}_\Lambda f^\varepsilon$ is finite since: it is finite in Γ_1^ε since Γ_1^ε is positively invariant and f is bounded from below and satisfies PS there, while Γ_2^ε is a subset of $\Sigma^* = \{u \in \Lambda: \|w_u\| \leqslant c^*, |Pu| > r^*\}$ and $\mathrm{cat}_{\Sigma^*} \Sigma^* = 2$ imply $\mathrm{cat}_\Lambda \Gamma_2^\varepsilon \leqslant \mathrm{cat}_\Lambda \Sigma^* \leqslant \mathrm{cat}_{\Sigma^*} \Sigma^* = 2$. We remark that $\mathrm{cat}_\Lambda \Lambda = +\infty$ implies (Λ). ∴

3. – Existence of critical points: a second case.

We deal here with a functional $f \in C^1(\Lambda, R)$ of the form

$$f(u) = \tfrac{1}{2}(Au, u) + g(u)$$

with A satisfying ($A1, 2$) and g satisfying ($g1, 2$), ($g4$) and

$g5$. $\forall c > 0$, $\exists r > 0$ such that for all $u \in \Lambda$ with $|Pu| > r$ and $\|w_u\| \leqslant c$ one has

$$g(u) < 0 \tag{3.1}$$

and

$$(g'(u), u) > 0\,. \tag{3.2}$$

As in § 2 we start investigating the PS condition.

LEMMA 3.1. – The PS condition holds in $\{f \geqslant \varepsilon\}$, $\forall \varepsilon > 0$, and in $f^{-\delta}$, $\forall \delta > 0$.

PROOF. – As in Lemma 2.6, if $u_n \in \Lambda$ satisfies (2.6) and (2.7) one finds $\|w_n\| \leqslant c_2$ (where $w_n = w_{u_n}$). If $|Pu_n| \to \infty$, one finds again that $w_n \to 0$. We can now use (3.1) of ($g5$) to get

$$f(u_n) = \tfrac{1}{2}(Au_n, u_n) + g(u_n) \leqslant \tfrac{1}{2}(Au_n, u_n)\,.$$

Hence $f(u_n) \to 0$, in contradiction with (2.6). Thus $|Pu_n| \leqslant \text{const.}$ and the conclusion follows as in Lemma 2.6. This proves that the PS holds in $\{f \geqslant \varepsilon\}$, $\forall \varepsilon > 0$. The same argument works for the second statement as well. ∴

As a consequence of (3.2) one has:

LEMMA 3.2. – For all $\varepsilon > 0$, $\exists R > 0$ such that

$$(f'(u), u) > 0 , \qquad \forall u \in f^{\varepsilon}, \ \|u\| \geqslant R .$$

In particular $f^{\varepsilon} \cap B_R$ is positively invariant under the steepest descent flow of f.

PROOF. – Arguing by contradiction, suppose there exist $\varepsilon' > 0$ and a sequence $u_n \in f^{\varepsilon'}$ such that $\|u_n\| \to +\infty$ and $(f'(u_n), u_n) \leqslant 0$. From $u_n \in f^{\varepsilon'}$ we infer readily that $\|w_n\| \leqslant$ $\leqslant$ const. Then $|Pu_n| \geqslant \|u_n\| - \|w_n\| \geqslant \|u_n\| - c$, hence $|Pu_n| \to +\infty$. Using (3.2) of ($g5$) one has

$$(f'(u_n), u_n) = \tfrac{1}{2}(Aw_n, w_n) + (g'(u_n), u_n) \geqslant (g'(u_n), u_n) > 0 ,$$

a contradiction. $\therefore$

We are in position to state:

THEOREM 3.3. – Suppose ($A1$, 2) and ($g1$, 2, 4, 5) hold and that $g \in C^2(\Lambda; \boldsymbol{R})$. Then:

(i) $Z(f) \neq \emptyset$;

(ii) if (Λ) holds, then $\exists u_k \in \Lambda$ such that $f'(u_k) = 0$ and $f(u_k) \to +\infty$.

PROOF. – (i) f is bounded from below on Λ and

$$\inf_{\Lambda} f \leqslant \inf_{X} f \leqslant \inf_{X} g < 0$$

because of (3.1). Since PS holds on $f^{-\delta}$, $\forall \delta > 0$ (Lemma 3.1), then f attains its minimum on some $\bar{u} \in \Lambda$:

$$f(\bar{u}) = \min \{f(u) \colon u \in \Lambda\} .$$

(ii) Fixed $\varepsilon > 0$, let us take $R > 0$ according to Lemma 3.2. Set

$$S(t, u) = \begin{cases} u & \text{if } \|u\| \leqslant R \\ (1 - t)u + tRu/\|u\| & \text{if } \|u\| \geqslant R . \end{cases}$$

We claim

LEMMA 3.4. – $f^{\varepsilon} \cap B_R$ is a deformation retract of f^{ε} through S.

PROOF. – By a direct calculation one has

$$\frac{d}{dt} f(S(t, u)) = \big(f'(S(t, u)), S(t, u)\big) \frac{(R/\|u\|) - 1}{1 + t[(R/\|u\|) - 1]}. \tag{3.3}$$

Since $\|S(t, u)\| \geqslant R$ whenever $\|u\| \geqslant R$, then, using Lemma 3.2, it follows that

$$\frac{d}{dt} f(S(t, u)) < 0 \qquad \text{whenever } \|u\| \geqslant R \text{ and } S(t, u) \in f^{\varepsilon}.$$

In particular $u \in f^{\varepsilon}$ implies that $(d/dt) f(S(t, u))|_{t=0} < 0$. Thus $S(t, u) \in f^{\varepsilon}$ for $t \in [0, \mu]$ for some $\mu > 0$. If the Lemma is not true, it would exists $\tau > 0$ such that $S(\tau, u) \in f^{\varepsilon}$ and $(d/dt) f(S(t, u))|_{t=\tau} = 0$. This is clearly a contradiction which proves the Lemma. ∴

PROOF OF THEOREM COMPLETED. – The proof is similar to that of Theorem 2.14. In fact also here we have that, $\forall b \leqslant +\infty$, $Z(f) \cap f^{b}$ is compact. This follows, essentially, from Lemma 3.2. More precisely, let $u_k \in Z(f) \cap f^{b}$. Take a subsequence u_k such that $f(u_k) \to c$. If $c \neq 0$, the precompactness follows from PS. If $f(u_k) \to 0$, from Lemma 3.2 we deduce that $\|u_k\| \leqslant R$ and it is easy to find a converging subsequence. The proof now follow by contradiction as in theorem 2.14, the only difference being that here $H_q(g^{\varepsilon/2}) \cong H_q(g^{\varepsilon/2} \cap B_R) \cong \{0\}$, $\forall q \geqslant q_2$ (g being, as before, a C^2 function, having only nondegenerate critical points, which coincide with f outside a neighborhood of the critical points). ∴

REMARK 3.5. – Also here, as in Remark 2.15, one can show that $\# Z(f) = +\infty$ provided $\operatorname{cat}_{\Lambda} \Lambda = +\infty$. ∴

REMARK 3.6. – Among papers dealing with lack of PS, [6] have studied functionals with strong resonance at infinity. They have used different methods based on linking arguments and obtained results which are different from ours in generality and form.

Different questions concerning the lack of PS are investigated in [4, 23, ...]. ∴

PART II: APPLICATIONS

4. – Applications: general framework.

We will apply the abstract results of Part I to find T-periodic solutions of n-dimensional second order systems. Precisely, let Ω be an open subset of $\mathbf{R}^n$, $n \geqslant 2$ (even if some of the results below will be true even when $n = 1$) and let $V: \mathbf{R} \times \Omega \to \mathbf{R}$

be such that

$$(V1) \qquad V \in C^1(\boldsymbol{R}\times\Omega)\,, \qquad V(t+T, y) = V(t, y)\,, \qquad \forall (t, y) \in \boldsymbol{R}\times\Omega\,.$$

Set $V'(t, y) = \nabla_y V(t, y)$. We look for T-periodic solutions of

$$-\ddot{y} = V'(t, y)\,. \tag{4.1}$$

Set $E = H^{1,2}(S^1, \boldsymbol{R}^n)$ where $S^1 = \boldsymbol{R}/[0, T]$, with scalar product

$$(u, v) = \int_0^T \langle \dot{u}, \dot{v}\rangle\, dt + \int_0^T \langle u, v\rangle\, dt$$

and norm

$$\|u\| = \int |\dot{u}|^2 + \int |u|^2 \ (^1)\,.$$

Here and below $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the Euclidean scalar product and norm in $\boldsymbol{R}^n$. Let $A: E \to E$ be defined by

$$(Au, v) = \int \langle \dot{u}, \dot{v}\rangle\,.$$

$(A1)$ trivially holds with $X = \operatorname{Ker} A = \boldsymbol{R}^n$. Remark that here

$$Pu = (1/T)\int u$$

and $W = \{u \in E: \int w = 0\}$. The Poincaré inequality implies that also $(A2)$ holds. Let

$$\Lambda = \{u \in E: u(t) \in \Omega, \forall t \in \boldsymbol{R}\}\,,$$

and define

$$g(u) = -\int V(t, u(t))\,.$$

The T-periodic solutions of (4.1) are the critical points of the functional

$$f(u) = \tfrac{1}{2}(Au, u) + g(u)\,.$$

Remark that f is C^1 (C^2) provided V is C^1 in y (resp. C^2 in y).

---

(1) From now on, we will write $\int$ for $\int_0^T dt$.

5. – Bounded potentials.

In this section we will take $\Omega = \mathbf{R}^n$ and V satisfying $(V1)$ and:

$(V2)$ $V(t, y) \to 0$ as $|y| \to +\infty$ uniformly in t and $\exists r_1 > 0$: $V(t, y) < 0$, $\forall |y| \geq r_1$:

$(V3)$ $V'(t, y) \to 0$ as $|y| \to +\infty$ uniformly in t.

We check the assumptions of theorem 2.10 and Corollary 2.11 by showing

LEMMA 5.1. – (i) If $(V2)$ holds, then $(g1)$ and $(g3)$ are true. (ii) If $(V3)$ holds, then $(g4)$ is true.

PROOF. – (i) $(g1)$ is trivial. As for $(g3)$, recall that $\forall w \in W$ one has

$$\|w\|_{L^\infty} \leq c_3 \|w\|_E .$$

Let r^* be such that $r^* - c_3 c^* > r_1$ and set

$$g_1(y) = (1/T) \min \left\{ -\int V(t, \xi) : t \in \mathbf{R},\ |y| - c_3 c^* \leq |\xi| \leq |y| + c_3 c^* \right\},$$

$$g_2(y) = (1/T) \max \left\{ -\int V(t, \xi) : t \in \mathbf{R},\ |y| - c_3 c^* \leq |\xi| \leq |y| + c_3 c^* \right\}.$$

Properties (2.1) and (2.2) of g_i are immediate consequence of $(V2)$. Moreover, from

$$-|Pu| + \|w_u\|_{L^\infty} \leq |u(t)| \leq \|w_u\|_{L^\infty} + |Pu|$$

it follows that $\forall u \in \Lambda$ with $\|w_u\| \leq c^*$

$$|Pu| - c_3 c^* \leq |u(t)| \leq |Pu| + c_3 c^*;$$

from which $(g3)$ immediately follows.

(ii) It is well known that g' is compact. Let $u_n = w_n + x_n$ be such that

$$\|w_n\|_E \leq \text{const}, \qquad |x_n| \to +\infty .$$

From this and

$$|u_n(t)| \geq |x_n| - |w_n(t)| \geq |x_n| - c_1 \|w_n\|_{L^\infty}$$

it follows that $|u_n(t)| \to +\infty$ uniformly. Then from $(V3)$ one deduces:

$$(g'(u_n), v) = -\int \langle V'(t, u_n), v \rangle \to 0 , \qquad \forall v \in E ,$$

and the lemma follows. $\therefore$

THEOREM 5.2. – Suppose $\Omega = \mathbf{R}^n$ and V satisfies ($V1, 2, 3$). Then (4.1) has at least one T-periodic solution.

PROOF. – We remark that in the present case ($\Omega = \mathbf{R}^n$), ($g2$) can be neglected. The result follows, taking into account the discussion in § 4 and Lemma 5.1, from Theorem 2.10. $\therefore$

The following is an example in which Corollary 2.11 applies:

THEOREM 5.3. – Suppose $\Omega = \mathbf{R}^n$ and V satisfies ($V1, 2, 3, 4$). In addition suppose

$$\exists \xi \in \mathbf{R}^n \colon \int V(t, \xi) \geqslant 0 \, . \tag{5.1}$$

Then (4.1) has at least 2 T-periodic solutions.

PROOF. – It suffices to note that $f(\xi) = -\int V(t, \xi) \leqslant 0$. Hence, by Remark 2.4 - (ii) $\xi \in \Gamma_1^\varepsilon$, $\forall \varepsilon > 0$. $\therefore$

REMARK 5.4. – According to Remark 2.12 one could improve the preceding result by showing that (4.1) has a third solution, provided V is C^2 in y, $n \geqslant 2$ and (4.1) has only non-degenerate solutions. $\therefore$

REMARK 5.5. – Periodic solutions for dynamical systems with bounded potentials have beeen studied in the following papers: [8, 11, 12, 22, 23, ...].

Papers [8, 11, 23] deal with even or periodic potential; they both use linking argument to prove existence of one (or more) solutions.

Papers [12, 22] deal with potential which are of the kind we have studied here, but they obtain results different from ours. In particular, [22] proves existence of only the solution corresponding to the minimum under the hypothesis of Theorem 5.3, while [12], using a method similar to the one used here, proves analogous results but under different assumptions. $\therefore$

6. – Strong forces.

Now we deal with with potentials V with singularities. Let

$$\Omega = \mathbf{R}^n \setminus K$$

with K compact. On the behaviour of V near K we will suppose:

(SF) there exist $\varepsilon > 0$ and $U \in C^1(\Omega; \mathbf{R})$ such that, setting $U' = \operatorname{grad} U$, one has

$$U(y) \to -\infty \quad \text{as} \quad y \to \bar{y} \in K, \; y \in \Omega; \tag{6.1}$$

$$V(t, y) \leqslant -|U'(y)|^2, \quad \forall t \in \mathbf{R}, \; \forall y \in K_\varepsilon \equiv \{y \in \Omega \colon \operatorname{dist}(y, K) < \varepsilon\} \, . \tag{6.2}$$

Condition (SF) (= Strong Force) has been first introduced by GORDON [16]. If $K = \{0\}$, (SF) implies that $V(t, y) \approx - |y|^{-\alpha}$ with $\alpha \geqslant 2$ as $y \to 0$.

LEMMA 6.1. – If (SF) holds, then ($g2$) is true.

PROOF. – [16], [18]. $\therefore$

In order to use Theorems 2.13, 2.14 and 3.3, we prove

LEMMA 6.2. – If $\Omega = \boldsymbol{R}^n \setminus K$, with K compact, then (Λ) holds.

PROOF. – The result is possibly well known, but we do not know a precise reference and thus we report here a sketch of the proof for completeness.

Let $p \in K$ and $R > 0$ be such that K is contained in B_R. Set $\Omega_1 = \boldsymbol{R}^n \setminus B_R$, $\Omega_2 = \boldsymbol{R}^n \setminus \{p\}$ and $\Lambda_1 = \{u \in E: u(t) \in \Omega_1, \forall t \in \boldsymbol{R}\}$, $\Lambda_2 = \{u \in E: u(t) \in \Omega_2, \forall t \in \boldsymbol{R}\}$. Clearly $\Lambda_2 \supset \Lambda \supset \Lambda_1$, and since Λ_1 is a deformation retract of Λ_2, Λ_1 is a retract of Λ. Then ([15, pag. 37])

$$H_q(\Lambda) \cong H_q(\Lambda_1) \oplus H_q(\Lambda, \Lambda_1) . \tag{6.3}$$

Since it is well known [7, (3.10)] that (Λ) holds for Λ_1, then (6.3) implies that (Λ) holds for Λ as well. $\therefore$

We can now prove

THEOREM 6.3. – Suppose $\Omega = \boldsymbol{R}^n \setminus K$ and let $V \in C^2(\boldsymbol{R} \times \Omega; \boldsymbol{R})$ satisfy ($V1$, 2, 3) and (SF). Then (4.1) has infinitely many T-periodic solutions.

PROOF. – Since V is bounded from above, ($g1$) holds. Moreover ($g3$, 4) continue to hold as in Lemma 5.1. Then the result follows from Lemmas 6.1, 6.2 and Theorem 2.13. $\therefore$

We can also prove

THEOREM 6.4. – Let the assumptions of Theorem 6.3 be satisfied. If, moreover, $\exists R$, $\delta > 0$ such that $|\xi| \geqslant R$, $|\eta| \leqslant \delta$ imply $\langle V'(t, \xi + \eta), \xi \rangle > 0$, then there exists a sequence u_k of T-periodic solutions of (4.1) such that $f(u_k) \to +\infty$.

PROOF. – It is easy to show, using arguments already used several times, that (2.20) holds and the result then follows from Theorem 2.14. $\therefore$

As a last application, we consider V satisfying ($V1$), ($V3$) and

($V4$) $\quad V(t, y) \to 0$ as $|y| \to +\infty$, uniformly in t, and $\exists r_2 > 0$: $\langle V'(t, y), y \rangle < 0$, $\forall |y| \geqslant r_2$, $\forall t$.

We show:

LEMMA 6.5. – If V satisfies $(V4)$, then $(g5)$ holds.

PROOF. – From $(V4)$ it follows readily that $V(t, y) > 0$, $\forall |y| \geqslant r_2$, $\forall t$. This together with arguments already used in Lemma 5.1, yields $(g5)$. ∴

THEOREM 6.6. – If $\Omega = \boldsymbol{R}^n \setminus K$ and $V \in C^2(\boldsymbol{R} \times \Omega; \boldsymbol{R})$ satisfies $(V1, 3, 4)$ and (SF), then (4.1) has infinitely many T-periodic solutions; moreover there exists a sequence u_k of such solutions such that $f(u_k) \to +\infty$.

PROOF. – Apply Theorem 3.3, taking into account Lemmas 6.1, 6.2 and 6.5. ∴

REMARK 6.7. – If

$$D = \{y \in \Omega \colon V'(t, y) = 0,\ \forall t \in \boldsymbol{R}\}$$

is not empty, each $y \in D$ is a (constant) solution of (4.1). If D is compact the arguments of Theorems 6.3, 6.4 and 6.6 can be carried over to show that (4.1) has actually infinitely many non-constant solutions. We remark that D is compact if $(V4)$ holds. ∴

REMARK 6.8. – Besides [16], already discussed in the introduction, papers [3, 5, 9, 10, 13, 14, 18] deal with singular potentials.

[3, 5] consider potentials defined in a bounded well Ω, with $V \to +\infty$ as $y \to \partial\Omega$.

[18] studies cases when V can have singularities both with $V \to +\infty$ and $V \to -\infty$. The existence of one T-periodic solution is proved, assuming further condition at $y = 0$.

[9, **10**] are close to [16] and either $n = 2$ or geometrical conditions are assumed which permit to avoid the lack of PS.

[13] studies potentials roughly of the type $|y|^{-2} - |y|^{-1}$, case which is different from ours because the corresponding functional is not bounded from below.

In all these papers, (SF) is assumed. The only work where (SF) is violated is [14], but Ω is a bounded well and $V \to -\infty$ as $x \to \partial\Omega$. For a discussion of the problems arising when (SF) does not hold, see [17]. ∴

We esplicitely remark that, in analogy with what seen in § 5, an existence result can be stated for bounded potentials verifying $(V4)$ insted of $(V2)$, namely:

THEOREM 6.9. – If V satisfies $(V1, 3, 4)$ and $\Omega = \boldsymbol{R}^n$, then (4.1) has at least one T-periodic solution.

PROOF. – Apply Theorem 3.3 - (i). Notice that in this case, however, f attains negative values and has a global (negative) minimum. ∴

7. – Autonomous systems.

If V does not depend on t, our problem becames to find *non-constant* periodic solutions of a given period T of the equation

$$-\ddot{y} = V'(y)\,. \tag{7.1}$$

As before, the T-periodic solution of (7.1) are the critical points of the functional

$$f_T(u) = \tfrac{1}{2}\int_0^T |\dot{u}|^2\, dt - \int_0^T V(u)\, dt\,.$$

All the setting and the results of the preceding sections apply to the present case, too. But now:

(i) the critical points of V on Ω (i.e. the $y \in \Omega$ such that $V'(y) = 0$) are also critical points of f_T on Λ corresponding to constant solution of (7.1) (hence periodic of any given period T);

(ii) if $u \in Z(f_T)$, then also $S^1 u = \{u(t+\theta), \theta \in S^1\}$ is contained in $Z(f_T)$. In particular $Z(f_T)$ is infinite whenever it contains a non-costant solution.

Thus additional arguments are required to deduce from the abstract results of § 2, 3 informations on the solutions of (7.1).

To overcome (i), one usually makes assumptions on the set of the critical pionts of V. In particular, the question arises if $\Omega = \boldsymbol{R}^n$ - case in which the existence of one or two solutions for (7.1) has been proved (Theorems 5.2, 5.3, 6.9). This kind of arguments have been discussed in [13, Step 4], where we refer to for statements and details.

If V has singularities, we can take advantage of the fact that now, for all given T, f_T has infinitely many critical points. Suppose that

$$Z(V) \quad \text{is compact}\,. \tag{7.2}$$

Then, using the same arguments of theorems 2.14 and 3.3, we can show that, for any fixed $T > 0$, f_T has at least one (actually infinitely many) critical points $u \notin Z(V)$. As solution of (7.1) u has *minimal* period $\tau = T/k$ for some interger $k \geqslant 1$. Take $T' = \tau/2$. As above, $f_{T'}$ has a critical point v which is a solution of (7.1) with period $T' = \tau/2$ hence also a T-periodic solution of (7.1). Moreover $\{v(t)\}_{t \in S^1} \neq \{u(t)\}_{t \in S^1}$ since τ was the minimal period of u. Repeating this argument, one obtains:

THEOREM 7.1. – Suppose $V \in C^2(\Omega; \boldsymbol{R})$ satisfies (SF), (7.2) and either ($V1$, 2, 3) or ($V1$, 3, 4). Then $\forall T > 0$ (7.1) has infinitely many, non-constant, distinct T-periodic solutions. ∴

REFERENCES

[1] A. AMBROSETTI, *Elliptic problems with jumping nonlinearities*, J. Math. Phys. Sci. (1984), pp. 1-10.

[2] A. AMBROSETTI, *Dynamical systems with singular potentials*, to appear in the Proc. of the meeting « Recent developments in Hamiltonian systems », L'Aquila, Italy, 1986.

[3] A. AMBROSETTI - V. COTI ZELATI, *Solutions with minimal period for Hamiltonian systems in a potential well*, to appear on Ann. I. H. P. « Analyse non linéaire ».

[4] A. BAHRI - J. M. CORON, *Vers une théorie des points critiques à l'infini*, Séminaire Bony-Söstrand-Meyer 1984-1985, Palaiseau, 1984.

[5] V. BENCI, *Normal modes of a Lagrangian system constrained in a potential well*, Ann. I. H. P. « Analyse non linéaire », **1** (1984), pp. 379-400.

[6] P. BARTOLO - V. BENCI - D. FORTUNATO, *Abstract critical point theorems and applications to some nonlinear problems with « strong » resonance at infinity*, Nonlinear Analysis, T.M.A., **7** (1983), pp. 981-1012.

[7] R. BOTT, *Lectures on Morse theory, old and new*, Bulletin (New Series) A.M.S., **7** (1982), pp. 331-358.

[8] A. CAPOZZI - D. FORTUNATO - A. SALVATORE, *Periodic solutions of Lagrangian systems with bounded potential*, preprint Università di Bari, 1985.

[9] A. CAPOZZI - C. GRECO - A. SALVATORE, *Lagrangian systems in presence of singularities*, preprint Università di Bari, 1985.

[10] A. CAPOZZI - A. SALVATORE, *Periodic solutions of Hamiltonian systems: the case of the singular potential*, Proc. NATO-ASI (Singh Ed.) (1986), pp. 207-216.

[11] A. CAPOZZI - A. SALVATORE, *Periodic solutions for nonlinear problems with strong resonance at infinity*, Comm. Math. Universitas Carolinae, **23** (1982), pp. 415-425.

[12] V. COTI ZELATI, *Periodic solutions of dynamical systems with bounded potential*, to appear in Journ. of Diff. Eq.

[13] V. COTI ZELATI, *Dynamical systems with effective-like potentials*, to appear in Nonlinear Analysis, T.M.A.

[14] V. COTI ZELATI, *Remarks on dynamical systems with weak forces*, preprint S.I.S.S.A., Trieste, 1986.

[15] A. DOLD, *Lectures on Algebraic Topology*, second edition, Springer-Verlag, Berlin, 1980.

[16] W. GORDON, *Conservative dynamical systems involving strong forces*, Trans. Am. Math. Soc., **204** (1975), pp. 113-135.

[17] W. GORDON, *A minimizing property of Keplerian orbits*, Am. Journ. of Math., **99** (1977), pp. 961-971.

[18] C. GRECO, *Periodic solutions of some nonlinear ODE with singular nonlinear part*, preprint Università di Bari, 1985.

[19] C. GRECO, *Periodic solutions of a class of singular Hamiltonian systems*, preprint Università di Bari, 1986.

[20] A. MARINO - G. PRODI, *Metodi perturbativi nella teoria di Morse*, Bollettino U.M.I. (4), **11**, Suppl. fasc. 3 (1975), pp. 1-32.

[21] R. PALAIS, *Lusternik-Schnirelman theory on Banach manifolds*, Topology, **5** (1969), pp. 115-132.

[22] K. THEWS, *T-periodic solutions of time dependent Hamiltonian systems with a potential vanishing at infinity*, Manuscripta Math., **33** (1981), pp. 327-338.

[23] J. R. WARD, *A boundary value problem with a periodic nonlinearity*, Nonlinear Analysis, T.M.A., **10** (1986), pp. 207-213.