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Nim-Type Games[1])

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For every partially ordered set $(A, \leqslant)$ we define a 2-person game $\Gamma(A, \leqslant)$ or $\Gamma(A)$ for short as follows: the first player P_{I} selects an element a_1 of A and then removes all elements a of A such that $a \geqslant a_1$. Player two, P_{II}, now picks a_2 from among the remaining elements of A and removes all a in A such that $a \geqslant a_2$. The play then reverts to P_{I} and continues in the same way until all elements have been removed. The player making the last move loses.

This general class of games include as special cases both Nim and *Gale*'s [1974] rectangular games (Gnim). Nim corresponds to the special case where A is the sum (disjoint union) of a finite number of totally ordered sets; Gale's rectangular games are products of two totally ordered sets.

We introduce now another subclass of such games which we call the $(n; k)$ games; Let $A_n^k = \{B \subset \{1, \ldots, n\} \mid |B| \leqslant k\}$. The set A_n^k is partially ordered by inclusion, i.e., $B_1 \leqslant B_2$ if and only if $B_1 \subset B_2$. Then the $(n; k)$ game is the game $\Gamma(A_n^k)$. Gale's argument showing that Gnim is a win for P_{I} applies to any set A which has a largest element, thus, the $(n; n)$ game is a win for P_{I}.

Question: Is the winning first move in the $(n; n)$ game to select the maximal element?

The cases where k equals 0 or 1 are trivial; $(n; 0)$ is a win for P_{II}, and $(n; 1)$ is a win for P_{II} if and only if $n = 0 \pmod 2$. For $k > 1$ we could characterize the winner only when $k = 2$; we have,

Theorem. The $(n; 2)$ game is a win for player two if and only if n is divisible by 3.

To summarize our results up to now we have: For $k \leqslant 2$, the game $(n; k)$ is a win for P_{II} if and only if $n = 0 \pmod{k+1}$. This leads us to,

Question: Is $(n; k)$ a win for P_{II} iff $n = 0 \pmod{k+1}$.

We are able to solve for the winner only in the following cases: $k = 2$, $k = n$, and

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the special games (4;3), (5;3), (6;3), (5;4), (6;4), and (7;4). We do not know who is the winner even in the games (6;5) and 7;3).

The following table shows for which $n \leqslant 7$ and k the conjecture holds:

k \ n	1	2	3	4	5	6	7
1	I	II	I	II	I	II	I
2	I	I	II	I	I	II	I
3	I	I	I	II	I	I	?
4	I	I	I	I	II	I	I
5	I	I	I	I	I	?	?
6	I	I	I	I	I	I	?

We would like to mention that whether $(n, n-1)$ is a win for P_{II} is equivalent to our first question as to whether selecting the largest element in $(n; n)$ is the winning move in the $(n; n)$ game.

Before starting with the proof of our theorem we introduce some notations. We denote by W_{I} (W_{II}) the set of all games which are a win for P_{I} (P_{II}). If A is a partially ordered set we denote by $A \ominus a$ the partially ordered subset of A obtained by removing all elements greater or equal to a.

Lemma 1. $(n; k) \in W_{\mathrm{I}}$ iff $\Gamma(A^k_{n+2} \ominus \{n+1, n+2\}) \in W_{\mathrm{I}}$.

Proof. Assume that $(n; k) \in W_{\mathrm{I}}$, and let σ be a winning strategy of P_{I}, in which his first move is to select an element a in A^k_n. We will describe a corresponding winning strategy $\bar{\sigma}$ for P_{I} in the game $\Gamma(A^k_{n+2} \ominus \{n+1, n+2\})$. His first move is to select the element $a \in A^k_n \subset A^k_{n+2} \ominus \{n+1, n+2\}$. All other moves will be defined by induction. Let D denote the set $\{1, \ldots, n\}$. Observe that no element in $A^k_{n+2} \ominus \{n+1, n+2\}$ contains both $n+1$ and $n+2$. Now, if at some move P_{II} selects $a \in A^k_{n+2}$ such that $a \not\subset D$, then P_{I} selects the element $\bar{a} \in A^k_{n+2}$ which is obtained by exchanging in a, $n+1$ and $n+2$. The symmetry of $n+1$ and $n+2$ at all positions following P_{I} moves, implies that this is always possible. Otherwise, if at some move P_{II} selects an element $a \subset D$, then P_{I} will move by selecting an element $a \subset D$ which is determined by his winning strategy σ in $\Gamma(A^k_n)$, as follows: P_{I} considers the sequence of elements $a_1, \ldots, a_l$ with $a_i \subset D$ that have already been selected. By induction, there is an element $a = \sigma(a_1, \ldots, a_l)$ in $A^k_n \ominus a_1 \ominus \ldots \ominus a_l$ such that $\Gamma(A^k_n \ominus a_1 \ominus \ldots \ominus a) \in W_{\mathrm{II}}$.

Therefore P_{I} will never select the last element and thus we described a winning strategy $\bar{\sigma}(\sigma)$ of P_{I} in $\Gamma(A^k_{n+2} \ominus \{n+1, n+2\})$.

The same arguments shows also that if $(n; k) \in W_{II}$ then $(A^k_{n+2} \ominus \{n+1, n+2\}) \in$
$\in W_{II}$.

Corollary 2: If $(n; k) \in W_{II}$ then $(n+2; k) \in W_I$, where $k \geqslant 2$.

Proof. If $(n; k) \in W_{II}$ then by lemma 1, $(A^k_{n+2} \ominus \{n+1, n+2\}) \in W_{II}$, which proves that $(n+2; k) \in W_I$.

Lemma 3: If $(n; k) \in W_{II}$ then $(n+1; k) \in W_I$, for every $k \geqslant 1$.

Proof. Follows from the assumption $(n; k) \in W_{II}$ and the identification of $A^k_{n+1} \ominus \{n+1\}$ with A^k_n.

Proof of the theorem. By induction on n. As both $A^2_1 \cong A^1_1$ and A^2_2 have largest elements, $(1;2) \in W_I$ and $(2;2) \in W_I$. Thus by corollary 2 and lemma 3 it is enough to prove that if $(n; 2) \in W_I$ and $(n+1; 2) \in W_I$ then $(n+2;2) \in W_{II}$. Assume that $(n;2) \in W_I$ and that $(n+1;2) \in W_I$. In order to prove that $(n+2;2) \in W_{II}$ we have to show that for every $a \in A^2_{n+2}$, $\Gamma(A^2_{n+2} \ominus a) \in W_I$. As $|a| = 1$ or $|a| = 2$ we may assume without loss of generality, that either $a = \{n+2\}$ or $a = \{n+1, n+2\}$. But $A^2_{n+2} \ominus \{n+2\} = A^2_{n+1}$ and thus as $(n+1;2) \in W_I$, $\Gamma(A^2_{n+2} \ominus \{n+2\}) \in W_I$. In the other case, as $(n; 2) \in W_I$ we deduce from lemma 1 that $\Gamma(A^2_{n+2} \ominus \{n+1, n+2\}) \in W_I$. This completes the proof of the theorem.

Observe that the proof also characterize the winning first move. If, $n = l \pmod 3$, $l = 1,2$, the only winning first move is to select an l-set. This suggests the following.

Question: if indeed, $(n; k) \in W_{II}$ iff $n = 0 \pmod{k+1}$, is the only winning first move in $(n; k)$, $n = l \pmod{k+1}$ $l = 1, \ldots, k$, to select an l-set.

We turn now to the determination of the winner in the special games (4;3), (5;3), (6;3), (5;4), (6;4) and (7;4). To prove that $(4;3) \in W_{II}$, we have to show that (i) $\Gamma(A^3_4 \ominus \{4\}) \in W_I$, (ii) $\Gamma(A^3_4 \ominus \{3,4\}) \in W_I$ and that (iii) $\Gamma(A^3_4 \ominus \{1,2,3\}) \in W_I$. (i) and (iii) follows from the identity $A^3_4 \ominus \{1,2,3\} \ominus \{4\} = A^2_3$ and the theorem which asserts in particular that $\Gamma(A^2_3) \in W_{II}$. To prove (ii), we observe that $\Gamma(A^3_4 \ominus \{1,2\} \ominus \{3,4\}) \in W_{II}$ by appealing to the same symmetry argument used in the proof of lemma 1. As $(4;3) \in W_{II}$ lemma 3 shows that $(5;3) \in W_I$, and by lemma 1, $(6;3) \in W_I$. To prove that $(5:4) \in W_{II}$, we have to show that (i) $\Gamma(A^4_5 \ominus \{5\}) \in W_I$, (ii) $\Gamma(A^4_5 \ominus \{1,2,3,4\}) \in W_I$, (iii) $\Gamma(A^4_5 \ominus \{4,5\}) \in W_I$ and that (iv) $\Gamma(A^4_5 \ominus \{1,2,3\}) \in W_I$. To prove (i) and (ii) it is enough to show that $\Gamma(A^4_5 \ominus \{5\} \ominus \{1,2,3,4\}) \in W_{II}$. This follows from the identity $A^4_5 \ominus \{5\} \ominus \{1,2,3,4\} = A^3_4$. By appealing to the same symmetry argument used in the proof of lemma 1, $\Gamma(A^3_5 \ominus \{4,5\} \ominus \{1,2,3\}) \in W_{II}$ and thus the identity $A^4_5 \ominus \{4,5\} \ominus \{1,2,3\} = A^3_5 \ominus \{4,5\} \ominus \{1,2,3\}$ implies (iii) and (iv). Again by lemma 3 it follows that $(6;4) \in W_I$ and by lemma 1 that $(7;4) \in W_I$.

References

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