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Some global aspects of compact space-times

By

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1. Introduction. As models of the universe, *compact* space-times are rather problematic since they necessarily contain closed timelike curves. The existence of a closed timelike curve signifies within the context of general relativity the most flagrant type of causality violation since it implies the ability of some observer to communicate with his past. In general relativity the positivity of energy density is expressed naturally, via the Einstein equations, in terms of conditions on the Ricci curvature of space-time. In his well known essay on global Lorentzian geometry [1], Avez considered the following questions (along with many others) which we paraphrase here: Do space-times obeying reasonable energy (i.e. Ricci curvature) conditions exist which are compact? If so, do they necessarily admit global compact spatial sections? (Avez's interest in the latter question is related to a result of Aufenkamp; see the discussion on p. 120 in [1]). Avez settled these questions by constructing a compact, stationary space-time whose energy momentum tensor is that associated with a perfect fluid and an electromagnetic field (see Section 8 of Chapter 1 in [1]). The Ricci curvature of the model obeys,

(1) $$\operatorname{Ric}(X, X) > 0, \quad \text{for all nonspacelike vectors } X \neq 0.$$

It follows from Aufenkamp's result (see Theorem (16, I), p. 127 in [1]) that this model admits no compact spatial sections. (This conclusion also follows from Avez's Theorem (17, I)). Thus if Avez's model admits a global spatial section it cannot be embedded as a *closed* subset. Avez also shows that through every point of his model there passes a closed timelike curve. Thus, in his model, the causality violation is total.

In this note we prove that the two global features: the nonexistence of compact spatial sections and the totality of causality violation, are in fact features of any compact space-time which obeys the curvature condition (1). Precise statements and proofs are given in the next section. The proofs make use of results in global Lorentzian geometry for which Beem and Ehrlich [2], Hawking and Ellis [6], and Penrose [7] are standard references.

2. Statements and Proofs. Let (M, g) denote an arbitrary space-time by which we mean a smooth connected manifold M of dimension $n \geqq 3$ equipped with a Lorentzian metric g having signature $(- + + \cdots +)$, with respect to which M is time oriented. It will be necessary in our work to extend the usual usage of the term "spacelike". Following Seifert [9], a subset S of M is said to be *spacelike* if there exists an open neighborhood U of S

such that S is acausal in U, i.e. such that any nonspacelike curve in U intersects S at most once. More generally, a subset S is said to be nontimelike if there exists an open neighborhood U such that S is achronal in U, i.e. such that any timelike curve in U intersects S at most one. A nontimelike set is said to be edgeless if it is edgeless relative to some (and hence any) open neighborhood in which it is achronal. Then, by a *global spacelike section* (respectively, *global nontimelike section*) we mean an edgeless spacelike (respectively, edgeless nontimelike) subset of M. A global nontimelike section S is necessarily a codimension one, C^{1-}, embedded submanifold of M (see [7], Lemma 3.17 and Proposition 5.8, and [6], Proposition 6.31). We are now prepared to state our main results.

Theorem 1. *Let M be a compact space-time in which the curvature condition* (1) *is satisfied. Then M does not admit any compact global spacelike* (*or global nontimelike*) *sections.*

Theorem 2. *Let M be a compact space-time in which the curvature condition* (1) *is satisfied. Then, through any two points of M there passes a closed timelike curve.*

Tipler has previously explored in detail the relationship between curvature and causality. In fact, in [10] Tipler presents a version of Theorem 2 under somewhat weaker curvature conditions (see Theorem 7). However, there appears to be an error in his proof. (Contrary to what Tipler claims, the arguments of his Proposition 3 are not applicable).

We remark that Theorem 1 is, of course, false if the curvature condition is dropped or weakened to: $\mathrm{Ric}(X, X) \geqq 0$ for nonspacelike X; consider the flat space-time torus. One can use the Reeb foliation of S^3 (see [8]) to construct a space-time having the leaves of this foliation as spacelike hypersurfaces. This model violates the conclusion of Theorem 2 (as well as that of Theorem 1), since it contains a compact, acausal spatial section. (The author is grateful to Ted Frankel for bringing this example of a space-time to his attention).

Although the conclusion of Theorem 2 seems stronger than the assertion that there exists a closed timelike curve through each point, it is easy to show (using properties of achronal boundaries [7]) that these statements are equivalent. Theorem 2 will follow easily from Theorem 1. Our proof of Theorem 1 relies crucially on the results derived on pages 295–298 in Hawking and Ellis [6], which we summarize in the statement of the following lemma. Recall, a partial Cauchy surface is a globally acausal edgeless subset of M.

Lemma 1. *Let M be a space-time satisfying* $\mathrm{Ric}(K, K) > 0$ *for all non-zero null vectors K. If S is a partial Cauchy surface then $H^+(S)$* (*similarly, $H^-(S)$*) *is either empty or noncompact.*

Briefly, the idea of the proof of Lemma 1 is to show that if the future (respectively, past) Cauchy horizon is compact and nonempty then the expansion $\hat{\theta}$ of the null geodesic generators of $H^+(S)$ (respectively, $H^-(S)$) must vanish. The Raychaudhuri equation for a null geodesic congruence then shows that the vanishing of $\hat{\theta}$ contradicts the curvature assumption.

Tipler has indicated how to extend Lemma 1 as follows (see [10], p. 19). Let B^+ be an n dimensional submanifold of M with boundary having the following property: If

$p \in H^+(S) \cap B^+$ then the past inextendible null geodesic generator η of $H^+(S)$ with future end point p is contained in $H^+(S) \cap B^+$. Define B^- time-dually. Then the conclusion of Lemma 1 becomes: $H^+(S) \cap B^+$ (similarly, $H^-(S) \cap B^-$) is either empty or noncompact. This extension shall be needed in the course of the proof of Theorem 1.

Proof of Theorem 1. We shall carry out the proof in two stages, first proving the theorem in the case that S is spacelike, then extending the theorem to the case that S is nontimelike.

Case 1 (*S spacelike*). The proof in this case is similar to the proof of Theorem 2 in [3]. If S is globally acausal the proof is immediate. Indeed, since $H^+(S)$ and $H^-(S)$ are closed and M is compact, Lemma 1 implies that $H(S) = H^+(S) \cup H^-(S) = \emptyset$. Thus, S is a Cauchy surface. But this implies that M does not contain any closed timelike curves, which contradicts the assumption that M is compact.

Suppose, then, that S is not acausal. In this case we can introduce the Geroch covering manifold, $\tilde{M}_S$, of M (see [5]). The description of this covering manifold most suitable for the present purposes is given in [4]. (The compactness of S permits its construction). We briefly list some of the pertinent features of this covering manifold. $\tilde{M}_S$ can be expressed as a union, $\tilde{M}_S = \bigcup_{i \in Z} M_i$, where for each i, (1) M_i is an n dimensional manifold with boundary (and is obtained by modifying a copy of M in a simple way), (2) M_i is compact (since M is) and $\pi(M_i) = M$, where π is the covering map, (3) M_i and M_{i+1} meet in a hypersurface S_i, with M_i lying to the past of S_i and M_{i+1} lying to the future, (4) $\pi: S_i \to S$ is a homeomorphism, and (5) S_i is acausal in the metric obtained in the usual way by lifting the metric of M to $\tilde{M}_S$ via the covering map.

We claim that S_0 is a Cauchy surface in $\tilde{M}_S$. It suffices to show that $H(S_0) = H^+(S_0) \cup H^-(S_0) = \emptyset$. Suppose $H^+(S_0) \neq \emptyset$. Then, for a sufficiently large integer J, $H^+(S_0) \cap B^+ \neq \emptyset$, where $B^+ = \bigcup_{i=1}^{J} M_i$. Since π is a local isometry the curvature condition (1) holds in $\tilde{M}_S$. Thus, by the extension of Lemma 1, $H^+(S_0) \cap B^+$ is noncompact. But this contradicts the fact that $H^+(S_0)$ is closed and B^+ is compact. Hence, $H^+(S_0) = \emptyset$, and similarly one argues that $H^-(S_0) = \emptyset$. Therefore, S_0 is Cauchy and $\tilde{M}_S$ is globally hyperbolic.

We now invoke a little lemma which is proved in [3].

Lemma 2. *Let M be a space-time satisfying* (1) *and let N be a compact subset of M. Then there exists a positive number δ such that* $\mathrm{Ric}(X, X) \geqq \delta$ *for all unit timelike vectors X applied at points of N.*

The proof of Lemma 2 is simple and uses the compactness of the nonspacelike direction bundle over N. (Note, however, that the unit timelike bundle is *not* compact).

By Lemma 2, there exists a positive number δ such that

(2) $$\mathrm{Ric}(X, X) \geqq \delta$$

for all unit timelike vectors applied at points of M_0. Since M_i is isometric to M_0 for all i, the inequality (2) holds for all unit timelike vectors on $\tilde{M}_S$. Then by the Lorentzian analogue of the Myers diameter theorem (see Beem and Ehrlich [2]; the global hyperbolicity of $\tilde{M}_S$ is needed here) $\tilde{M}_S$ admits no timelike curve having length greater than $\pi\sqrt{n-1}/\delta$. We arrive at a contradiction, since $\tilde{M}_S$ must contain arbitrarily long timelike curves. To see this, consider a closed timelike curve γ in M. By lifting $n\gamma$ (γ traversed n times) to $\tilde{M}_S$ we obtain arbitrarily long timelike curves in $\tilde{M}_S$.

Case 2 (*S nontimelike*): The idea now is to perturb the Lorentzian metric g of M so that in the resulting space-time condition (1) still holds and S is spacelike. Case 2 then reduces to Case 1.

Let $\hat{g}$ be another Lorentzian metric on M. Then, recall, g is said to be *wider* than $\hat{g}$ if and only if for all nonzero vectors X, X is g-timelike ($g(X,X) < 0$) whenever X is $\hat{g}$-nonspacelike ($\hat{g}(X,X) \leqq 0$).

Lemma 3. *Let* (M,g) *be a space-time satisfying*: $\mathrm{Ric}_g(X,X) > 0$ *for all g-nonspacelike vectors* $X \neq 0$. *Then for any compact subset N of M, there exists a Lorentzian metric* $\hat{g}$ *defined on M such that*

1) *g is wider than $\hat{g}$, and*
2) $\mathrm{Ric}_{\hat{g}}(X,X) > 0$ *for all $\hat{g}$-nonspacelike vectors $X \neq 0$ applied at points of N.*

Proof of Lemma 3. Let $\mathcal{N}$ denote the subset of the tangent bundle TM consisting of those vectors applied at points of $\mathcal{N}$. Let u be an arbitrary but fixed unit future timelike vector field on M. Define

$$K = \{X \in N: X \text{ is } g\text{-nonspacelike and } g(X,u) = -1\}.$$

K is a compact subset of TM, and since $\mathrm{Ric}_g > 0$ on K, there is a number $\delta > 0$ such that

$$\mathrm{Ric}_g(X,X) \geqq \delta \quad \text{for all} \quad X \in K. \tag{3}$$

Let v be the covector: $v(X) = g(X,u)$, and let $X^\perp$ denote the projection of the vector X onto the subspace orthogonal to u. Consider the following one-parameter family of bilinear forms defined on M,

$$g_\lambda = -v \otimes v + \lambda h,$$

where for all vectors X, Y, $h(X,Y) = g(X^\perp, Y^\perp)$. For $\lambda > 0$, g_λ is a Lorentzian metric and $g_1 = g$. Furthermore, one easily checks that g is wider than g_λ if and only if $\lambda > 1$.

Let $\mathrm{Ric}_\lambda = \mathrm{Ric}_{g_\lambda}$. Since the mapping,

$$(X,\lambda) \to \mathrm{Ric}_\lambda(X,X)$$

is continuous and K is compact, it follows easily from (3) that $\mathrm{Ric}_\lambda(X,X) > 0$ for all X in K and all λ sufficiently close to one. Setting $\hat{g} = g_\lambda$ for any such λ greater than one, yields the desired Lorentzian metric.

Having proved Lemma 3, we see that Case 2 of Theorem 1 follows immediately from Case 1 of Theorem 1 by observing that if S is nontimelike in (M,g) then S is spacelike in $(M,\hat{g})$.

Proof of Theorem 2. It suffices to prove that for any point $p \in M$, $I^+(p) = M$ and $I^-(p) = M$ (where $I^+(p)$ is the chronological future and $I^-(p)$ is the chronological past of p). Suppose $I^+(p) \neq M$. Then $\partial I^+(p) \neq \emptyset$. But $\partial I^+(p)$ is globally achronal and edgeless (see Definition 3.13 and Corollary 5.9 in [7]). Hence, it is a compact, global nontimelike section. But this conclusion violates Theorem 1. Thus, $I^+(p) = M$, and similarly $I^-(p) = M$.

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