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Dynamics of Cooperative Games[1])

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Abstract: Systems of differential equations are exhibited, the solutions of which converge to optimal points, some of which are shown to coincide with classical solution concepts, to wit, the core, the Shapley value, and, under certain conditions, the Nucleolus.

Introduction

An important part of the study of cooperative game theory is the development of models whereby the dynamics of negotiation among the players can be investigated. One approach to this problem concentrates on the use of discrete transfer schemes to study how players might arrive at a desirable outcome. A parallel approach employs systems of differential equations whose solutions represent a continuous transfer of payoff over time. It is the intention of this paper to further research in this latter area.

The advantages of such an approach are multifold. Not only does it enable us to view game theory in terms of the actions of individual players or coalitions of players, but it also enables us to characterize solution concepts, many of them well known, in terms of systems of differential equations which can be interpreted as representing a rational model of action or "behavior." Having done so, it is possible to ask which points of a solution concept are attainable from initial points exterior to the solution concept; which are stable and in what sense; how a final point is reached over time and so forth.

Stearns [1968] exhibited a sequence of discrete transfers of payoff which converged to points of the kernel of *Davis* and *Maschler* [1965]. *Billera* [1972] smoothed these transfer sequences to obtain a system of differential equations whose solutions represented a continuous transfer of payoff and which also converged to the kernel. *Kalai, Maschler* and *Owen* [1973] reproved the above convergence results using different approaches and also answered some stability questions. *Wang* [1974] showed that a modification of the relaxation method of *Agmon* [1954] could provide a discrete

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transfer sequence which converged to the core [*Gillies*, 1959] of a game. This "core sequence" however could not be smoothed in the manner that *Billera* smoothed *Stearns'* "kernel sequence."

In this paper, we exhibit several systems of differential equations which represent possible behavior patterns for the players. The solutions of these equations are shown to converge to a number of solution concepts, among them the core, the Shapley value [*Shapley*, 1953], and, in certain instances, the nucleolus [*Schmeidler*, 1969]. This is accomplished by defining for games classes of optimal "centroids" and "nuclei" which fall into the class of "convex nuclei" was defined by *Charnes* and *Kortanek* [1970], since they minimize certain convex functions. These centroids and nuclei are the (stable) critical points of the various systems of differential equations and it is shown under what conditions the centroids and nuclei coincide with classical solution concepts.

This work is divided into two chapters. Chapter I establishes most of the mathematical foundation for the rest of the paper and also provides some geometrical insight into the processes discussed. Chapter II applies these results to cooperative games with sidepayments and also proves some results peculiar to this formulation. The symbol □ will signify an end of proof.

I. Systems of Differential Equations with Polyhedral Stable Sets

§1. Geometric Considerations

Let $\{a^i\}$ $i = 1, \ldots, m$ be a fixed set of (Euclidean norm) unit vectors in R^n where R^n is Euclidean n-space. For $b \in R^m$ with components $\{b_1, b_2, \ldots, b_m\}$ and $x \in R^n$ define the functions

$$g^i(x, b) = \langle a^i, x \rangle + b_i.$$

Here, $\langle\ ,\ \rangle$ is the standard inner product on R^n, and we will also denote by $\|\cdot\|$ the Euclidean norm on the appropriate space. Also define

$$P^i(b) = \{x \mid g^i(x, b) = 0\} \qquad i = 1, \ldots, m$$

$$\text{core}(b) = \{x \mid g^i(x, b) \leqslant 0, \qquad i = 1, \ldots, m\}.$$

Each $P^i(b)$ is a hyperplane in R^n while core (b), if nonempty, is a possibly unbounded polyhedron in R^n since it is the intersection of half-spaces. Here, as in the rest of this work, "polyhedron" will be synonymous with "convex polyhedron." The following two facts are elementary results from analytic geometry:

a) The normal (perpendicular) Euclidean distance from any point $x \in R^n$ to $P^i(b)$ is $|g^i(x, b)|$ (where $|\cdot|$ is absolute value).
b) The normal vector from any point $x \in R^n$ to $P^i(b)$ is $-g^i(x, b)\, a^i$.

Let $R_+^m = \{k \in R^m \mid k_i > 0,\ i = 1, \ldots, m\}$, i.e., R_+^m is the strictly positive orthant in R^m. For $k \in R_+^m$, consider the following system of differential equations:

$$\dot{x} = D(x, b, k) \equiv -\sum_{i=1}^{m} k_i [g^i(x, b)]^+ a^i$$

where

$$\dot{x} = \frac{dx}{dt} \qquad \text{(I.a)}$$

and

$$[\,\cdot\,]^+ = \max\{\cdot, 0\}.$$

Proposition I.1: For any $b \in R^m$, $k \in R_+^m$, $x_0 \in R^n$, there exists a unique solution $\gamma(t, x_0, b, k)$ to (I.a), continuous in t for $t \in (-\infty, \infty)$ and such that $\gamma(0, x_0, b, k) = x_0$.

Proof: This is an immediate consequence of the fact that $D(x, b, k)$ is continuous and locally Lipschitz in x. The reader is referred to *Coddington* and *Levinson* [1957], or *Hale* [1969] for results on systems of differential equations. □

Geometrically, one can imagine the half-space

$$\{x \mid g^i(x, b) > 0\}$$

to be the "wrong side" of hyperplane $P^i(b)$. All other points will constitute the "right side." At any point $x \in R^n$, consider all those i such that x is on the wrong side of $P^i(b)$. Let us call such a $P^i(b)$ an "offended" hyperplane. Take a positive linear combination of the normal vectors from x to the offended hyperplanes to obtain

$$-\sum_{i=1}^{m} k_i [g^i(x, b)]^+ a^i.$$

Thus, the solutions of system (I.a) tend to move toward the offended hyperplanes as t increases, ignoring the others, so it might be expected that, along solutions, the distance to offended hyperplanes would tend to decrease. This notion will be made rigorous and proven later.

§2. Centroids

With $\{a^i\}$, b, and k as above, we can define $C(b, k)$, the set of "k-centroids of b (with vectors $\{a^i\}$)" to be

$$\{x \in R^n \mid \Phi(x, b, k) = \inf_{y \in R^n} \Phi(y, b, k)\}$$

where

$$\Phi(y, b, k) = \sum_{i=1}^{m} k_i ([g^i(y, b)]^+)^2.$$

Observe that (1) if core (b) is nonempty, then core (b) is precisely $C(b, k)$, and (2) $C(b, k)$ is, in this case, independent of k. In general, however, $C(b, k)$ is not independent of k.

Proposition I.2: For any $b \in R^m$, and $k \in R_+^m$, $C(b, k) \neq \emptyset$.

Proof: Observe that the problem

$$\inf_{y \in R^n} \Phi(y, b, k) \quad \text{can be written}$$

$$\inf_{\substack{z \in R^m \\ y \in R^n}} \sum_{i=1}^{m} k_i z_i^2$$

subject to

$$\left.\begin{array}{l} z_i \geqslant 0 \\ z_i \geqslant g^i(y, b) \end{array}\right\} \quad i = 1, \ldots, m.$$

The objective function of the rewritten problem is a convex quadratic function, bounded below, and the constraints define a nonempty polyhedral convex set. The proposition then follows from Corollary 27.3.1 of *Rockafellar* [1970]. (The author is grateful to the referee for indicating this proof.) □

Since $[\cdot]^+$ is a convex, nonnegative, and nondecreasing function on R, and $(\cdot)^2$ is convex while $g^i(x, b)$ is an affine function of x, it follows that $\Phi(x, b, k)$ is also a convex function in x. Observe also that $([\cdot]^+)^2$ is continuously differentiable with

$$\frac{d}{ds}([s]^+)^2 = 2[s]^+.$$

Thus, $\Phi(x, b, k)$ is continuously differentiable on R^n.

Let $\dot{x} = f(x)$ be any system of differential equations on R^n. A "critical point" of the system is any point y such that $f(y) = 0$.

Proposition I.3: x_0 is a k-centroid of b if and only if $\nabla \Phi(x, b, k)|_{x_0} = 0$, where ∇ is the gradient operator with respect to x.

Proof: This follows from the observation that Φ is convex and continuously differentiable (see *Fleming* [1965], section 2–5). □

Proposition I.4: x_0 is a k-centroid of b if and only if x_0 is a critical point of System (I.a).

Proof: $\frac{\partial}{\partial x_j}(\Phi(x, b, k)) = 2 \sum_{i=1}^{m} k_i [\langle a^i, x\rangle + b_i]^+ a_j^i$

Hence, $\nabla \Phi(x, b, k) = -2D(x, b, k)$, so x_0 is a critical point if and only if $D(x, b, k) = 0$ if and only if $\nabla \Phi(x, b, k)|_{x_0} = 0$ if and only if x_0 is a k-centroid of b.

§3. Properties of $C(b, k)$

We will now establish certain properties of $C(b, k)$. An easy observation is that if core $(b) \neq \emptyset$, then the set of k-centroids of b is a polyhedron. This is true even if core $(b) = \emptyset$.

Proposition I.5: $C(b, k)$ is a closed polyhedron.

Proof: Let x_0, x_1 be k-centroids of b. Then

$$0 = \sum_{i=1}^{m} k_i [g^i(x_1, b)]^+ a^i$$

so
$$0 = \sum_{i=1}^{m} k_i [g^i(x_1, b)]^+ \langle a^i, x_0 - x_1 \rangle$$

$$= \sum_{i=1}^{m} k_i [g^i(x_1, b)]^+ (g^i(x_0, b) - g^i(x_1, b)).$$

Similarly

$$0 = \sum_{i=1}^{m} k_i [g^i(x_0, b)]^+ (g^i(x_0, b) - g^i(x_1, b)).$$

Subtracting we obtain

$$0 = \sum_{i=1}^{m} k_i ([g^i(x_1, b)]^+ - [g^i(x_0, b)]^+)(g^i(x_0, b) - g^i(x_1, b))$$

$$= \sum_{i=1}^{m} k_i \{-([g^i(x_1, b)]^+)^2 - ([g^i(x_0, b)]^+)^2$$

$$+ [g^i(x_1, b)]^+ (g^i(x_0, b)) + [g^i(x_0, b)]^+ (g^i(x_1, b))\}$$

$$\leqslant \sum_{i=1}^{m} k_i \{-([g^i(x_1, b)]^+)^2 - ([g^i(x_0, b)]^+)^2 + 2[g^i(x_1, b)]^+ [g^i(x_0, b)]^+\}$$

$$= -\sum_{i=1}^{m} k_i ([g^i(x_1, b)]^+ - [g^i(x_0, b)]^+)^2 \leqslant 0.$$

Therefore, $[g^i(x_0, b)]^+ = [g^i(x_1, b)]^+$ $i = 1, \ldots, m$; moreover, if x_2 is any point in R^n such that $[g^i(x_2, b)]^+ = [g^i(x_0, b)]^+$, then x_2 must also be a k-centroid of b since $\Phi(x_2, b, k) = \Phi(x_0, b, k)$. Therefore, knowing that there exists at least one k-centroid of b, x_0, $C(b, k)$ can be rewritten as

$$\{x \in R^n \mid g^i(x, b) \leqslant 0 \quad \text{for all } i \text{ for which } g^i(x_0, b) \leqslant 0\}$$

$$\cap \{x \in R^n \mid g^i(x, b) = g^i(x_0, b) \quad \text{for all } i \text{ for which } g^i(x_0, b) > 0\}$$

which is the finite intersection of half-space and hyperplanes and is therefore a polyhedron.

The following fact which appears in the previous proof bears emphasizing:

Corollary I.6: $[g^i(x, b)]^+$ is constant over $C(b, k)$ for $i = 1, \ldots, m$.
Geometrically, this means that all k-centroids of b not only "offend" the same hyperplanes, but lie equidistant from each of them.

Corollary I.7: If x_0 and x_1 are distinct k-centroids of b, then $\langle x_1 - x_0, a^i \rangle = 0$ for all i such that $g^i(x_0, b) > 0$.

Corollary I.8: Let x_0 be a k-centroid of b. If $\{a^i | g^i(x_0, b) > 0\}$ span R^n, then x_0 is the unique k-centroid of b.

It would be of interest to know how the set $C(b, k)$ changes with b and k. Unfortunately, this is still primarily an open question as of this writing, although partial answers can be given. In particular, when core $(b) \neq \emptyset$, $b \in$ {interior $\{b | \text{core } b \neq \emptyset\}$} then small changes in b affect $C(b, k) = \text{core}(b)$ only slightly. To show this, we first establish some terminology in the manner of *Dantzig*, et al. [1967].
Let $\{A_n\}$ be a sequence of subsets of some metric space X (in our case, X will be R^n).

Define

$$\overline{\lim}\, A_n = \{x \in X | x = \lim_{i \to \infty} x_{n_i} \text{ where } \{n_i\} \text{ is an infinite sequence of integers and } x_{n_i} \in A_{n_i}\}.$$

$$\underline{\lim}\, A_n = \{x \in X | x = \lim_{n \to \infty} x_n \text{ where } x_n \in A_n \text{ for all but a finite number of } n\}.$$

If $\underline{\lim}\, A_n = \overline{\lim}\, A_n$, then we say $\lim A_n$ exists and we set $\lim A_n = \underline{\lim}\, A_n = \overline{\lim}\, A_n$.

Lemma I.9 (*Dantzig* et al.): Let X be a metric space and let $\{A_n\}$ be a sequence of connected subsets of X. Let U be an open subset of X with compact boundary. If $\underline{\lim}\, A_n$ is nonempty and $\overline{\lim}\, A_n \subset U$, then $A_n \subset U$ for all sufficiently large n.

Lemma I.10 (*Dantzig* et al.): Let $\{b^n\}$ be a sequence in R^m, where $b^n \to b$ and suppose core $(b) \neq \emptyset$, core $(b^n) \neq \emptyset$ for all n, then $\lim(\text{core}(b^n)) = \text{core}(b)$.
We would like to be able to quantify this notion by putting a metric on subsets of R^n. To do this, first define for any $x \in R^n$, and any set $A \subseteq R^n$,

$$d(x|A) = \inf_{y \in A} \| x - y \|.$$

For two sets A and B in R^n define

$$\mu(A, B) = \max \left(\sup_{x \in A} d(x|B), \sup_{x \in B} d(x|A) \right) .$$

This is a metric on the space of compact subsets of R^n and is commonly called the Hausdorff metric. The following proposition establishes the continuity of core (b) in the Hausdorff metric. This has already been observed by *Sondermann* [1972] in the case of games.

Proposition I.11: Suppose $b^n \to b$, core $(b^n) \neq \emptyset$ for all n, core $(b) \neq \emptyset$ and core (b) is compact. Then for all $\epsilon > 0$, there exists N s.t. μ (core (b), core $(b^n)) < \epsilon$ whenever $n \geqslant N$.

Proof: Suppose not, then there exists an $\epsilon > 0$ and a subsequence $n_i \to \infty$ such that (core (b^{n_i}), core $(b)) \geqslant \epsilon$. This can happen in either (or both) of two ways.

i) There exists subsequence $n_j \to \infty$, $x_{n_j} \in$ core (b^{n_j}) and $d(x_{n_j}|$core $(b)) \geqslant \epsilon$ for all j.

ii) There exists subsequence $n_k \to \infty$, $x_{n_k} \in$ core (b) and $d(x_{n_k}|$core $(b^{n_k})) \geqslant \epsilon$ for all k.

Suppose i) occurs, then by Lemma I.9, $\{x_{n_j}\}$ must have a convergent subsequence, so without loss of generality we may assume $\{x_{n_j}\}$ converges to some point x_0. By definition, $x_0 \in \overline{\lim}$ core (b^n), hence $x_0 \in$ core (b) by Lemma I.10. But $d(x_{n_j}|$core $(b)) \geqslant \epsilon$ implies $d(x_0|$core $(b)) \geqslant \epsilon$, a contradiction.

Now suppose ii) occurs. By the compactness of core (b), we can assume $x_{n_k} \to x_0 \in$ core (b). But $x_0 \in$ core (b) if and only if $x_0 \in \underline{\lim}$ core (b^n) so $x_0 = \lim_{k \to \infty} y_{n_k}$ where $y_{n_k} \in$ core (b^{n_k}) for all but finitely many k. Pick k sufficiently large so that

$$\| x_{n_k} - x_0 \| < \epsilon/2$$

and

$$\| y_{n_k} - x_0 \| < \epsilon/2 .$$

Therefore

$$\| x_{n_k} - y_{n_k} \| < \epsilon$$

so that

$$\epsilon > \| x_{n_k} - y_{n_k} \| \geqslant d(x_{n_k} \mid \text{core } (b^{n_k})) .$$

But we assumed $d(x_{n_k} \mid$ core $(b^{n_k})) \geqslant \epsilon$ so we are left with another contradiction. □

§4. Convergence of Solutions of (I.a)

We have already shown that the k-centroids of b were precisely the critical points of System (I.a). The next Proposition will show the relationship between solutions of (I.a) and $C(b, k)$.

Proposition I.14: For any $x_0 \in R^n$, $b \in R^m$ and $k \in R_+^m$, the solution $\gamma(t, x_0, b, k)$ of (I.a) with $\gamma(0, x_0, b, k) = x_0$ is bounded for $t \geqslant 0$ and further, as $t \to \infty$, $\gamma(t, x_0, b, k)$ converges to a k-centroid of b.

Proof: Let $\hat{x}$ be any k-centroid of b. For any $x \in R^n$ define

$$Z(x) = \frac{1}{2} \| x - \hat{x} \|^2 .$$

Thus, along any solution to (I.a), i.e., where

$$x = x(t) = \gamma(t, x_0, b, k),$$

$$\frac{d}{dt}(Z(x)) = \left\langle \frac{dx}{dt}, x - \hat{x} \right\rangle = - \sum_{i=1}^{m} k_i [g^i(x, b)]^+ \langle a^i, x - \hat{x} \rangle$$

$$= \sum_{i=1}^{m} k_i [g^i(x, b)]^+ \langle a^i, \hat{x} - x \rangle = \sum_{i=1}^{m} k_i [g^i(x, b)]^+ (g^i(\hat{x}, b) - g^i(x, b)) .$$

We saw in the proof of Proposition I.5 that

$$\sum_{i=1}^{m} k_i [g^i(\hat{x}, b)]^+ (g^i(\hat{x}, b) - g^i(x, b)) = 0.$$

Therefore, by subtracting

$$\frac{d}{dt} Z = \sum_{i=1}^{m} k_i ([g^i(x, b)]^+ - [g^i(\hat{x}, b)]^+)(g^i(\hat{x}, b) - g^i(x, b)) \tag{I.b}$$

$$\leqslant - \sum_{i=1}^{m} k_i ([g^i(x, b)]^+ - [g^i(\hat{x}, b)]^+)^2 \leqslant 0,$$

i.e.,

$$\frac{d}{dt} \| \gamma(\tau, x_0, b, k) - \hat{x} \|^2 \Big|_{\tau = t} \leqslant 0 \quad \text{for all } t \geqslant 0 \quad \text{so}$$

$$\| \gamma(t, x_0, b, k) - \hat{x} \| \leqslant \| x_0 - \hat{x} \| \quad \text{for all } t \geqslant 0. \tag{I.b'}$$

Moreover, (I.b) and uniqueness of solutions imply that if x_0 is *not* a k-centroid of b, then

$$\frac{d}{dt} \| \gamma(\tau, x_0, b, k) - \hat{x} \|^2_{\tau = t} < 0 \quad \text{for all } t \geqslant 0.$$

Hence, $Z(x)$ is a Lyapunov function on R^n for System (I.a) and it follows from standard results (see *Hale* [1969], p. 296) that the ω-limit set of $\gamma(t, x_0, b, k)$ is contained in $C(b, k)$, where the ω-limit set of $\gamma(t, x_0, b, k)$ is the set of limit point in R^n of $\gamma(t, x_0, b, k)$ as $t \to \infty$. All that remains to show is that $\gamma(t, x_0, b, k)$ converges to a single k-centroid of b. Suppose there were two distinct points, $\bar{x}$ and $\tilde{x}$ in the ω-limit set of $\gamma(t, x_0, b, k)$. Let $\epsilon > 0$ be such that $\|\bar{x} - \tilde{x}\| > 2\epsilon$. By the definition of ω-limit set, there exists $T > 0$ s.t. $\|\gamma(T, x_0, b, k) - \bar{x}\| < \epsilon$, but $\|\gamma(t, x_0, b, k) - \bar{x}\|$ is a decreasing function of t, so for all $t \geqslant T$, $\|\gamma(t, x_0, b, k) - \bar{x}\| < \epsilon$ so $\|\gamma(t, x_0, b, k) - \tilde{x}\| > \epsilon$, contradicting the assertion that $\tilde{x}$ was in the ω-limit set of $\gamma(t, x_0, b, k)$. □

Note: In the case that core $(b) \neq \emptyset$, it is possible to show the following more general result. For $i = 1, \ldots, m$, let $f^i(s)$ be a continuous and locally Lipschitz function on R such that $f^i(s) > 0$ if $s > 0$, $f^i(s) = 0$ if $s \leqslant 0$. Then if $\gamma(t, x_0, b, f)$ is a solution to the system

$$\dot{x} = -\sum_{i=1}^{m} f^i(g^i(x, b))\, a^i$$

then as $t \to \infty$, $\gamma(t, x_0, b, f)$ converges to a point of core (b). For $f^i(\cdot) = k_i[\cdot]^+$, this result is contained in Proposition I.14.

We will denote the limit point of $\gamma(t, x_0, b, k)$ by $\gamma(\infty, x_0, b, k)$. It is evident from equation (I.b') that all k-centroids of b are stable (in the sense of Lyapunov) points of System (I.a). It clearly follows that System (I.a) has no unstable critical points.

Convergence, as has been seen, is straightforward. For any initial point x_0, the solution $\gamma(t, x_0, b, k)$ approaches each k-centroid of b simultaneously as $t \to \infty$ and converges to a particular one.

Convergence can be viewed in another way, however. Since the k-centroids of b were characterized as the minimizing points of $\Phi(x, b, k)$, it is of interest to investigate

$$\Phi(\gamma(t, x_0, b, k), b, k)$$

as $t \to \infty$. Recall that in the proof of Proposition I.4 we showed that

$$\nabla \Phi = -2D(x, b, k).$$

Thus we immediately see that

$$\frac{d}{dt}\Phi(\gamma(t, x_0, b, k), b, k) = \langle \nabla\Phi, \frac{d}{dt}(\gamma(t, x_0, b, k)\rangle = -2\,\|D(x, b, k)\|^2,$$

that is, Φ is decreasing along solutions of (I.a). Moreover, since System (I.a) can be rewritten

$$\dot{x} = -\frac{1}{2}\nabla\Phi(x, b, k),$$

the solutions of (I.a) follow the negative gradient of the function Φ. In other words, at any point x, the solutions of (I.a) tend in the direction most optimal to minimize Φ. In general, however, it is not the case that the solutions follow a shortest path (in the sense of arclength) from x_0 to $C(b, k)$, nor is $\gamma(\infty, x_0, b, k)$ necessarily the closest k-centroid of b to x_0.

§5. Cocentroids

The set $CC(b, k)$ of "k-cocentroids of b" is the set

$$\{x \in R^n \mid \psi(x, b, k) = \inf_{y \in R^n} \psi(y, b, k)\}$$

where

$$\psi(x, b, k) = \sum_{i=1}^{m} k_i ([-g^i(x, b)]^+)^2 .$$

Note that the k-cocentroids of b (with vectors $\{a^i\}$) are the k-centroids of $-b$ (with vectors $\{-a^i\}$). Hence such observations as $CC(b, k)$ is a polyhedron and $[-g^i(x, b)]^+$ is constant over $CC(b, k)$ and so forth are obvious. Moreover, it immediately follows that solutions of

$$\dot{x} = \sum_{i=1}^{m} k_i [-g^i(x, b)]^+ a^i \tag{I.c}$$

converge to k-cocentroids of b. We will say more about cocentroids later on.

§6. Continuity of Limit Points

We can consider $\gamma(\infty, x_0, b, k)$ as a function from $R^n \times R^m \times R^m_+$ to $C(b, k)$. This section will investigate some of the continuity properties of $\gamma(\infty, \cdot, \cdot, \cdot)$. Note that any such result is also dependent on the continuity of $C(b, k)$. We will need the following lemma which is a standard result of the theory of ordinary differential equations.

Lemma I.15: Let $\gamma(t, x_0, b_0, k_0)$ be a solution of System (I.a) for some (x_0, b_0, k_0) in $R^n \times R^m \times R^m_+$. For (x, b, k) in an open neighborhood of (x_0, b_0, k_0) (in the product space), there is a solution $\gamma(t, x, b, k)$ of System (I.a). Moreover $\gamma(t, x, b, k)$ is continuous in (t, x, b, k) at (t_0, x_0, b_0, k_0) for all t_0.

Proof: This follows from the continuity of $D(x, b, k)$ in (x, b, k) and also from the uniqueness of solutions of System (I.a). (cf. *Hale* [1969], Theorem I.3.4). □

Proposition I.16: For any $(b, k) \in R^m \times R^m_+$ $\gamma(\infty, x_0, b, k)$ is continuous in x_0.

Proof: Pick $\epsilon > 0$, any $x_0 \in R^n$. Pick T so large that $\| \gamma(T, x_0, b, k) - \gamma(\infty, x_0, b, k) \| < \epsilon/4$. Choose δ s.t. $\| x - x_0 \| < \delta$ implies $\| \gamma(T, x_0, b, k) - \gamma(T, x, b, k) \| < \epsilon/4$ which we can do by the previous lemma. Therefore $\| \gamma(T, x, b, k) - \gamma(\infty, x_0, b, k) \| < \epsilon/2$, but by Equation I.b'

$$\| \gamma(t, x, b, k) - \gamma(\infty, x_0, b, k) \| < \epsilon/2$$

for all $t \geqslant T$. Since for some $T' \geqslant T$

$$\| \gamma(t, x, b, k) - \gamma(\infty, x, b, k) \| < \epsilon/2$$

for all $t \geqslant T'$ it follows that

$$\| \gamma(\infty, x, b, k) - \gamma(\infty, x_0, b, k) \| < \epsilon . \quad \square$$

The continuity of $\gamma(\infty, x_0, b, k)$ in (x_0, b, k), as mentioned before is dependent on the continuity of $C(b, k)$ and can only be established, therefore, in those cases where the continuity of $C(b, k)$ is known.

Let $W = \{b \in R^m | \text{core}(b) \neq \emptyset \text{ and core}(b) \text{ is compact}\}$. Let D be a compact subset of R^n, and E a compact subset of W. Observe that by Proposition I.11, core (b) can be viewed as a continuous mapping from W to the space of compact subsets of R^n. Hence, over E, the continuity is uniform, i.e., for all $\eta > 0$, there exists $\delta > 0$ such that $\mu(\text{core}(b), \text{core}(b')) < \eta$ whenever, $b, b' \in E$, $\| b - b' \| < \delta$. Let B be a compact subset of $D \times E \times R^m_+$.

Lemma I.17: Let $\epsilon > 0$. Then there exists N s.t. $\| \gamma(t, x, b, k) - \gamma(\infty, x, b, k) \| < \epsilon$ for all $t \geqslant N$ and all $(x, b, k) \in B$.

Proof: Let $T_n(b_0) = \{(x, b, k) \in R^n \times W \times R^m_+ | d(\gamma(n, x, b, k) \mid \text{core}(b_0)) < \epsilon/4\}$

$$\text{for } n = 1, 2, \ldots \qquad \text{and all } b_0 \in W$$

and pick δ such that for all $b, b' \in E$, $\| b - b' \| < \delta$ implies $\mu(\text{core}(b), \text{core}(b')) < \epsilon/4$. Let $V(b) = \{b \in W | \|b - \bar{b}\| < \delta\}$ for all $b \in W$. Now set $U(b) = R^n \times V(b) \times R^m_+$.

$T_n(b)$ is an open set in $R^n \times W \times R^m_+$ since it is the inverse image of an open set under the continuous map $\gamma(n, \cdot, \cdot, \cdot)$. Also it is clear that $U(b)$ is open in $R^n \times W \times R^m_+$.

Let

$$S_n(b) = T_n(b) \cap U(b), \qquad n = 1, 2, \ldots \quad b \in W,$$

and let

$$S_n = \bigcup_{b \in E} S_n(b) \qquad n = 1, 2, \ldots .$$

Each $S_n(b)$ is open in $R^n \times W \times R^m_+$ and thus so is each S_n. Moreover, for all

$(x, b, k) \in B, (x, b, k) \in S_n$ for some n since for some n, $d(\gamma(n, x, b, k) \mid \text{core}(b)) < \epsilon/4$, and, of course, $(x, b, k) \in U(b)$. Thus $\{S_n\}$ is an open cover of B, B is compact, hence there is a finite subcover $S_{n_1}, \ldots, S_{n_k}$ of B. Let

$$(x, b, k) \in S_{n_j} \cap B, \text{ then } (x, b, k) \in S_{n_j}(b_0)$$

for some $b_0 \in E$, i.e.,

$$(x, b, k) \in T_{n_j}(b_0) \cap U(b_0).$$

But if so, then

$$d(\gamma(n_j, x, b, k)) \mid \text{core}(b_0)) < \epsilon/4$$

and

$$\| b - b_0 \| < \delta \text{ which implies } \mu(\text{core}(b), \text{core}(b_0)) < \epsilon/4.$$

Therefore

$$d(\gamma(n_j, x, b, k) \mid \text{core}(b)) < \epsilon/2.$$

From Equation (I.b'), it follows that

$$\| \gamma(n_j, x, b, k) - \gamma(\infty, x, b, k) \| < \epsilon.$$

But since any $(x, b, k) \in B$ lies in some S_{n_j}, setting $N = \max_{1 \leqslant i \leqslant k} \{n_i\}$ will satisfy the requirement of the hypothesis. □

Note that continuity of γ in k was not explicitly used in the above proof. Indeed, the variable k was merely carried along in the notation (except in the assertion that $T_n(b)$ was open). The reason for this is that if core $(b) \neq \emptyset$, then, as we have seen, $C(b, k)$ is idependent of k. To complete the continuity section we show:

Proposition I.18: $\gamma(\infty, x, b, k)$ is jointly continuous in (x, b, k) for $(x, b, k) \in R^n \times W \times R^m_+$.

Proof: Let $\{x^j\}, \{b^j\}, \{k^j\}$ be sequences in R^n, W, and R^m_+ respectively and suppose there exists $(x, b, k) \in R^n \times W \times R^m_+$ such that $x^j \to x$, $b^j \to b$, and $k^j \to k$. Since $\left(\bigcup_{j=1}^{\infty} (x^j, b^j, k^j) \right) \cup (x, b, k)$ is compact, then by Lemma I.17 there exists a T such that $\| \gamma(T, x^j, b^j, k^j) - \gamma(\infty, x^j, b^j, k^j) \| < \epsilon/3 \qquad j = 1, 2, \ldots$

$$\| \gamma(T, x, b, k) - \gamma(\infty, x, b, k) \| < \epsilon/3.$$

By Lemma I.15 it is possible to choose an M so large that

$$\| \gamma(T, x^j, b^j, k^j) - \gamma(T, x, b, k) \| < \epsilon/3 \quad \text{for all } j \geqslant M.$$

Therefore, for all $j \geq M$,

$$\| \gamma(\infty, x^j, b^j, k^j) - \gamma(\infty, x, b, k) \| \leq \| \gamma(\infty, x^j, b^j, k^j) - \gamma(T, x^j, b^j, k^j) \|$$

$$+ \| \gamma(T, x^j, b^j, k^j) - \gamma(T, x, b, k) \| + \| \gamma(T, x, b, k) - \gamma(\infty, x, b, k) \| < \epsilon. \quad \square$$

It is conjectured that $\gamma(\infty, x, b, k)$ is continuous in (x, b, k) over $R^n \times R^m \times R_+^m$, but this has not as yet been proven.

§7. Nuclei

Recall that for System (I.a), there were no restrictions on the vectors $\{a^i\}$ other than they be unit vectors. Hence, in particular, there is no requirement that they be linearly independent. Suppose, given $\{a^i \mid i = 1, \ldots, m\}$, $a^i \in R^n$, $b \in R^m$, $k \in R_+^m$, we generate a new set of vectors: $\{\bar{a}^i \mid i = 1, \ldots, 2m\}$, $\bar{a}^i \in R^n$, $\bar{b} \in R^{2m}$, $\bar{k} \in R_+^m \times R_+^m = R_+^{2m}$ in the following way:

$$\bar{a}^i = -\bar{a}^{m+i} = a^i \qquad i = 1, \ldots, m$$

$$\bar{b}_i = -\bar{b}_{m+i} = b_i \qquad i = 1, \ldots, m$$

$$\bar{k}_i = \bar{k}_{m+i} = k_i \qquad i = 1, \ldots, m.$$

Using these vectors, we can exhibit the analogue of System (I.a):

$$\dot{x} = -\sum_{i=1}^{2m} \bar{k}_i [\langle \bar{a}^i, x\rangle + \bar{b}_i]^+ \bar{a}_i$$

$$= -\sum_{i=1}^{m} k_i \{[g^i(x, b)]^+ a_i - [-g^i(x, b)]^+ a^i\} \tag{I.d}$$

or

$$\dot{x} = -\sum_{i=1}^{m} k_i (g^i(x, b)) a^i. \tag{I.d'}$$

Similarly, we can define the $\bar{k}$-centroids of $\bar{b}$ (with vectors $\{\bar{a}^i\}$) to be the minimizing points of

$$\Theta(x) = \sum_{i=1}^{2m} \bar{k}_i ([\langle \bar{a}^i, x\rangle + \bar{b}_i]^+)^2 = \sum_{i=1}^{m} \bar{k}_i (g^i(x, b))^2.$$

We will define $N(b, k)$, the set of "k-nuclei of b (with vectors $\{a^i\}$)", to be the set of $\bar{k}$-centroids of $\bar{b}$ (with vectors $\{\bar{a}^i\}$). This definition, while introducing perhaps redundant terminology, stresses the differences between $C(b, k)$ and $N(b, k)$ while indicating that the k-nuclei of b are themselves centroids of a different, albeit related, set of vectors.

It is therefore to be expected that the set of k-nuclei of b would share many of the properties of $C(b, k)$ and this is indeed so. These are listed below for completeness.

Corollary I.19: For any $x_0 \in R^n$, $b \in R^m$, $k \in R^m_+$, there exists a unique solution to System (I.d') which converges to a k-nucleus of b. The set $N(b, k)$ is precisely the set of critical points of (I.d').

Corollary I.20: The set of k-nuclei of b is nonempty and polyhedral. Moreover $(\langle a^i, x\rangle + b_i)$ is constant as x ranges over $N(b, k)$ for $i = 1, \ldots, m$.

Corollary I.21: The set $N(b, k)$ comprises a unique point if $\{a^i | i = 1, \ldots, m\}$ spans R^n.

There is a slightly more general continuity result.

Proposition I.22: Let $\zeta(t, x_0, b, k)$ be a solution of (I.d') with limit point $\zeta(\infty, x_0, b, k)$. If the $\{a^i\}$ span R^n, then $\zeta(\infty, x_0, b, k)$ is continuous in (x_0, b) over $R^n \times R^m$.

Proof: Since $\{a^i\}$ span R^n, the k-nucleus of b is unique for all b. Thus, $\zeta(\infty, x_0, b, k)$ is independent of x_0. Letting A be the matrix with rows $\sqrt{k_i}\, a^i$, we know that the k-nucleus of b, $\zeta(\infty, x_0, b, k)$, is $A^+\beta$ where $\beta \in R^m$, $\beta_i = \sqrt{k_i}\, b_i$ and A^+ is the generalized (pseudo-) inverse of A. The conclusion follows from the observations the $A^+\beta$ is a continuous function of b.

§8. Relationships among Centroids, Cocentroids and Nuclei

We conclude this chapter with a number of observations on the relationships among centroids, cocentroids, and nuclei.

Proposition I.23: If x is an element of any two of $C(b, k)$, $CC(b, k)$, $N(b, k)$, then it is an element of the third.

Proof: Note that

$$-\sum_{i=1}^{m} k_i (\langle a^i, x\rangle + b^i)\, a^i = -\sum_{i=1}^{m} k_i [\langle a^i, x\rangle + b_i]^+ a^i + \sum_{i=1}^{m} k_i [-\langle a^i, x\rangle - b_i]^+ a^i \tag{I.e}$$

so if any two of the summations vanishes, so must the third. □

Therefore, a k-centroid of b is a k-nucleus of b if and only if it is also a k-cocentroid of b, and so on.

Finally, we note some relations among the solutions of Systems (I.a), (I.c) and (I.d). Let $\gamma(t, x_0, b, k)$ be the solution of (I.a) with initial point x_0, $\bar{\gamma}(t, x_0, b, k)$ the solution of System (I.c) with initial point x_0 and $\zeta(t, x_0, b, k)$ be the solution of System (I.d') with initial point x_0. We will say that two functions of t, say $\alpha(t)$, $\beta(t) \in R^n$ are "negatively tangent" at x_0 if $\alpha(0) = \beta(0) = x_0$ and if

$$\frac{d}{dt}(\alpha(t))\,|_{t=0} = -\frac{d}{dt}(\beta(t))\,|_{t=0}.$$

Similarly, $\alpha(t)$ and $\beta(t)$ are "positively tangent" at x_0 if $\alpha(0) = \beta(0) = x_0$ and

$$\frac{d}{dt}(\alpha(t))\,|_{t=0} = \frac{d}{dt}(\beta(t))\,|_{t=0}.$$

The following are simple consequences of Equation (I.e).

Proposition I.24:

a) $x_0 \in C(b, k)$ if and only if $\bar{\gamma}(t, x_0, b, k)$ and $\zeta(t, x_0, b, k)$ are positively tangent at x_0.

b) $x_0 \in CC(b, k)$ if and only if $\gamma(t, x_0, b, k)$ and $\zeta(t, x_0, b, k)$ are positively tangent at x_0.

c) $x_0 \in N(b, k)$ if and only if $\gamma(t, x_0, b, k)$ and $\bar{\gamma}(t, x_0, b, k)$ are negatively tangent at x_0.

II. Applications to Cooperative Game Theory

§1. Cooperative Games with Sidepayments

The concept of an "n-person cooperative game with sidepayments" was introduced in *von Neumann* and *Morgenstern* [1953]. It consists of:

a) $N = \{1, 2, \dots, n\}$, a set of players.

b) $2^N - \emptyset = \{S \neq \emptyset \mid S \subseteq N\}$, all "coalitions" of the players.

c) v: $2^N - \emptyset \to R$, a "characteristic function".

d) Some "set of payoffs" in R^n.

We will define below precisely those sets of payoffs in which we are interested. A game is denoted (N, v), or simply v, with the set N understood.

A payoff $x \in R^n$ represents a potential or actual distribution of some transferable commodity among the players where each player i receives x_i. Certainly not all $x \in R^n$ are logical payoffs. If we denote $\sum_{i \in S} x_i$ by $x(S)$, then among the more reasonable payoff concepts are the following:

Feasible payoffs:	$\{x \in R^n \mid x(N) \leqslant \mathrm{v}(N)\}$
Efficient payoffs:	$\{x \in R^n \mid x(N) = \mathrm{v}(N)\} \equiv E(\mathrm{v})$
S-rational payoffs:	$\{x \in R^n \mid x(S) \geqslant \mathrm{v}(S)\}$
Imputations:	$\{x \in R^n \mid x(N) = \mathrm{v}(N), x_i \geqslant \mathrm{v}(\{i\})$ for all $i = 1, \ldots, n\}$.

Since v (N) represents the amount of the commodity which the entire set of players N can obtain by cooperating, it is not surprising that efficient payoffs are desirable if the game is to result in some sort of stable outcome with all players participating. Each coalitions S, however, is most interested in an end result which is S-rational, and therein often lies the conflict among coalitions over what the final payoff should be. Infeasible points, i.e., those which are not feasible, may be thought of as unattainable by the grand coalition N.

In order to quantify in some way the satisfaction or dissatisfaction of coalition S with a payoff x, denote by $e_S(x)$ the quantity

$$\mathrm{v}(S) - x(S).$$

This quantity is sometimes called the "excess of S at x".

Presumably, the smaller $e_S(x)$, the more satisfied is coalition S with payoff x. Let us also define at this time the "efficient excess of S at x" for $S \neq N, \emptyset$ to be

$$\hat{e}_S(x) = \langle -A^S, x\rangle + \left(\frac{|N|\,\mathrm{v}(S)}{|S| \quad (|N| - |S|)} - \frac{\mathrm{v}(N)}{|N| - |S|} \right)$$

where:

$|S|$ is the cardinality of S,

$|N| = n$, and

$A^S \in R^n$ such that

$$A_i^S = \begin{cases} \dfrac{1}{|S|} & i \in S \\[2ex] \dfrac{-1}{|N| - |S|} & i \notin S. \end{cases}$$

The purpose of this efficient excess will be come clear shortly.

§2. Solution Concepts

A solution concept is a payoff or a set of payoffs which is either (1) equitable with respect to certain axioms of fairness or optimality, or (2) is "stable" with respect to some type of bargaining procedure. Two well-known solution concepts are appropriate to the results of this chapter.

The "core" is the set of efficient points which are S-rational for all S. Explicitly,

$$\text{core}(\mathrm{v}) = \{x \in E(\mathrm{v}) \mid e_S(x) \leqslant 0 \ \text{ for all } \ S \in 2^N - \emptyset\}.$$

The Shapley value is a solution concept which falls into the category of "fair" points. The Shapley value, usually denoted $\phi\,[\mathrm{v}]$, is determined uniquely over the class of all n-person games by the following three axioms.

I. A carrier for a game v is a coalition T such that for all S, $\mathrm{v}(S) = \mathrm{v}(S \cap T)$. the first axiom requires that for any carrier T of v, $\phi\,[\mathrm{v}]\,(T) = \mathrm{v}(T)$.

II. Let π be a permutation on $\{1, \ldots, n\}$. Let $\pi\mathrm{v}$ be the game such that $\pi\mathrm{v}(S) = \mathrm{v}(\pi S)$. For any vector $x \in R^n$ let πx be the vector such that $(\pi x)_i = x_{\pi i}$, $i = 1, \ldots, n$. Then the second axiom requires that

$$\phi(\pi\mathrm{v}) = \pi\phi(\mathrm{v}) \ \text{ for all permutations } \ \pi \ \text{ and all games } \ \mathrm{v}.$$

III. If u and v are two n-person games, let the game $u + \mathrm{v}$ be the game $(u + \mathrm{v})(S) = u(S) + \mathrm{v}(S)$. The third axiom then requires that $\phi_i\,[u + \mathrm{v}] = \phi_i\,[u] + \phi_i\,[\mathrm{v}]$.

Axioms I and II have several well-known consequences which substantiate the notion that the Shapley value is a fair division point. Let us briefly mention two which we will recall later. First, call player i a "dummy" if, for all coalitions S shich do not contain i, $\mathrm{v}(S \cup \{i\}) = \mathrm{v}(S) + \mathrm{v}(\{i\})$. It follows then that $\phi_i\,[\mathrm{v}] = \mathrm{v}(\{i\})$. Second, let us say two players, i and j, are "symmetric" if $\mathrm{v}(\{i\}) = \mathrm{v}(\{j\})$ and for all coalitions S containing neither i nor j, $\mathrm{v}(S \cup \{i\}) = \mathrm{v}(S \cup \{j\})$. Then, by Axiom II, $\phi_i\,[\mathrm{v}] = \phi_j\,[\mathrm{v}]$.

§3. Efficient Bargaining Systems

For $\{A^S \in R^n \mid S \in 2^N - \emptyset\}$ and efficient excesses $\{\hat{e}_S(x) \mid x \in R^n, S \in 2^N - \{\emptyset, N\}\}$ as defined previously, we define an "efficient bargaining system" to be a system of differential equations of the following form:

$$\dot{x} = \sum_{S \in 2^N - \{\emptyset, N\}} k_S \left[\frac{\hat{e}_S(x)}{\|A^S\|}\right]^+ \frac{A^S}{\|A^S\|} \qquad \text{(II.a)}$$

where $\dot{x} = \dfrac{dx}{dt}$ and

$k_S \in R_+$ for all $S \in 2^N - \{\emptyset, N\}$.

Note that we have substituted $2^N - \{\emptyset, N\}$ for a set of integers as the index set of the summation. The set $\{k_S > 0 \mid S \in 2^N - \{\emptyset, N\}\}$ will be called the set of "coalitional weights". $R_+^{2n\text{-}2}$ is clearly the set of all such. The variable t may be considered to stand for time.

It is apparent that System (II.a) is of the same form as System (I.a) so that for any point x_0, there exists a continuous (in t) solution $\gamma(t, x_0, \mathrm{v}, k)$ such that $\gamma(0, x_0, \mathrm{v}, k) = x_0$. Note that along solutions of (II.a)

$$\frac{d}{dt} \sum_{i=1}^{n} \gamma_i(t, x_0, \mathrm{v}, k) = 0$$

so that we can state:

Lemma II.1: If initial point x_0 is efficient, then $\gamma(t, x_0, \mathrm{v}, k)$ is efficient for all t.

Simple manipulation shows

Lemma II.2: For all $S \neq N, \emptyset$, all $x \in E(\mathrm{v})$

$$\frac{\hat{e}_S(x)}{\| A^S \|^2} = e_S(x).$$

It follows that core (v) $= \{x \in E(\mathrm{v}) \mid e_S(x) \leqslant 0 \quad \text{for all } S \neq N\}$

$= \{x \in E(\mathrm{v}) \mid \hat{e}_S \leqslant 0 \quad \text{for all } S \neq N\}$.

Lemmas II.1 and II.2 yield:

Proposition II.3: If initial point $x_0 \in E(\mathrm{v})$ then $\gamma(t, x_0, \mathrm{v}, k)$ with $\gamma(0, x_0, \mathrm{v}, k) = x_0$ is a solution of System (II.a) if and only if it is a solution of the following system:

$$\dot{x} = \sum_{S \neq N} k_S [e_S(x)]^+ A^S. \tag{II.b}$$

It is informative to give an intuitive interpretation of System (II.b) in terms of possible actions of the players in the game. We will, in general, refer to such an interpretation as a "behavior". It should be noted that, in this context, "behavior" is not intended to be a rigorous concept, but only an aid to intuition.

Suppose, during negotiation among the players to determine the final distribution of the payoff, some efficient payoff x is offered. Since the players participate in the game through coalitions, it is the task of the coalitions, through demands or some other tactic, to alter x to obtain a more desirable payoff. Let us assume coalition S evaluates x by observing $e_S(x)$, and on that basis decides whether to demand more from its complimentary set, i.e., the remaining players. If $e_S(x) \leqslant 0$, coalition S is receiving at least as much as it is worth (according to the characteristic function) and therefore cannot enforce a demand on $N - S$. If $e_S(x) > 0$, however, we will permit S to extract payment from $N - S$ at a rate proportional to $e_S(x)$. It is understood, of course, that $N - S$ will be permitted to extract payment from S if $e_{N\text{-}S}(x) > 0$. The term $k_S [e_S(x)]^+$ in (II.b) represents the rate of payment from $N - S$ to S. The multiple k_S is just the constant of proportionality. Since all members of a coalition participate equally in the activities of that coalition, each member of S receives $\frac{1}{|S|} k_S [e_S(x)]^+$ while each member of $N - S$ pays $\frac{1}{|N| - |S|} k_S [e_S(x)]^+$. This ensures that the total payoff $x(N)$ remains constant. Summing all these payments over all coalitions of $2^N - \{N, \emptyset\}$, the total rate of redistribution of payoff is clearly

$$\sum_{S \neq N} k_S \, [e_S(x)]^+ A^S.$$

The grand coalition N is excluded from the summation since there is no one from whom N can extract payment. In addition, by choosing efficient initial points, the coalition N always receives satisfactory payment.

In light of the previous discussion, it would not be unreasonable to view the coalitional weights as some measure of a coalition's ability to extract payment from its complementary coalition; in other words, its "influence." Such heuristic interpretations will be given from time to time although no attempt will be made in this work to make these more rigorous. The coalitional weights will be studied later as a means by which certain notions of fairness in bargaining can be enforced.

§4. Centroids for Games

We will define k-centroids of a game v in a somewhat more restrictive way than in Chapter I. The added constraint will be seen to cause no great difficulty.

Let v be an n-person game, and $k \in R_+^{2^n-2}$. Define $C(\text{v}, k)$, the set of "k-centroids of v" to be the set

$$\{x \in E(\text{v}) \mid \Phi'(x, \text{v}, k) = \inf_{y \in E(\text{v})} \Phi'(y, \text{v}, k)\}$$

where

$$\Phi'(x, \text{v}, k) = \sum_{S \neq N} k_S \left(\left[\frac{\hat{e}_S(x)}{\| A^S \|} \right]^+ \right)^2 .$$

Had we defined the k-centroid of v as in Chapter I, that is by omitting the constraint $x(N) = \text{v}(N)$, the nature of $\{A^S\}$ would make it clear that the set of unconstrained centroids would be precisely $\{C(\text{v}, k) + \lambda u \mid -\infty < \lambda < \infty\}$ where u is the unit vector normal to $E(\text{v})$; i.e., $C(\text{v}, k)$ is the projection of the set of unconstrained centroids onto $E(\text{v})$. This is because $\langle A^S, u \rangle = 0$ for all $S \neq N$. We can therefore drop the inf and substitute min from now on.

Proposition II.4: $\bar{x}$ is a k-centroid of v if and only if $\bar{x}$ minimizes

$$\Phi(x, \text{v}, k) = \sum_{S \neq N} k_S \, \| A^S \|^2 \, ([e_S(x)]^+)^2$$

over $E(\text{v})$.

Proof: Lemma II.2 shows that over $E(\text{v})$, $\Phi = \Phi'$. □

For $x \in E(\text{v})$, let us call $k_S \, \| A^S \|^2 \, ([e_S(x)]^+)^2$ the "dissatisfaction of S at x", and $\Phi(x, \text{v}, k)$ the "total dissatisfaction at x".

The set $\{S \mid e_S(x) > 0\}$ will be the "set of dissatisfied coalitions". Using this terminology, $C(\text{v}, k)$ is the set of efficient payoffs which minimize total dissatisfaction, while core (v) consists of those efficient points at which total dissatisfaction is 0. As in Chapter I, if core (v) $\neq \emptyset$, core (v) $= C(\text{v}, k)$.

Lemma II.5: For all $S \neq N$, the dissatisfaction of S at x is constant as x ranges over $C(\text{v}, k)$.

Proof: See Corollary I.6. □

Therefore, a dissatisfied coalition S is indifferent to variations of payoff over $C(\text{v}, k)$ since $e_S(x)$ will remain constant. It is interesting that the set of dissatisfied coalitions is the same for all k-centroids of v for a given k, i.e., it is impossible to satisfy any such S without raising the total dissatisfaction.

Under this interpretation, the coalitional weights could be viewed as measures of the coalitions' sensitivities to *not* receiving their values – the larger k_S, the more dissatisfied is S at any given payoff.

Proposition II.6: $C(\text{v}, k)$ is a nonempty closed polytope.

Proof: By Proposition I.5, $C(\text{v}, k)$ is a closed polyhedron. Suppose it is not compact, then it contains some half line $\{y_0 + ru \mid r \geqslant 0,\ y_0 \in C(\text{v}, k),\ u \neq 0\}$. Since $C(\text{v}, k) \subset E(\text{v})$, it follows that $\sum_{i=1}^{n} u_i = 0$.

By Lemma II.5

$$[e_S(y_0 + ru)]^+ = [e_S(y_0)]^+ \qquad \text{for all } r \geqslant 0 \text{ and all } S \in 2^N - \{N, \emptyset\}$$

equivalently

$$[e_S(y_0) - ru(S)]^+ = [e_S(y_0)]^+ \text{ for all } r \geqslant 0 \text{ and } S \in 2^N - \{N, \emptyset\}.$$

Therefore

$$u(S) \geqslant 0 \text{ for all } S \text{ such that } e_S(y_0) \leqslant 0$$

$$u(S) = 0 \text{ for all } S \text{ such that } e_S(y_0) > 0$$

or in any case

$$u(S) \geqslant 0 \text{ for all } S \in 2^N - \{N, \emptyset\}.$$

This combined with $u(N) = 0$ implies $u \equiv 0$ contradicting the previous assumption that $u \neq 0$. □

We complete this section with a characterization of the collection of dissatisfied coalitions at a k-centroid.

Shapley [1967] defined the notion of a balanced collection of sets. Given a collection $\mathcal{S}$ of subsets S of a set N, $\mathcal{S}$ is said to be balanced if there exists $\{c_S > 0 | S \in \mathcal{S}\}$ such that $\sum_S c_S a^S = a^N$ where $(a^S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$. *Shapley* noted that a balanced collection could be considered a generalized partition.

Proposition II.7: Let $\mathcal{S}$ be a collection of subsets S of a set N. Then $\mathcal{S}$ is balanced if and only if there exist $\{d_S > 0 \mid S \in \mathcal{S}\}$ such that $\sum_{S \in \mathcal{S}} d_S A^S = 0$.

Proof: $\mathcal{S}$ is balanced if and only if there exists $\{c_S > 0 \mid S \in \mathcal{S}\}$ such that $\sum_S c_S a^S = a^N$. Note that $\sum_S c_S a^S$ can never be 0 whenever the family $\mathcal{S}$ is nonempty. Thus $\mathcal{S} \neq \emptyset$ is balanced if and only if there exists $\{c_s > 0 \mid S \in \mathcal{S}\}$ such that

$$\sum_S c_S a^S - \left\langle \sum_S c_S a^S, \frac{a^N}{\sqrt{|N|}} \right\rangle \frac{a^N}{\sqrt{|N|}} = 0$$

But

$$\sum_S c_S a^S - \left\langle \sum_S c_S a^S, \frac{a^N}{\sqrt{|N|}} \right\rangle \frac{a^N}{\sqrt{|N|}} = \sum_S c_S \left(a^S - \langle a^S, a^N \rangle \frac{a^N}{|N|} \right)$$

$$= \sum_S c_S \left(a^S - \frac{|S|}{|N|} a^N \right) = \sum_S c_S \frac{1}{\|A^S\|^2} A^S.$$

So, by putting $d_S = \dfrac{c_S}{\|A^S\|^2}$, we can see that $\mathcal{S}$ is balanced if and only if there exists $\{d_S > 0 \mid S \in \mathcal{S}\}$ such that $\sum_S d_S A^S = 0$. □

Corollary II.8: The collection of dissatisfied coalitions at a k-centroid is balanced.

Proof: In the above proposition, put $d_S = k_S\, [e_S(x)]^+$ for all dissatisfied S, where x is any k-centroid of v.

§5. Convergence

Let us restate the convergence results of Chapter I in terms of games.

Proposition II.9: Let v be a game and $\{k_S\}$ any set of coalitional weights. For any $x_0 \in E(\mathrm{v})$, there exists a solution $\gamma(t, x_0, \mathrm{v}, k)$, continuous in t such that $\lim_{t \to \infty} \gamma(t, x_0, \mathrm{v}, k)$, exists and is a k-centroid of v.

As before, denote this limit point by $\gamma(\infty, x_0, \mathrm{v}, k)$. Thus bargaining as described above in which dissatisfied coalitions extract payment from complementary coalitions results in a redistribution of the total payoff $\mathrm{v}(N)$ over time in such a way that, as $t \to \infty$, the distribution converges to one which minimizes total dissatisfaction. Recall that this convergence is such that $\gamma(t, x_0, \mathrm{v}, k)$ approaches all k-centroids of v simultaneously as t increases, and also follows the negative gradient of $\Phi'(x, \mathrm{v}, k)$. $\nabla \Phi(x, \mathrm{v}, k)$, on the other hand, does not, in general, lie in the hyperplane $\{x \mid x(N) = 0\}$ as does $\nabla \Phi'$. However, a simple computation demonstrates that for any $x \in E(\mathrm{v})$, $\nabla \Phi'(x, \mathrm{v}, k)$ is the projection of $\nabla \Phi(x, \mathrm{v}, k)$ onto $\{x \mid x(N) = 0\}$. In this sense, $\gamma(t, x_0, \mathrm{v}, k)$ follows the negative gradient of the total dissatisfaction function. Therefore, while this type of behavior may not result in a "shortest route" in Euclidean distance to a k-centroid, which would translate into "minimum total exchange of payoff", it is optimal in the sense that it produces, at any x, a rate of redistribution which is most effective in reducing total dissatisfaction locally, i.e., in small enough neighborhoods of x. Hence, players employing an efficient bargaining system arrive at a global optimum by acting in a locally optimal manner.

Also, with respect to efficient bargaining systems, it is clear that, individually, each k-centroid of v is a stable point and, if we define a set to be asymptotically stable if all points of the set are stable, and if all trajectories converge to a point of the set then $C(\mathrm{v}, k)$ is asymptotically stable. In particular, the core, if nonempty, is asymptotically stable with respect to this system.

§6. Cocentroids

In the manner of Chapter I, we will define k-cocentroids of a game v. While it may appear in the model we are using that cocentroids are highly nonoptimal and therefore perhaps uninteresting, it will become evident that, in some cases, these "worst" points will bear an important relationship to the optimal centroids and certain "fair" points.

Given a game v, coalitional weights $\{k_S\}$, and some efficient point x, we will call

$$k_S \, \| A^S \|^2 \left([-e_S(x)]^+ \right)^2$$

the "satisfaction" of S at x, and we will also call

$$\Psi(x, \mathrm{v}, k) = \sum_{S \neq N} k_S \, \| A^S \|^2 \left([-e_S(x)]^+ \right)^2$$

the "total satisfaction" at x. $\{S \mid e_S(x) \leqslant 0\}$ will be the set of "satisfied coalitions" at x. The set of "k-cocentroids of v", $CC(\mathrm{v}, k)$ is the set

$$\{x \in E(\mathrm{v}) \mid \Psi(x, \mathrm{v}, k) = \min_{y \in E(\mathrm{v})} \Psi(y, \mathrm{v}, k)\}.$$

Although cocentroids are those points which minimize total satisfaction, it does not necessarily follow that total dissatisfaction is large over $CC(\mathrm{v}, k)$, since we will see in

Section §12 of this Chapter that $C(\mathrm{v}, k)$ and $CC(\mathrm{v}, k)$ can, under certain conditions, coincide.

Clearly, it is possible to display a system of differential equations

$$\dot{x} = -\sum_{S \neq N} k_S \, [-e_S(x)]^+ A^S, \tag{II.c}$$

the solutions of which, for any efficient initial point, converge to a k-cocentroid of v. A behavior for such a system would be one in which satisfied coalitions are donating payoffs to their complements at a rate proportional to $k_S\,[-e_S(x)]^+$ while dissatisfied coalitions are silent, achieving, in the limit, a final distribution which minimizes total satisfaction.

An argument entirely similar to that of Proposition II.6 yields

Proposition II.10: $CC(\mathrm{v}, k)$ is a nonempty closed polytope.

It is also clear that $e_S(x)$ is constant over $CC(\mathrm{v}, k)$ for all satisfied coalitions S.

§7. Continuity

Let $x_0 \in E(\mathrm{v})$, and let $\gamma(t, x_0, \mathrm{v}, k)$ be a solution of System (II.b). We have already shown that as $t \to \infty$, this solution converges to a point $\gamma(\infty, x_0, \mathrm{v}, k) \in C(\mathrm{v}, k)$. Propositions I.16 and I.18 establish the following results for games.

Proposition II.11: For any game v and any set of coalitional weights $\{k_S\}$, $\gamma(\infty, x_0, \mathrm{v}, k)$ is continuous in x_0 over $E(\mathrm{v})$.

Proposition II.12: Let

$$W = \{\mathrm{v} \mid \mathrm{core}(\mathrm{v}) \neq \emptyset\},$$

then $\gamma(\infty, x_0, \mathrm{v}, k)$ is continuous in (x_0, v, k) over

$$X = \{(x, \mathrm{v}, k) \mid x \in E(\mathrm{v}),\ \mathrm{v} \in W,\ k \in R_+^{2^n-2}\}$$

Proof: Note the added restriction that $x_0 \in E(\mathrm{v})$, and also $\mathrm{core}(\mathrm{v}) \subset E(\mathrm{v})$. Thus the proof of Proposition I.18 must be modified slightly using the observation that if $\{\mathrm{v}^n\} \to \mathrm{v}$ then $\mathrm{core}(\mathrm{v}^n) \to \mathrm{core}\ \mathrm{v}$ from *Dantzig*, et al. [1967] and also, despite $E(\mathrm{v}^n)$ not being compact, $\mu(E(\mathrm{v}^n), E(\mathrm{v})) \to 0$. Then the proof essentially goes as that for Proposition I.18. □

§8. Allocation Systems and Nuclei

Suppose for a game v and set of coalitional weights $\{k_S\}$, we were to combine the two systems (II.b) and (II.c), much as we did in Chapter I, to obtain

$$\dot{x} = \sum_{S \neq N} k_S (e_S(x)) A^S \qquad \text{(II.e)}$$

such a system will be called an "efficient allocation system". The behavior it represents is straightforward: satisfied coalitions are giving to their complements their excess payoff while dissatisfied coalitions are extracting payment from their complements. Note that in general a coalition S being dissatisfied does not necessarily imply that $N-S$ is satisfied or conversely. However, in the case that core (v) $\neq \emptyset$, it is true that $e_S(x) > 0$ implies $e_{N\text{-}S}(x) < 0$ (for proof, see *Wang* [1974], Lemma 2.1) so that dissatisfied coalitions are always demanding payment from coalitions who "can afford it".

We define $N(\mathrm{v}, k)$ to be the set of k-nuclei of v which is the set

$$\{x \in E(\mathrm{v}) \mid \Theta(x, \mathrm{v}, k) = \min_{y \in E(\mathrm{v})} \Theta(y, \mathrm{v}, k)\}$$

where

$$\Theta(x, \mathrm{v}, k) = \sum_{S \neq N} k_S \, \| A^S \|^2 \, (e_S(x))^2 .$$

We will call $\Theta(x, \mathrm{v}, k)$ the total "disorder" of the game at x, and it is clear that total disorder is the sum of total satisfaction and total dissatisfaction. A k-nucleus of v is therefore a point which minimizes total disorder. As with centroids and cocentroids, the k-nuclei fall into the class of "convex nuclei" proposed by *Charnes* and *Kortanek* [1970].

Proposition II.13: Let $\zeta(t, x_0, \mathrm{v}, k)$ be a solution of System (II.e) with efficient initial point x_0. Then as $t \to \infty$ $\zeta(t, x_0, \mathrm{v}, k)$ converges to a k-nucleus of v.

Proof: This follows from Corollary I.19. □

Further it should be apparent that total disorder will decrease along solutions of (II.e).

From Corollary I.20, $e_S(x)$ is constant as x ranges over $N(\mathrm{v}, k)$ for all $S \neq N$. Since any given set of excesses determines a unique payoff we have

Proposition II.14: For any game v, and any of coalitional weights $\{k_S\}$, $N(\mathrm{v}, k)$ contains a unique point.

By Proposition I.23, we can state the following.

Proposition II.15: Let $x \in E(\mathrm{v})$. Then x being in any two of $C(\mathrm{v}, k)$, $CC(\mathrm{v}, k)$, and $N(\mathrm{v}, k)$ implies x is in the third.

So if x minimizes both total dissatisfaction and total disorder, then x must minimize total satisfaction also.

The sets $C(\mathrm{v}, k)$, $CC(\mathrm{v}, k)$ and $N(\mathrm{v}, k)$ can also be characterized by the tangency of solutions of the Systems (II.b), (II.c), and (II.e) as in Proposition I.24. Such a result gives information on the various behaviors of the players at payoffs in these sets. For instance, players with a distribution $x \in CC(\text{v}, k)$, i.e., where total satisfaction is minimized, will act in the same way, instantaneously at x, as if to arrive ultimately at $C(\text{v}, k)$ or $N(\text{v}, k)$, although the trajectories will diverge as soon as they leave $CC(\text{v}, k)$.

§9. Coalitional Weights

Some possible interpretations of the coalitional weights have been already mentioned, and it is not difficult to list more, e.g., k_S could be the probability of coalition S forming, giving the term $k_S \| A^S \| ([e_S(x)]^+)^2$ a possible interpretation of "expected dissatisfaction." Similar interpretations have been used by other writers with respect to other weighting schemas. See, for example, *Owen* [1968]. Unfortunately, notions such as "influence" of "sensitivity" or "probability of a coalition forming" are difficult to quantify. Suppose instead, we view the coalitional weights as a mechanism whereby we can impose some concept of "fairness" on the bargaining. In this section, this idea of fairness will be made rigorous by axioms, not unlike those in the definition of the Shapley value. Necessary and sufficient conditions on the coalitional weights will be deduced in order for these axioms to hold. In this manner, we will obtain a set of "universal" coalitional weights, i.e., weights which are not functions of the game v. Note that this has tacitly been assumed in the previous sections of this work although it would be of interest to see what sort of results one could derive if k_S were a function of v, e.g., if $k_S \approx \text{v}(S)$. Such an analysis will not be undertaken here.

Let $\dot{x} = D(x, \text{v})$ be either (II.b) or (II.e). (The result also holds for System (II.c), but this fact is not of much interest.) We would like to enforce the notion that bargaining depends only on the characteristic function, rather than on the labelling of the players. We can do that with the following axiom. Recall that for $x \in R^n$, we denote by πx the vector in R^n such that $(\pi x)_i = x_{\pi i}$, $i = 1, \ldots, n$.

A. If π is any permutation on $\{1, \ldots, n\}$, then we require

$$D(\pi x, \pi \text{v}) = \pi D(x, \text{v})$$

for all n-person games v and all efficient points x.

Proposition II.16: A necessary and sufficient condition for Axiom A to hold is that $k_S = k_T$ whenever $|S| = |T|$. Such a set of coalitional weights will be denoted $\{k_{|S|}\}$.

Proof: We will prove this result for efficient bargaining systems only. The proof for efficient allocation systems is entirely analogous.

Necessity: Pick any $\gamma \in R^n$, and $S_0 \neq N$. Let v be the game given by $\text{v}(S) = \gamma(S)$ for all $S \neq S_0$ and $\text{v}(S_0) = \gamma(S_0) + \alpha$, for some $\alpha > 0$. Let π be any permutation on $\{1, \ldots, n\}$, then

$$D(\gamma, \mathrm{v}) = \sum_{S \neq N} k_S [\mathrm{v}(S) - \gamma(S)]^+ A^S = (k_{S_0} \cdot \alpha) A^{S_0}$$

$$D(\pi\gamma, \pi\mathrm{v}) = \sum_{T \neq N} k_T [\pi\mathrm{v}(T) - \pi\gamma(T)]^+ A^T.$$

The only non-zero term in this latter sum is for $\pi T = S_0$ or $T = \pi^{-1} S_0$, i.e.,

$$D(\pi\gamma, \pi\mathrm{v}) = (k_{\pi^{-1} S_0} \cdot \alpha) A^{\pi^{-1} S_0}.$$

Note that $\pi^{-1} A^{\pi^{-1} S_0} = A^{S_0}$, so if Axiom A is to hold, $k_{\pi^{-1} S_0} = k_{S_0}$. Observe that for all permutations π, $| \pi^{-1} S_0 | = | S_0 |$. Thus since S_0 was arbitrary, necessity must follow.

Sufficiency: Let v be any game, and x any point in $E(\mathrm{v})$.
Then

$$D(x, \mathrm{v}) = \sum_{S \neq N} k_{|S|} [\mathrm{v}(S) - x(S)]^+ A^S$$

$$D(\pi x, \pi\mathrm{v}) = \sum_{T \neq N} k_{|T|} [\pi\mathrm{v}(T) - \pi x(T)]^+ A^T.$$

In the latter sum let $T = \pi^{-1} S$, so

$$D(\pi x, \pi\mathrm{v}) = \sum_{\pi^{-1} S \neq N} k_{|\pi^{-1} S|} [\pi\mathrm{v}(\pi^{-1} S) - \pi x(\pi^{-1} S)]^+ A^{\pi^{-1} S}$$

$$= \sum_{\pi^{-1} S \neq N} k_{|S|} [\mathrm{v}(S) - x(S)]^+ A^{\pi^{-1} S}$$

$$= \sum_{S \neq N} k_{|S|} [\mathrm{v}(S) - x(S)]^+ A^{\pi^{-1} S} \quad \text{so therefore}$$

$$\pi^{-1} D(\pi x, \pi\mathrm{v}) = \sum_{S \neq N} k_{|S|} [\mathrm{v}(S) - x(S)] A^{\pi^{-1} S} = D(x, \mathrm{v}). \quad \square$$

This result has pleasant consequences for symmetric players. For convenience, let us adopt the following convention: given two players i and j, let us call player i "as powerful as" player j (denote by $i \gg j$) if $\mathrm{v}(\{i\}) \geqslant \mathrm{v}(\{j\})$ and for all S containing neither i nor j, $\mathrm{v}(S \cup \{i\}) \geqslant \mathrm{v}(S \cup \{j\})$.

Lemma II.17: Given coalitional weights $\{k_{|S|}\}$, if $i \gg j$ and $x \in R^n$ such that $x_i \leqslant x_j$, then $D_i(x, \mathrm{v}) \geqslant D_j(x, \mathrm{v})$.

Proof: Again, the proof is for efficient bargaining systems only. For allocation systems the proof is similar.

$$D(x, \mathrm{v}) = \sum_{\{S | i \notin S, \ldots\}} k_{|S|} [e_S(x)]^+ A^S$$

$$+ k_{|S|+1} [e_{S\cup\{i\}}(x)]^+ A^{S\cup\{i\}}$$

$$+ k_{|S|+1} [e_{S\cup\{j\}}(x)]^+ A^{S\cup\{j\}}$$

$$+ k_{|S|+2} [e_{S\cup\{i\}\cup\{j\}}(x)]^+ A^{S\cup\{i\}\cup\{j\}}$$

$$+ k_2 [e_{\{ij\}}(x)]^+ A^{\{ij\}} + k_1 [e_{\{i\}}(x)]^+ A^{\{i\}} + k_1 [e_{\{j\}}(x)]^+ A^{\{j\}}.$$

Therefore

$$D_i(x, \mathrm{v}) - D_j(x, \mathrm{v}) = \sum_{\substack{\{S|i\notin S\\ j\notin S\}}} k_{|S|+1} \left\{ [e_{S\cup\{i\}}(x)]^+ \left(\frac{1}{S+1}\right)\right.$$

$$- [e_{S\cup\{j\}}(x)]^+ \left(\frac{1}{|N|-|S|-1}\right) - [e_{S\cup\{j\}}(x)]^+ \left(\frac{1}{|S|+1}\right)$$

$$\left. + [e_{S\cup\{i\}}]^+ \left(\frac{1}{|N|-|S|-1}\right)\right\}$$

$$+ k_1 [-x_i + \mathrm{v}(\{i\})]^+ \left(1 - \frac{1}{|N|}\right)$$

$$- k_1 [-x_j + \mathrm{v}(\{j\})]^+ \left(1 - \frac{1}{|N|}\right)$$

$$= \sum_{\substack{\{S|i\notin S\\ j\notin S\}}} k_{|S|+1} \left(\frac{1}{|S|+1} + \frac{1}{|N|-|S|-1}\right) ([-x(S)-x_i+\mathrm{v}(S\cup\{i\})]^+$$

$$- [-x(S) - x_j + \mathrm{v}(S\cup\{j\})]^+)$$

$$+ k_1 \left(1 - \frac{1}{|N|-1}\right) ([-x_i + \mathrm{v}(\{i\})]^+ - [-x_j + \mathrm{v}(\{j\})]^+).$$

But we assumed $\quad -x_i + \mathrm{v}(\{i\}) \geqslant -x_j + \mathrm{v}(\{j\})$

and

$$-x_i + \mathrm{v}(S\cup\{i\}) \geqslant -x_j + \mathrm{v}(S\cup\{j\})$$

for all S such that $i \notin S$ and $j \notin S$,

so $\quad D_i(x, \mathrm{v}) - D_j(x, \mathrm{v}) \geqslant 0.$ □

Proposition II.18: Suppose $i \gg j$ and $x_0 \in E(\mathrm{v})$ such that $(x_0)_i \geqslant (x_0)_j$. If $\gamma(t, x_0)$ is a solution of $\dot{x} = D(x, \mathrm{v})$ with initial point x_0, then

$$\gamma_i(t, x_0) \geqslant \gamma_j(t, x_0) \quad \text{for all } t \geqslant 0,$$

and in particular $\gamma_i(\infty, x_0) \geqslant \gamma_j(\infty, x_0)$.

Proof: Suppose that for some $t' < \infty, \gamma_i(t', x_0) < \gamma_j(t', x_0)$.
Let $t_0 = \max \{0 \leqslant t \leqslant t' | \gamma_i(t, x_0) \geqslant \gamma_j(t, x_0)\}$. Since γ is continuous in t, it follows from the Mean Value Theorem that there exists a t_1 in the open interval $(t_0, t'$ such that

$$\frac{d}{dt}[\gamma_i(t, x_0) - \gamma_j(t, x_0)] \Big|_{t=t_1} = D_i(\gamma(t_1, x_0), \mathrm{v}) - D_j(\gamma(t_1, x_0), \mathrm{v})) < 0.$$

But $\gamma_i(t_1, x_0) < \gamma_j(t_1, x_0)$ by choice of t_0, so by Lemma II.17, $D_i(\gamma(t_1, x_0), \mathrm{v}) - D_j(\gamma(t_1, x_0), \mathrm{v}) \geqslant 0$. This contradiction invalidates the assumption on the existence of t'. □

So, if a player i is as powerful as a player j, and receives at least as much at the outset of bargaining as j, then at no time in bargaining (or allocation) will player i do worse than player j.

Corollary II.19: Given coalitional weights $\{k_{|S|}\}$, if players i and j are symmetric, and $(x_0)_i = (x_0)_j$, then $\gamma_i(t, x_0) = \gamma_j(t, x_0)$ for all $t \geqslant 0$. In particular $\gamma_i(\infty, x_0) = \gamma_j(\infty, x_0)$.

Thus, Axiom A preserves symmetric payoffs to symmetric players, and, when enforced, results in solutions of efficient bargaining systems or efficient allocation systems which reflect the power of the players as indicated by their marginal effect on coalitional strength.

Now suppose we have a dummy player i, who, at some payoff x_0, receives $\mathrm{v}(\{i\})$. There would not seem to be any reason for i to receive any more or less than $\mathrm{v}(\{i\})$ at any future point in the bargaining. This is the essence of Axiom B.

B. For any game v, if i is a dummy player and $x \in E(\mathrm{v})$ where $x_i = \mathrm{v}(\{i\})$, then $D_i(x, \mathrm{v}) = 0$.

Proposition II.20: A necessary and sufficient condition for Axiom B to hold for efficient bargaining or allocation systems is that for all S such that $i \notin S \neq N - \{i\}$.

$$\frac{k_{S \cup \{i\}}}{|S| + 1} = \frac{k_S}{|N| - |S|}.$$

Proof: Again, we give the proof only for bargaining systems.
Necessity: Pick $\gamma \in R^n$ and some $S_0 \in 2^N - N$, where $i \notin S_0 \neq N - \{i\}$.
Let v be the game

$$\mathrm{v}(S_0) = \gamma(S_0) + \alpha \qquad \text{for some } \alpha > 0$$

$$\mathrm{v}(S_0 \cup \{i\}) = \gamma(S_0 \cup \{i\}) + \alpha \quad \text{and}$$

$$\mathrm{v}(S) = \gamma(S) \qquad\qquad \text{for all other } S.$$

For B to hold we must have

$$0 = D_i(\gamma, \mathrm{v}) = k_{S_0}\,[\alpha]^+ A_i^{S_0} + k_{S_0 \cup \{i\}}[\alpha]^+ A_i^{S_0 \cup \{i\}}$$

$$= (k_{S_0} \cdot \alpha) - \left(\frac{1}{|N| - |S_0|}\right) + (k_{S_0 \cup \{i\}} \cdot \alpha)\left(\frac{1}{|S_0| + 1}\right)$$

so

$$\frac{k_{S_0}}{|N| - |S_0|} = \frac{k_{S_0 \cup \{i\}}}{|S_0| + 1}.$$

But S_0 was arbitrary, and B must hold for all games v, so this part of the proof is complete.

Sufficiency: Let v be any game with dummy player i, $x \in E(\mathrm{v})$ such that $x_i = \mathrm{v}(\{i\})$.

Then

$$D(x, \mathrm{v}) = \sum_{\{S : i \notin S \neq N - \{i\}\}} \{k_S\,[\mathrm{v}(S) - x(S)]^+ A^S$$

$$+ k_{S \cup \{i\}}[\mathrm{v}(S \cup \{i\}) - x(S \cup \{i\})]^+ A^{S \cup \{i\}}\}$$

$$+ k_{\{i\}}[\mathrm{v}(\{i\}) - x_i]^+ A^{\{i\}}$$

$$+ k_{N - \{i\}}[\mathrm{v}(N - \{i\}) - x(N - \{i\})]^+ A^{N - \{i\}}.$$

Note that since x is efficient and i is a dummy

$$\mathrm{v}(N - \{i\}) - x(N - \{i\}) = \mathrm{v}(N) - \mathrm{v}(\{i\}) - x(N) + x(\{i\}) = 0,$$

so that

$$D(x, \mathrm{v}) = \sum_{\{S : i \notin S \neq N - \{i\}\}} \left\{ - k_S\,[\mathrm{v}(S) - x(S)]^+ \left(\frac{1}{|N| - |S|}\right) \right.$$

$$\left. + k_{S \cup \{i\}}[\mathrm{v}(S \cup i) - x(S) - x_i]^+ \left(\frac{1}{|S| + 1}\right) \right\}$$

$$= \sum_{\{S : i \notin S \neq N - \{i\}\}} \frac{k_{S \cup \{i\}}}{|S| + 1} \{[\mathrm{v}(S) + \mathrm{v}(i) - x(S) - x(i)]^+ - [\mathrm{v}(S) - (S)]^+\}. \tag{II.f}$$

When $x_i = \mathrm{v}(\{i\})$, this sum is zero. □

The next proposition give us some indication of how dummies fare along trajectories.

Proposition II.21: Suppose v is a game with dummy i, $x \in E(\mathrm{v})$. Then

$$x_i \geqslant \mathrm{v}(\{i\}) \text{ implies } D_i(x, \mathrm{v}) \leqslant 0$$

$$x_i \leqslant \mathrm{v}(\{i\}) \text{ implies } D_i(x, \mathrm{v}) \geqslant 0.$$

Proof: This follows directly from Equation (II.f). □

So, along trajectories, the amount received by a dummy will tend to decrease monotonically, if it is more than the dummy's value, or will increase monotonically if it is less.

Corollary II.22: Let $\gamma(t, x_0)$ be a solution to $\dot{x} = D(x, \mathrm{v})$ with initial point x_0. If i is a dummy and $(x_0)_i = \mathrm{v}(\{i\})$, then $\gamma_i(t, x_0) = \mathrm{v}(\{i\})$ for all $t \geqslant 0$. In particular $\gamma_i(\infty, x_0) = \mathrm{v}(\{i\})$.

Suppose we wish to have both Axioms A and B hold. Then we can inductively construct the coalitional weights as follows (where we denote k_S by k_α when $|S| = \alpha$):

$$k_1 = w \quad \text{for some } w > 0$$

$$k_2 = w \cdot \frac{2}{|N| - 1}$$

$$k_3 = w \cdot \frac{2}{|N| - 1} \cdot \frac{3}{|N| - 2}$$

clearly

$$k_{|S|} = w \; \frac{|S|! \; (|N| - |S|)!}{(|N| - 1)!} \, .$$

If we set $c = \dfrac{w}{|N|}$ we have

Proposition II.23: A necessary and sufficient condition for Axioms A and B to hold is that for all $S \neq N$ or $\emptyset$, $k_S = c \begin{pmatrix} |N| \\ |S| \end{pmatrix}^{-1}$, for some $c > 0$.

The constant c only determines the speed of convergence of the solutions, which can be taken into account by a change in the time variable. Therefore the constant c will be omitted henceforth.

§10. The Shapley Value as a k-Nucleus of v

Recall that the Shapley value is an efficient payoff which reflects the symmetry of the game and which gives dummies their marginal values. In light of the above discussion, it is apparent that the Shapley value is an excellent choice as an initial point for many bargaining systems. This is particularly true in those cases where the Shapley value is not a point of $C(\mathrm{v}, k)$. Then, by applying the bargaining system with the above coalitional weights, the limit distribution of payoff will be one reflecting the same desirable symmetries and payoffs to dummies as the Shapley value, but with lower total dissatisfaction. Note that this proves the existence of such a point.

The allocation system converges to a point which minimized total entropy. We will now show the relationship between the Shapley value and the k-nucleus of v for the "fair" coalitional weights

"fair" coalitional weights $\left\{\binom{|N|}{|S|}\right\}$ We first need the following result of *Keane* [1969] (Section 7):

Lemma II.24: The Shapley value is the unique efficient point minimizing

$$\sum_{S \neq N} \binom{|N|-2}{|S|-1}^{-1} (e_S(x))^2 \quad \text{subject to} \quad x(N) = \mathrm{v}(N).$$

Proposition II.25: The Shapley value $\phi[\mathrm{v}]$ is the unique k-nucleus of v, if for all $S \neq N$ or $\emptyset$

$$k_S = \binom{|N|}{|S|}^{-1}.$$

Proof: This follows immediately from the observation that

$$\binom{|N|}{|S|}^{-1} \|A^S\|^2 = \frac{1}{|N|-1} \binom{|N|-2}{|S|-1}^{-1} \qquad \text{for all } S \neq N. \ \square$$

Hence, for any efficient initial point, the solutions of an allocation system with coalitional weights $\left\{\binom{|N|}{|S|}\right\}^{-1}$ converge to the Shapley value, demonstrating that the Shapley value is asymptotically stable with respect to this system.

The difference between the dynamics of the bargaining and allocation systems provides insight into the difference between $C(\mathrm{v}, k)$ (or core (v)) and the Shapley value. $C(\mathrm{v}, k)$ is, in essence a "greedy" solution concept, since the information about negative excesses is supressed. Coalitions act only to minimize dissatisfaction, ignoring how much over their values certain coalitions may be receiving at any point. The Shapley value, on the other hand, arises when coalitions seek payoffs as close to their values as possible, with the coalitional weights $\binom{|N|}{|S|}^{-1}$ determining which coalitions must be the closest.

Proposition II.15 yields a condition for the Shapley value to be a centroid.

Proposition II.26: $\phi\,[\mathrm{v}] \in C(\mathrm{v}, k)$ for $k = \binom{|N|}{|S|}^{-1}$ if and only if $\phi\,[\mathrm{v}] \in CC(\mathrm{v}, k)$.

Suppose core (v) $\neq \emptyset$ and $\phi\,[\mathrm{v}]$ is in the core. Then it is the unique core point which *minimizes* total satisfaction. Since the core is compact, however, there is a point which maximizes total satisfaction over the core. Such a "maximin" point might be of interest to players of an actual game.

§11. The Two-Center of Spinetto

Other choices of the coalitional weights can be justified on the basis of which sets of points become optimal when those weights are used. *Spinetto* [1974], defined the *two-center* to be the point minimizing.

$$\sum_{S \neq N} (e_S(x))^2 \quad \text{over all } x \in E(\mathrm{v})$$
$$\text{subject to } x_i \geqslant 0 \text{ for all } i.$$

Letting $k_S = \| A^S \|^{-2} = \dfrac{|S|(|N| - |S|)}{|N|}$, the k-nucleus of v is precisely the two-center whenever the k-nucleus is an imputation. Using this fact, a condition for the two-center to be in $C(\mathrm{v}, k)$ or core (v) can be deduced. Note that these weights satisfy the symmetry condition.

§12. Constant Sum Games

Constant sum games are those games for which $\mathrm{v}(S) + \mathrm{v}(N - S) = \mathrm{v}(N)$ for all $S \neq N$. For this class of games, a particular limitation on the coalitional weights yields an interesting relationship among the solutions of the various systems already encountered.

Proposition II.27: Let v be a constant sum game. If $k_S = k_{N\text{-}S}$ for all S then there exists a unique point x such that $\{x\} = C(\mathrm{v}, k) = CC(\mathrm{v}, k) = N(\mathrm{v}, k)$. Furthermore, for any initial point x_0, the orbits through x_0 for the bargaining and allocation systems (and also System (II.c)) coincide.

Note: If $\gamma(t, x_0)$ is a solution to a system of differential equations, the *orbit* through x_0 is $\{\gamma(t, x_0) \mid t \geqslant 0\}$. Also note that the condition on the coalitional weights in Proposition II.27 is satisfied by $k_S = \binom{|N|}{|S|}^{-1}$ and by $k_S = \|A^S\|^{-2}$, among others.

Proof: For $x \in E(\mathrm{v})$, $\mathrm{v}(S) - x(S) = -(\mathrm{v}(N-S) - x(N-S))$

so
$$[e_S(x)]^+ = [-e_{N\text{-}S}(x)]^+.$$

Hence by the choice of coalitional weights

$$k_S\,[e_S(x)]^+ = k_{N\text{-}S}\,[-e_{N\text{-}S}(x)]^+.$$

But observe, $A^S = -A^{N\text{-}S}$
so
$$\sum_{S \neq N} k_S\,[e_S(x)]^+ A^S = -\sum_{S \neq N} k_{N\text{-}S}\,[-e_{N\text{-}S}(x)]^+ A^{N\text{-}S}$$
$$= -\sum_{S \neq N} k_S\,[-e_S(x)]^+ A^S.$$

This shows also that

$$2\sum_{S \neq N} k_S\,[e_S(x)]^+ A^S = \sum_{S \neq N} k_S\,(e_S(x))\,A^S.$$

Therefore, if $\gamma(t, x_0, \mathrm{v}, k)$ is a solution to

$\dot{x} = \sum_{S \neq N} k_S\,[e_S(x)]^+ A^S$, then it is a solution to

$\dot{x} = -\sum_{S \neq N} k_S\,[-e_S(x)]^+ A^S$ and if $\zeta(t, x_0, \mathrm{v}, k)$ is a solution to

$$\dot{x} = \sum_{S \neq N} k_S\,(e_S(x))\,A^S$$

then $\gamma(2t, x_0, \mathrm{v}, k) = \zeta(t, x_0, \mathrm{v}, k)$. So the orbits coincide. The coincidence of $C(\mathrm{v}, k)$, $CC(\mathrm{v}, k)$, and $N(\mathrm{v}, k)$ follows, or can be seen from the fact that in all three cases, the same function is minimized. □

§13. The Nucleolus as *k*-Centroid of v

For any $x \in E(\mathrm{v})$, let $Q(x)$ be the vector in R^{2^n-2} whose components are the excesses $e_S(x)$ arranged in decreasing order. We will define the "nucleolus of the set of efficient points," $\nu^*(\mathrm{v})$, to be any point of $E(\mathrm{v})$ for which $Q(x)$ is lexicographically least over the hyperplane $E(\mathrm{v})$. Similarly, "the nucleolus of the game v," $\nu(\mathrm{v})$, is generally considered to be that imputation for which $Q(x)$ is lexicographically least over the set of imputations for v. It has been shown that both $\nu^*(\mathrm{v})$ and $\nu(\mathrm{v})$ are unique points (for a further discussion of the nucleolus, see *Schmeidler* [1969] and *Kohlberg* [1970]). Clearly, if $\nu^*(\mathrm{v})$ is an imputation, then $\nu^*(\mathrm{v})$ and $\nu(\mathrm{v})$ coincide.

Proposition II.28: Let v be any game.

a) If core (v) $\neq \emptyset$, then ν (v) = ν^* (v) and ν (v) is a k-centroid of v for any choice of coalitional weights.

b) If core (v) = $\emptyset$, then there exist coalitional weights $\{k_S\}$ such that ν^* (v) is a k-centroid of v.

Proof: Part a) follows directly from the observation that if core v $\neq \emptyset$, then for any $k_+^{2^n-2}$, core v = C (v, k) and ν^* (v) $\in$ core (v).

Part b) follows from a minor modification of an argument of *Kohlberg* [1970] which yields the result that the set

$$\mathcal{B} = \{S \mid e_S (\nu^* (\mathrm{v}))\} > 0$$

is balanced. By Proposition II.7, therefore, there exist positive constants $\{d_S \mid S \in \mathcal{B}\}$ such that

$$\sum_{\mathcal{B}} d_S A^S = 0$$

$$\text{let} \quad k_S = \begin{cases} \dfrac{d_S}{e_S (\nu^* (\mathrm{v}))} & S \in \mathcal{B} \\ \text{any positive value} & S \notin \mathcal{B}. \end{cases}$$

Then

$$\sum_{S \neq N} k_S [e_S (\nu^* (\mathrm{v}))]^+ A^S = 0$$

proving the result. □

Corollary II.29: Let v be any game. If ν^* (v) is an imputation, then ν (v) is a k-centroid of v for some set of coalitional weights.

Corollary II.30: Let v be any game. If ν (v) is in the interior of the set of imputations for v, then ν (v) is a k-centroid of v for some set of coalitional weights.

Proof: If ν^* (v) is an imputation then ν^* (v) = ν (v) and the result follows. If not, then in a neighborhood of ν (v) lying in the imputation set, there is a point y on the open line segment (ν^* (v), ν (v)) for which Q (y) is lexicographically less than Q (ν (v)), contradicting the definition of ν (v). □

It is not difficult to show that if v is a 0-monotonic game, then ν^* (v) is an imputation (see, for example, the proof of Theorem 2.4 in *Maschler*, et al. [1972]. This paper also gives a definition of 0-monotonic games.). Therefore, we have

Corollary II.31: If v is a 0-monotonic game, then ν (v) is a k-centroid of v for some set of coalitional weights.

§14. Examples

The first example is a case where the core, the Shapley value, and the k-cocentroid do not coincide.

Example 1: $v(123) = 1 \quad v(12) = 7/8 \quad v(13) = 3/4 \quad v(23) = 3/8$

$v(1) = v(2) = v(3) = 0$

$\text{Core}(v) = \{(5/8, 1/4, 1/8)\}$

$\text{Shapley value} = \left(\frac{23}{48}, \frac{14}{48}, \frac{11}{48}\right)$

$k\text{-cocentroid of } v = \left(\frac{18}{40}, \frac{11}{40}, \frac{11}{40}\right) \quad \text{for } k_S = \binom{|N|}{|S|}^{-1}$.

The second example exhibits some solutions to

$$\dot{x} = \sum_{S \neq N} k_S \, [e_S(x)]^+ A^S$$

for $k_S = \binom{|N|}{|S|}^{-1}$. The trajectories are drawn in the set of imputations displayed in barycentric coordinates.

Example 2: Consider the game

$v(123) = 1 \quad v(12) = 1/3 \quad v(13) = 1/5 \quad v(23) = 1/2$

$v(1) = v(2) = v(3) = 0.$

Figure 1 depicts several of the orbits of System (II.b) for $k_S = 1$ for all S.

It is not difficult to see what is happening along these trajectories; for instance, along the trajectory marked (a), player 2 is making payment to 1 and 3 equally until core (v) is reached. Along (b), 2 is again making payment to 1 and 3 until coalition {23} finds itself with too little, at which point player 1 must also pay 2 and 3 to correct this imbalance. Over the trajectory, player 2's share decreases, 3's increases and 1's initially increases and then decreases.

§15. Discussion

A number of valid objections can be raised concerning the systems of this paper. The players must agree to act according to the behavior modelled by these systems in order for the results to apply to a game situation and hence no information can be gained about what would happen if a player or coalition changed its behavior unilaterally. This type of normative approach is not, however, uncommon in game theory. Also, because all the systems are autonomous, they cannot be used to model situations

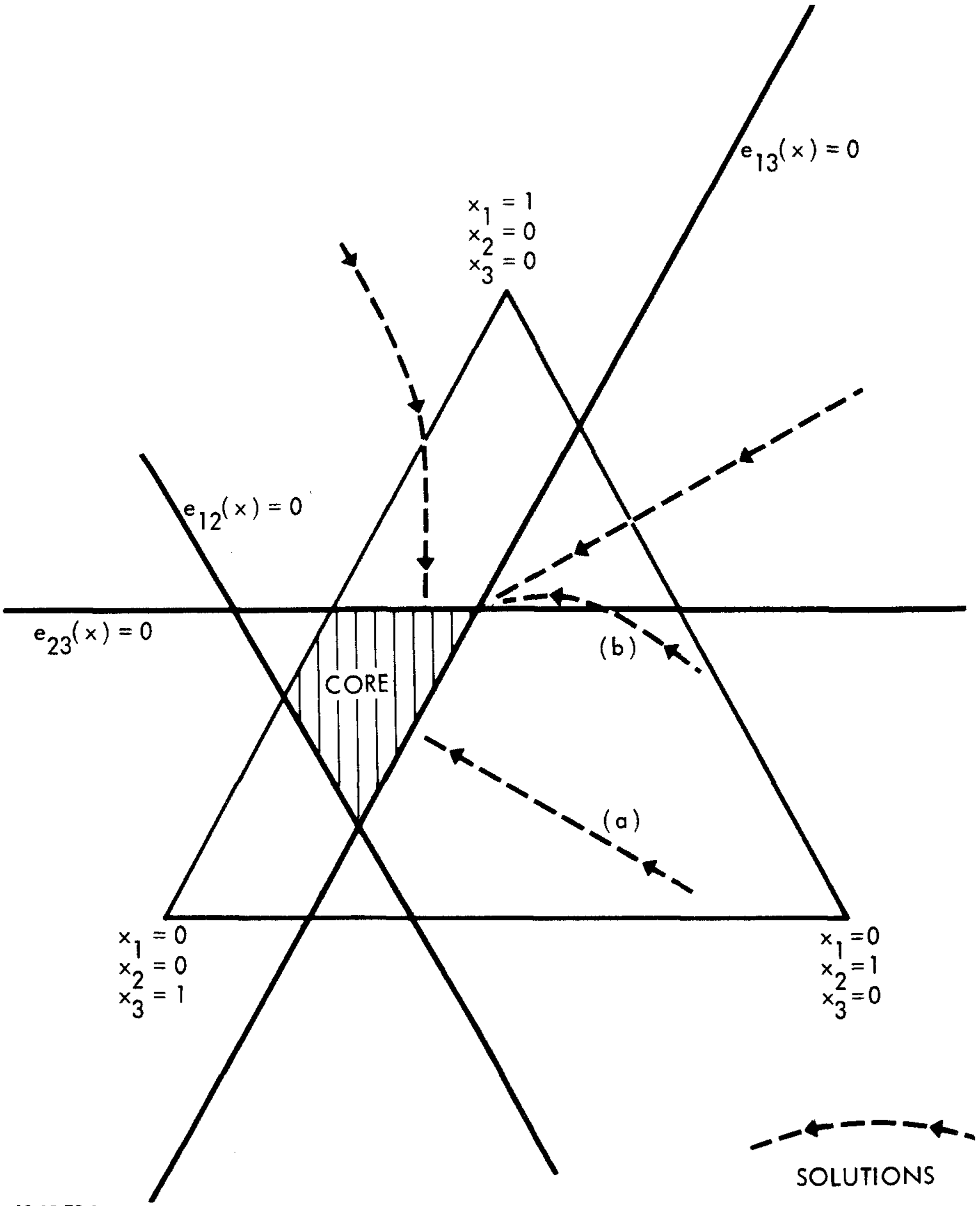

FIGURE 1
For Example 2

Figure 1

in which the satisfaction or dissatisfaction of the players is a function of time as well as payoff. Such questions are of great interest and await further investigation.

Nevertheless, this differential approach to cooperative game theory has numerous benefits, among them the characterization of several of the better known solution concepts as stable points (in a well defined sense) of systems of differential equations with reasonable behavioral interpretations. In addition, the conditions under which different behavior (as defined by the systems) lead to different solution concepts (as determined by the critical points) may enable one to choose a solution concept to fit a particular situation by observing which behavior seems to dominate.

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