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Coalition Formation in Simple Games with Dominant Players[1])

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Abstract: A player, in a proper and monotonic simple game, is dominant if he holds a "strict majority" within a winning coalition. A (non-dictatorial) simple game is dominated if it contains exactly one dominant player. We investigate several possibilities of coalition formation in dominated simple games, under the assumption that the dominant player is given a mandate to form a coalition. The relationship between the various hypotheses on coalition formation in dominated games is investigated in the first seven sections. In the last section we classify real-life data on European parliaments and town councils in Israel.

1. Introduction

Let $G = (N, W)$ be a proper and monotonic simple game (see (2.1) and (2.2)). A player $i \in N$ is *dominant* if he holds a "strict majority" within a winning coalition (see Definition 2.3). We investigate the size of the set of dominant players and show, in particular, that if G is a weighted majority game then it may contain at most one dominant player (see Corollary 2.7). G is (non-trivially) *dominated* if a) G is non-dictatorial (see Definition 2.20), and b) G contains exactly one dominant player (see Definition 2.21). An examination of *De Swaan's* [1973] data on 9 democracies reveals that about 80 percent of the assemblies in those countries were dominated (see Table 2.1). This fact explains the relevancy of the theory of dominated simple games to the analysis of coalition formation in parliaments (see Remark 2.22). A winning coalition in a dominated simple game is *ordinary* if it contains the dominant player (see Definition 3.1). A further examination of De Swaan's data shows that about 80 percent of the coalitions that formed in dominated assemblies were ordinary (see Table 3.1). Further aspects of formation of ordinary coalitions in dominated simple games are discussed in Section 3. Starting in Section 4, we investigate 4 possibilities of coalition formation in dominated games, under the assumption that the dominant player is given a mandate to form a coalition (see (A) in Section 4). Our first hypothesis is that the dominant player seeks to maintain a "simple majority" within the coalition which he forms. (See (H) and the discussion which follows it in Section 4.) In Section 5 we investigate the possibility of the formation of an ordinary and determining coalition (see (D) in

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Section 5). The relationship between (H) and (D) is completely clarified by means of several theorems and examples (see, especially, Theorems 5.7 and 5.10). The possibility that the dominant player chooses a coalition in order to maximize his Shapley value is discussed in Section 6. (See (SV) in Section 6 and the discussion which follows it.) We show by means of an example that (SV) may be incompatible with both (H) and (D) (see Example 6.5). Finally, in Section 7, we examine the possibility that the dominant player chooses a coalition in order to maximize his payoff according to the nucleolus. (See (NUC) in Section 7 and discussion which follows it.) In the last Section, Section 8, we give a complete classification of 67 ordinary coalitions which formed in Denmark, Israel, Italy, the Netherlands and Sweden (see Tables 8.1 and 8.2). We reach the conclusion that hypotheses (H) and (NUC) are acceptable (for European parliaments), while (D) and (SV) should be rejected (see Remarks 8.7–8.9). We conclude with a (summary of) similar classification of ordinary coalitions in town councils in Israel (see Table 8.3). For this last class of committees (NUC), (H) and (D) seem to be sustainable, while (SV) still has to be rejected.

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2. Dominant Players and Dominated Games

In this section the notion of a dominant player in a simple game (i.e., a player who holds a strict majority within a winning coalition), is defined in a precise way. Then we proceed to investigate the size of the set of dominant players of a simple game. In particular, it is shown that a proper weighted majority game may contain at most one dominant player. Also, the importance of the theory of dominant players to political science is discussed in the light of some real-life data. For the sake of completeness we recall the necessary definitions concerning simple games at the appropriate places. For a comprehensive study of simple games the reader is referred to *Shapley* [1962].

A *simple game* is an ordered pair $G = (N, W)$, where $N = \{1, \ldots, n\}$ is the set of *players*, and W is a set of coalitions (i.e., subsets of N), whose members are the *winning coalitions*. Let $G = (N, W)$ be a simple game. G is *monotonic* if

$$[S \in W \text{ and } T \supset S] \Rightarrow T \in W. \tag{2.1}$$

Let $G = (N, W)$ be a monotonic simple game. The following desirability relation for coalitions is derived from G.

Definition 2.1: A coalition S is *at least as desirable as* a coalition T (with respect to G), written $S \succsim T$, if for every $B \subset N$ such that $B \cap (S \cup T) = \emptyset$

$$B \cup T \in W \Rightarrow B \cup S \in W.$$

If $S \succsim T$, but $T \succsim S$ does not hold, then we write $S \succ T$. If $S \succsim T$ and $T \succsim S$ then we denote $S \sim T$.

For a recent discussion of Definition 2.1 the reader is referred to Section 6 of *Peleg* [1980].

Definition 2.2: Let $G = (N, W)$ be a monotonic simple game (see (2.1)), and let S be a coalition. A player $i \in S$ *(weakly) dominates* S if $\{i\} \succ S - \{i\}$ ($\{i\} \succsim S - \{i\}$) (see

Definition 2.1). If i (weakly) dominates S then we also say that S is *(weakly) dominated* by i.

Let $G = (N, W)$ be a monotonic simple game. If S is a coalition, $i \in S$ and i dominates S then, intuitively, i holds a "strict majority" within S. [See Remark 6.4 in *Peleg*.]

Let $G = (N, W)$ be a simple game. G is *proper* if

$$S \in W \Rightarrow N - S \notin W. \tag{2.2}$$

We are now able to state the definition of the central concept of this work.

Definition 2.3: Let $G = (N, W)$ be a proper and monotonic simple game (see (2.1) and (2.2)). A Player $i \in N$ is *dominant* (with respect to G), if there exists a coalition $S \in W$ such that i dominates S (see Definition 2.2). The set of dominant players of G is denoted by $h(G)$.

The first question that we should answer is: how big $h(G)$ can be? In particular, we are interested in conditions on G which imply that $|h(G)| \leqslant 1$, where, here and in the sequel, if B is a finite set then $|B|$ denotes the number of members of B. The following lemma leads to a satisfactory solution of the above problem.

Lemma 2.4: Let $G = (N, W)$ be a proper and monotonic simple game and let $i \in N$. If there exists $j \in N, j \neq i$, such that $j \succsim i$ (i.e., $\{j\} \succsim \{i\}$; see Definition 2.1), then $i \notin h(G)$ (see Definition 2.3).

Proof: Assume, on the contrary, that $i \in h(G)$. Thus, there exists $S \in W$ such that $i \in S$ and $\{i\} \succ S - \{i\}$. Since $j \succsim i$, by Corollary 6.7 of *Peleg* [1980] $j \notin S$. Let $T = N - (S \cup \{j\})$. As G is proper and $S \in W$, $\{j\} \cup T = N - S \notin W$. Hence, since $j \succsim i$, $\{i\} \cup T \notin W$. Let now $B \subset N$ satisfy $B \cap S = \emptyset$. If $\{i\} \cup B \in W$ then, since G is monotonic and $\{i\} \cup T \notin W$, $j \in B$. Using once more our assumption that $j \succsim i$ we obtain that $(S - \{i\}) \cup \{j\} \in W$. Hence, by monotonicity, $(S - \{i\}) \cup B \in W$. Thus, $S - \{i\} \succsim \{i\}$, which is the desired contradiction.

Lemma 2.4 has several important corollaries. In order to formulate them we first need the following remark and definition.

Remark 2.5: The following notation will be useful in the sequel. Let $N = \{1, \ldots, n\}$ be a set with n members. If $x = (x^1, \ldots, x^n)$ is an n-tuple of real numbers and if S is a coalition (i.e., $S \subset N$), then we denote $x(S) = \sum_{i \in S} x^i$. Also, $x(\emptyset) = 0$.

Definition 2.6: Let $G = (N, W)$ be a simple game. G is a *weighted majority game* if there exist a *quota* $q > 0$ and *weights* $w^1 \geqslant 0, \ldots, w^n \geqslant 0$, such that

$$S \in W \Leftrightarrow w(S) \geqslant q. \qquad \text{(See Remark 2.5.)}$$

The $(n + 1)$-tuple $[q; w^1, \ldots, w^n]$ is called a *representation* of G, and we write $G = [q; w^1, \ldots, w^n]$.

Clearly, a weighted majority game is monotonic.

Corollary 2.7: If $G = [q; w^1, \ldots, w^n]$ is a proper weighted majority game then $|h(G)| \leqslant 1$.

Proof: If S and T are coalitions and $w(S) \geqslant w(T)$ then, obviously, $S \succcurlyeq T$. Let i be a player with maximum weight, i.e., $w^i \geqslant w^j$ for all $j \in N$. By the above observation and Lemma 2.4, $h(G) \subset \{i\}$.

Remark 2.8: As most of the simple games that arise in applications of game theory to politics are proper weighted majority games, Corollary 2.7 justifies and enhances the importance of the notion of a dominant player to political science.

The next corollary is preceded by the following definition.

Definition 2.9: Let $G = (N, W)$ be a monotonic simple game. G is *weak* if

$$V = \cap \{S \mid S \in W\} \neq \emptyset .$$

The members of V are called *veto* players.

Corollary 2.10: If $G = (N, W)$ weak then $|h(G)| \leqslant 1$.

Proof: If i is a veto player (see Definition 2.9), then $i \succcurlyeq j$ for all $j \in N$. Hence, by Lemma 2.4, $h(G) \subset V$ and, furthermore, if $|V| \geqslant 2$ then $h(G) = \emptyset$.
Thus, $|h(G)| \leqslant 1$.

A single vetoer may or may not be dominant as is shown by the following examples.

Example 2.11: Let $G = [3; 2, 1, 1]$ (see Definition 2.6). 1 is a vetoer. 1 is also a dominant player since $\{1, 2\} \in W$ and $1 \leftarrow 2$.

Example 2.12: Let $G = [4; 2, 1, 1, 1]$. Then $V = \{1\}$, but 1 is not dominant since $\{1\} \sim S$ if $S \subset \{2, 3, 4\}$ and $|S| \geqslant 2$.

We now present an example of a proper and monotonic simple game with two dominant players. First we need the following remark.

Remark 2.13: Let $G = (N, W)$ be a monotonic simple game (see (2.1)). We denote by W^m the set of *minimal* winning coalitions. Clearly, W is completely determined by W^m. Also, G is proper if and only if for every pair of coalitions $S, T \in W^m$, $S \cap T \neq \emptyset$.

Example 2.14: Let $N = \{1, 2, 3, 4, 5\}$. We define a proper and monotonic simple game G by specifying

$$W^m = \{\{1, 2\}, \{1, 3, 4\}, \{2, 4, 5\}\}.$$

(One can check that if $S, T \in W^m$ then $S \cap T \neq \emptyset$.) We claim that $h(G) = \{1, 2\}$. Indeed, $\{1, 3, 4\} \in W$ and $\{1\} \leftarrow \{3, 4\}$ (see Definition 2.1). Similarly, $\{2, 4, 5\} \in W$ and $\{2\} \leftarrow \{4, 5\}$. Thus, $h(G) = \{1, 2\}$.

Remark 2.15: Let $G = (N, W)$ be a proper and monotonic simple game. If $|N| \leqslant 4$ then $|h(G)| \leqslant 1$ (see Corollary 5.14).

Let $G = (N, W)$ be a simple game. G is *strong* if

$$S \notin W \Rightarrow N - S \in W. \tag{2.3}$$

In order to construct an example of a proper, monotonic and strong simple game G such that $|h(G)| \geqslant 2$, it is convenient to make use of the following definition.

Definition 2.16: Let $G = (N, W)$ be a proper and monotonic simple game. The *strong extension* of G is the simple game $G^* = (N \cup \{z\}, W^*)$, where $z \notin N$, and $S \in W^*$ if and only if

a) $z \notin S$ and $S \in W$, or
b) $z \in S$ and $N - S \notin W$.

Remark 2.17: G^* is proper, monotonic and strong; it is usually called the "zero-sum extension" of G [see *Aumann/Peleg/Rabinowitz*, p. 547].

Lemma 2.18: Let $G = (N, W)$ be a proper and monotonic simple game and let $G^* = (N \cup \{z\}, W^*)$ be the strong extension of G (see Definition 2.16).
Then $h(G^*) \supset h(G)$ (see Definition 2.3).

The proof, which is straightforward, is omitted.

Corollary 2.19: There exists a proper, monotonic and strong six-person game G^* such that $|h(G^*)| \geqslant 2$. Indeed, the strong extension G^* of the game G of Example 2.14 has all the above properties.

I do not know whether there exists a proper, monotonic and strong five-person game G such that $|h(G)| \geqslant 2$.

It is time now to inquire how important is the notion of a dominant player to political science. More specifically, we want to know how often do simple games with dominant players occur in real-life situations. In order to render the above question completely precise we introduce the following definitions.

Definition 2.20: Let $G = (N, W)$ be a simple game. A player $j \in N$ is a *dictator* if $[S \in W \Leftrightarrow j \in S]$. G is *essential* if it is non-dictatorial (i.e., there exists no dictator in N).

Definition 2.21: Let $G = (N, W)$ be a proper and monotonic simple game. G is (non-trivially) *dominated* if a) G is essential (see Definition 2.20), and b) $|h(G)| = 1$ (see Definition 2.3).

We shall now determine, in several cases, the relative frequency of dominated games (see Definition 2.21), among essential political (simple) games. We shall refer to two sources of information: *De Swaan* [1973] and *Peleg* [1980]. Using the information contained in Chapters 9–11 of *De Swaan* [1973] we obtain Table 2.1.

Nation	Period considered	Essential assemblies	Dominated assemblies
Denmark	1918–1971	21	18
Finland	1919–1972	16	8
France	1946–1957	16	9
Israel	1949–1970	7	7
Italy	1946–1972	9	9
The Netherlands	1918–1972	15	13
Norway	1933–1969	5	5
Sweden	1917–1970	10	9
Weimar	1919–1930	6	3
Total	–	105	81

Tab. 2.1: Dominated Assemblies in 9 Democracies

Remark 2.22: The relevancy of the theory of dominated simple games to the analysis of coalition formation in European democracies is clear from Table 2.1. In particular, three cases should be distinguished: Sweden was continuously dominated during the years 1921–1970 by the Social Democrats. Italy was continuously dominated during the period 1946–1972 by the Christian Democrats. Finally, Israel was dominated by the Labor Party during the years 1949–1977. (It is now dominated by the Likud, a right-wing party.)

We now turn to our second source of information.

Remark 2.23: 54 out of the 78 town councils in Table 8.1 of *Peleg* [1980] are essential. 43 out of the 54 essential councils are dominated. Thus, again, the dominated games consist of a clear majority of the relevant games.

3. Ordinary and Exceptional Coalitions

Let $G = (N, W)$ be a dominated simple game (see Definition 2.21). If $h(G) = \{i\}$ then we write $i = h(G)$.

Definition 3.1: Let $G = (N, W)$ be a dominated simple game and let $i = h(G)$. Let further S be a winning coalition, i.e., $S \in W$. S is *ordinary* if $i \in S$; otherwise, i.e., if $i \notin S$, S is *exceptional.*

Thus, if G is a dominated game then a winning coalition which includes the dominant player is ordinary, while a majority which excludes the dominant player is exceptional. Note that a minority coalition is neither ordinary nor exceptional. The numbers of ordinary and exceptional coalitions in the nine countries investigated by *De Swaan* [1973], are recorded in Table 3.1.

Nation	Period considered	Ordinary coalitions	Exceptional coalitions
Denmark	1918–1971	13	2
Finland	1919–1972	5	4
France	1946–1957	2	7
Israel	1949–1970	14	0
Italy	1946–1972	16	0
The Netherlands	1918–1972	16	2
Norway	1933–1969	4	2
Sweden	1917–1970	8	0
Weimar	1919–1930	5	2
Total	–	83	19

Tab. 3.1: Ordinary and Exceptional Coalitions in Dominated Assemblies

A careful examination of De Swaan's data reveals that exceptional coalitions may form (almost) only when the dominant player represents an "extreme" ideology (see Remark 3.4). To put it more precisely we need the following definition.

Definition 3.2: Let $G = (N, W)$ be a simple game. A *policy order* is a weak ordering (i.e., a complete and transitive binary relation) R on the set of players N.

Let $G = (N, W)$ be a committee, i.e., a simple game. A policy order for G is, intuitively, an ordering of the players according to their "positions" with respect to the major issues which confront G. For example, if G is a parliament then, usually, there is a well defined "left to right" ordering of the members of G (i.e., the parties which are represented in the parliament). The reader is referred to *De Swaan* [1973] for a deeper discussion of the notion of a policy scale.

We are now able to state:

Definition 3.3: Let $G = [q; w^1, \ldots, w^n]$ be an essential and proper weighted majority game (see Definition 2.20, (2.2) and Definition 2.6). A player $e \in N$ (with a "non-negligible" weight w^e), is *extreme* with respect to a policy order R, if $e\,R\,i$ for all $i \in N$, or $i\,R\,e$ for all $i \in N$ (i.e., if e is either the first or the last player in the order R).

Remark 3.4: During the period 10/1947 – 3/1951 the Communist Party of France was a dominant player in seven assemblies. However, since it was extreme according to (De Swaan's) policy order, it was excluded from the cabinet. Similar phenomena occured in Norway in 1965 and 1969, and in Finland during the years 1919–1936. However, in almost all the dominated assemblies where the dominant party occupied a "central position" according to De Swaan's policy order, an ordinary coalition formed.

Remark 3.5: Among the 43 dominated councils of Table 8.1 in *Peleg* [1980] *only two* exceptional coalitions formed. We conclude from this observation that, in the absence of a well defined policy order, ordinary coalitions are very likely to form in dominated games.

4. Weakly Dominated and Connected Coalitions

Henceforth we shall devote our investigation to formation of ordinary coalitions (see Definition 3.1), in dominated simple games (see Definition 2.21).

Let $G = (N, W)$ be a dominated simple game and let $i = h(G)$. We assume that i is given a mandate to form a coalition. For example, in the case of a parliament such a mandate is usually given, to some party, by an official authority like a king or a president. However, the assumption that the task of erecting a coalition is assigned to the dominant party consists of a non-trivial restriction on the process of coalition formation (see Section 3, especially Remark 3.4). We now restate the above assumption for the sake of easy reference.

(A) *The dominant player is given a mandate to form a coalition.*

Clearly, under Assumption (A) only ordinary coalitions are formed. Throughout the rest of this paper we investigate several hypotheses on coalition formation by dominant players. Also, further analysis of the data of *De Swaan* [1973] and Table 8.1 of *Peleg* [1980] is given.

Let, again, $G = (N, W)$ be a dominated simple game and let $i = h(G)$. Denote by $H = H(G)$ the set of all winning coalitions which are weakly dominated by i (see Definition 2.2). Formally,

$$H = H(G) = \{S \mid S \in W,\ i \in S \text{ and } \{i\} \succeq S - \{i\}\}. \tag{4.1}$$

Clearly, every coalition in H is ordinary. The first hypothesis that we investigate is

(H) *Let G be a dominated simple game. Under Assumption* (A) *only coalitions in $H = H(G)$* (see (4.1)) *are formed.*

The reasoning behind (H) is very simple: the dominant player seeks to maintain a "simple majority" within the coalition which he forms. Since G is dominated, $H(G) \neq \emptyset$. Hence, it seems as if nothing can prevent i (where $i = h(G)$), from erecting a coalition $S \in H$. However, as soon as the members of G can be "ideologically ordered", i.e., some policy order (see Definition 3.2) is known to exist, combinatorial obstructions may prevent formation of any member of H as we shall see below. First we need the following notation and definition.

Remark 4.1: Let $G = (N, W)$ be a simple game and let R be a policy order on N. If $i, j \in N$, $i R j$ but $j R i$ does not hold, then we write $i P j$. If $i R j$ and $j R i$, then we denote $i I j$.

Definition 4.2: Let $G = (N, W)$ be a simple game and let R be a policy order on N. A coalition S is *connected* (with respect to R), if it satisfies the following condition:

$$[i, k \in S, j \in N \text{ and } i P j P k] \Rightarrow j \in S.$$

Remark 4.3: Definition 4.2 is due to *Axelrod* [1970]. Connected coalitions are called "closed" by *De Swaan* [1973, p. 70].

In *Axelrod* [1970] it is claimed that, in the presence of a well defined policy order, only connected coalitions form. Axelrod's hypothesis is strongly supported by De Swaan's empirical research [see *De Swaan*, p. 159]. The following examples show that Axelrod's assumption may be incompatible with (H).

Example 4.4: Consider the French Assembly of January 1947 [see *De Swaan*, p. 182]. It is (incompletely) described by the weighted majority game [310; 182, 104, 167, 43, 26, 28, 35]. (The description is incomplete since the parties with less than 2.5 percent of the seats are omitted.) The policy order R (of De Swaan), is given by: $1P2P3P4I5P6P7$, (see Remark 4.1). Now, $\{1, 2, 5\} \in W$ and $\{1, 3\} \in W$, while $\{2, 3, 5\} \notin W$. Hence, player 1 (the Communists in this case), is dominant (see Definition 2.3). Now, if $S \in W$ is connected (see Definition 4.2), and $1 \in S$ then $S \supset \{1, 2, 3\}$, and therefore $S \notin H$.

In Example 4.4 the dominant player is an extreme player (see Definition 3.3). In the following example he is a median player according to the given policy order.

Example 4.5: Let $G = [17; 1, 1, 7, 1, 1, 1, 9, 1, 1, 1, 7, 1, 1]$ and let $R = (1, 2, 3, \ldots, 13)$ be a policy order for G. Then G is a strong weighted majority game and $7 = h(G)$. Also, if S is a connected and winning coalition and $7 \in S$, then either $S \supset \{3, 4, 5, 6\}$ or $S \supset \{8, 9, 10, 11\}$. Hence, there exists no connected coalition in $H(G)$.

5. Determining Coalitions

Definition 5.1: Let $G = (N, W)$ be a proper and monotonic simple game. A coalition $S \in W$ is *determining* (in the game G) if

$$[i \in S \text{ and } S - \{i\} \leftarrow \{i\}] \Rightarrow S - \{i\} \in W. \tag{5.1}$$

The set of all determining coalitions in G is denoted by $D(G)$.

Remark 5.2: Determining coalitions were introduced in *Peleg* [1980]. Indeed, Remark 7.9 in that paper is identical to our present Definition 5.1. For motivation and a detailed discussion of Definition 5.1 the reader is referred to *Peleg* [1980]. Here we shall be content with the following remark.

Remark 5.3: Let $G = (N, W)$ be a proper and monotonic simple game and let $S \in D(G)$. Condition (5.1) says that if $i \in S$ is not a "vetoer with respect S" (i.e., $S - \{i\} \leftarrow \{i\}$), then he is not a swinger with respect to S (i.e., $S - \{i\} \in W$). It implies the following stability property of S. If $T \subset S$ consists of a "simple majority" within S (i.e., $T \leftarrow S - T$), then no player i of the "internal opposition" $S - T$ is a swinger [see Corollary 6.7 in *Peleg*]. Therefore, if S decides on its course of action by "simple majority" then no single player $i \in S - T$ can deprive T from its control of a majority *within N.*

We now formulate our second hypothesis about coalition formation in dominated games.

(D) *Let G be a dominated simple game. Under Assumption* (A) (see Section 4) *only determining and ordinary coalitions* (see Definitions 5.1 and 3.1), *are formed in G.*

An immediate question is whether (H) (see Section 4), is compatible with (D). Before we consider two counter examples we need the following definition.

Definition 5.4: Let $G = [q; w^1, \ldots, w^n]$ be a weighted majority game (see Definition 2.6). $[q; w^1, \ldots, w^n]$ is a *homogeneous representation* of G if

$$S \in W^m \Rightarrow w(S) = q \quad \text{(see Remarks 2.5 and 2.13).}$$

A weighted majority game G is *homogeneous* if it has a homogeneous representation.

Example 5.5: Let $G = [5; 3, 2, 2, 1]$. Then G is homogeneous. Furthermore, G is dominated and, obviously, $1 = h(G)$. As the reader can easily verify, $D(G) = \{\{1, 2, 3, 4\}\}$. Clearly, since $\{2, 3, 4\}$ is winning, $\{1, 2, 3, 4\} \notin H(G)$. Thus, $H(G) \cap D(G) = \emptyset$. Note, however, that G is *not* strong (see (2.3)).

Example 5.6: Consider the strong weighted majority game $G = [9; 5,3,3,3,1,1,1]$. $\{1, 2, 5\} \in W$ and $\{1, 3, 6\} \in W$, while $\{2, 3, 5, 6\} \notin W$. Hence $\{1\} \leftarrow \{2, 5\}$. Thus, G is dominated and, of course, $1 = h(G)$. As the reader can easily verify if $1 \in S$ and $S \in D(G)$ then $|S \cap \{2, 3, 4\}| \geqslant 2$. Hence $S \notin H(G)$. Thus, again, $D(G) \cap H(G) = \emptyset$. Note, however, that G is *not* homogeneous [see *Isbell*, p. 28].

The following theorem deals with the remaining possibility.

Theorem 5.7: Let $G = (N, W)$ be a strong and proper homogeneous weighted majority game. If G is dominated then $H(G) \cap D(G) \neq \emptyset$.

Proof: Let $[q; w^1, \ldots, w^n]$ be a homogeneous representation of G. We start with the following claim.

Claim 5.8: Let $S = \{k_1, \ldots, k_s\}$ be a winning coalition. If $w^{k_1} \geqslant \ldots \geqslant w^{k_s}$ then there exists t, $2 \leqslant t \leqslant s$, such that $\{k_1, \ldots, k_t\} \in W^m$.

Claim 5.8 follows immediately from the fact that $[q; w^1, \ldots, w^n]$ is a representation of G. So we continue with the proof of Theorem 5.7. Without loss of generality we may assume that $w^1 \geqslant \ldots \geqslant w^n$. Thus, $1 = h(G)$. By Claim 5.8 there exist m and r such that $2 \leqslant m < r$, and $S = \{1, \ldots, m\}$ and $T = \{1, m+1, \ldots, r\}$ are minimal winning. Let $S^* = S - \{1\}$ and $T^* = T - \{1\}$. Since 1 is a dominant player (see Definition 2.3) $w^1 > q/2$. Hence $w(S^*) + w(T^*) < q$ (see Remark 2.5).

Thus, $U_1 = N - (S^* \cup T^*) \in W$. We now distinguish the following possibilities:

a) $U_1 - \{r+1\} \in W$.

In this case let $U_2 \subset U_1 - \{r+1\}$ be a minimal winning coalition, and let further $U = U_2 \cup \{r+1\}$.

b) $U_1 - \{r+1\} \notin W$.

In this case let $U_2 \subset U_1$ be a minimal winning coalition, and let further $U = U_2 \cup \{r\}$.

Clearly, in both cases $1 \in U_2 \subset U$. We claim that, in both cases, $U \in H(G) \cap D(G)$. Let $i \in U$, $i \neq 1$. Then, in case a) $w^i \leqslant w^{r+1}$, and in case b) $w^i \leqslant w^r$. Thus, in both cases, $U - \{i\} \in W$. Hence, in order to prove that $U \in H(G) \cap D(G)$, it is sufficient to show that $\{1\} \succsim U - \{1\}$ (see (4.1) and Definition 5.1). In order to prove this we first observe that since G is homogeneous and $w^{r+1} \leqslant w^k$ for all $k \in S^* \cup T^*$, the inequality $w(S^*) + w(T^*) + w^{r+1} \leqslant q$ is true. We now distinguish the following possibilities:

c) $w(S^*) + w(T^*) + w^{r+1} < q$.

In this case $N - (S^* \cup T^* \cup \{r+1\}) \in W$, i.e., a) is true. Now c) is equivalent to $2(q - w^1) + w^{r+1} < q$, or $q - w^1 + w^{r+1} < w^1$. But, the last inequality is equivalent to $w(U_2 - \{1\}) + w^{r+1} = w(U - \{1\}) < w^1$, which implies that $\{1\} \succsim U - \{1\}$.

d) $w(S^*) + w(T^*) + w^{r+1} = q$.

In this case b) is true. Assume now, on the contrary, that $U - \{1\} \succ \{1\}$. Denote $U^* = U - \{1\}$. Then there exists $B \subset N$ such that $B \cap U = \emptyset$, $B \cup U^* \in W$ and $\{1\} \cup B \notin W$. Let $R = N - (S^* \cup T^* \cup U_2)$.

Since $U_1 - \{r+1\} \notin W$, $w(R) < w^{r+1}$. Hence

$$w(B) + w(U^*) = w(B) + w^r + q - w^1 \geqslant q$$

implies that

$$w(B - R) + w^r \geqslant w^1 - w(R) > w^1 - w^{r+1}. \tag{5.2}$$

d) is equivalent to

$$q - w^1 + w^{r+1} = w^1. \tag{5.3}$$

(5.2) and (5.3) imply that

$$w^1 + w(B - R) + w^r > q. \tag{5.4}$$

Let $C = \{1\} \cup (B - R) \cup \{r\}$. Then, by (5.4), $C \in W$. However, $C - \{r\} = \{1\} \cup (B - R) \subset \{1\} \cup B$ and is therefore losing. Since $w^r \leqslant w^k$ for all $k \in C$, C does not contain a subset C^* such that $w(C^*) = q$, contradicting the fact that $[q; w^1, \ldots, w^n]$ is a homogeneous representation. Hence, $U - \{1\} \succ \{1\}$ is impossible. Since $\succsim$ is complete (as G is a weighted majority game), $\{1\} \succsim U - \{1\}$.

We shall now prove that if G is a dominated strong game, and if $i = h(G)$ is "almost winning", then $H(G) \cap D(G) \neq \emptyset$. We start with a precise definition of an "almost winning" player.

Definition 5.9: Let $G = (N, W)$ be an essential, proper and monotonic simple game. A player i is *almost winning* if a) $i \in h(G)$, and b) there exists $j \in N$ such that $\{i, j\} \in W$.

Theorem 5.10: Let $G = (N, W)$ be a dominated strong game. If $i = h(G)$ is almost winning (see Definition 5.9), *then $H(G) \cap D(G) \neq \emptyset$.*

First we need the following lemma.

Lemma 5.11: Let $G = (N, W)$ be a proper and monotonic simple game. If $i \in h(G)$ and $S \subset N$, $|S| = 2$ and $i \notin S$, then $S \notin W$.

We postpone the proof of Lemma 5.11 and start with a

Proof of Theorem 5.10: Since i is almost winning there exists $j \in N$, $j \neq i$, such that $\{i, j\} \in W$. Let $S = N - \{j\}$. Since G is strong and essential $S \in W$. Also, since $\{i, j\} \in W$, $\{i\} \succsim S - \{i\}$ (see Definition 2.1). Hence $S \in H(G)$ (see (4.1)). Let now $k \in S$, $k \neq i$. By Lemma 5.11, $\{j, k\} \notin W$. Since G is strong, $S - \{k\} \in W$. Hence $S \in D(G)$ (see Definition 5.1).

Proof of Lemma 5.11: Without loss of generality $i = 1$ and $S = \{2, 3\}$. Since $1 \in h(G)$ there exists $T \in W$ such that $1 \in T$ and $\{1\} \leftarrow T - \{1\}$ (see Definition 2.3). Assume now, on the contrary, that $S \in W$. Since G is proper $T \cap S \neq \emptyset$; also, $\{1\} \notin W$. Hence, since $\{1\} \leftarrow T - \{1\}$, $|T \cap S| = 1$. Again, without loss of generality, $\{2\} = S \cap T$. Let $B \subset N$ such that $B \cap T = \emptyset$. If $\{1\} \cup B \in W$ then, since G is proper and $S \in W$, $3 \in B$. Since $2 \in T - \{1\}$ and $3 \in B$, $(T - \{1\}) \cup B \in W$. Thus, $T - \{1\} \succsim \{1\}$ (see Definition 2.1), which is the desired contradiction.

Since many political games (e.g., parliaments), involve only a small number of players, the following corollary might be useful.

Corollary 5.12: If $G = [q; w^1, \ldots, w^n]$ is a dominated strong weighted majority game, and if $n \leqslant 6$ (i.e., G has at most six players), then $H(G) \cap D(G) \neq \emptyset$.

Indeed, by direct inspection of Isbell's list of strong weighted majority games [see *Isbell*, p. 27], one finds that if G satisfies the conditions of Corollary 5.12 and if $i = h(G)$, then i is almost winning. Hence the corollary follows from Theorem 5.10.

Corollary 5.13: If $G = (N, W)$ is a proper and monotonic simple game then N contains at most two almost winning players.

Corollary 5.13 is a direct consequence of Definition 5.9 and Lemma 5.11. (Note that the game G of Example 2.14 has two almost winning players.)

Corollary 5.14: Let $G = (N, W)$ be a proper and monotonic simple game. If $|N| \leqslant 4$ then $|h(G)| \leqslant 1$ (see Remark 2.15).

Proof: We consider the case $|N| = 4$. (The case $|N| = 3$ is left to the reader.) Suppose, on the contrary, that $|h(G)| \geqslant 2$. Without loss of generality let $1, 2 \in h(G)$.

Since $1 \in h(G)$ there exists a coalition $S \in W$ such that $1 \in S$ and $\{1\} \leftarrow S - \{1\}$. By Corollary 6.7 in *Peleg* [1980] and Lemma 2.4, $2 \notin S$. Hence, by Lemma 5.11, $S = \{1, 3, 4\}$. Similarly, $\{2, 3, 4\} \in W$. Hence, $\{3, 4\} \leftrightharpoons \{1\}$, which is the desired contradiction.

We conclude this section with the following remark concerning weak games.

Remark 5.15: If $G = (N, W)$ is a dominated weak game (see Definition 2.9), then $H(G) \cap D(G) \neq \emptyset$.

Proof: Indeed, if $v = h(G)$ then, since v is a vetoer, $\{v\} \leftrightharpoons N - \{v\}$. Hence, $N \in H(G)$. Also, by Corollary 7.10 in *Peleg* [1980], $N \in D(G)$. Thus, $N \in H(G) \cap D(G)$.

6. The Shapley Value Approach

Let $G = (N, W)$ be a monotonic simple game (see (2.1)). A *linear order* on N is a transitive, complete and antisymmetric binary relation on N (i.e., it is an antisymmetric policy order; see Definition 3.2). We denote by L the set of all linear orders on N. Let $R \in L$ and $i \in N$. We denote

$$B_i = \{j \mid j \in N, j \neq i \text{ and } j \, R \, i\}.$$

Thus, B_i is the set of players that precede i in the order R. We call i a *pivot* (with respect to the order R), if $B_i \notin W$ while $B_i \cup \{i\} \in W$. Clearly, since G is monotonic, there exists *at most* one pivot with respect to R (note that G may be null, i.e., $W = \emptyset$). We denote

$$\pi_i = \{R \mid R \in L \text{ and } i \text{ is a pivot with respect to } R\}.$$

The *Shapley value* of player i (in the game G) is $\varphi^i(G) = \varphi^i = |\pi_i| / n!$, where $n = |N|$. Thus, φ^i is the probability of i being pivotal when all the (linear) policy orders are equally probable. (See *Shapley* [1977] for a recent discussion of the Shapley value.) Let S be a coalition (i.e., $S \subset N$). The *subgame* of G which is determined by S is the simple game $G \mid S = (S, W \cap 2^S)$, where 2^S is the set of all subsets of S. Let now $i \in S$. The Shapley value of i *with respect to the coalition* S is $\varphi^i(S) = \varphi^i(G \mid S)$ [see *Aumann/Dreze*, p. 220].

Let now $G = (N, W)$ be a dominated game (see Definition 2.21), and let $i = h(G)$. Under Assumption (A) (see Section 4), it is quite reasonable (or, at least, possible) that i chooses S in order to maximize $\varphi^i(S)$. Indeed, by choosing S with maximum $\varphi^i(S)$, i maximizes his "power" *within the formed coalition* (provided, of course, that after the formation of S the members of the opposition, $N - S$, have no influence on the process of decision making by S). For a discussion of the possibility and plausibility of such a behaviour in parliamentary coalitions, the reader is referred to *De Swaan* [1973, 127–129].

The above discussion leads us to the following hypothesis. First we need the following notation. Let $G = (N, W)$ be a monotonic simple game and let $k \in N$. We denote

$$\theta_k = \{S \mid S \subset N \text{ and } k \in S\},$$

and

$$\mathrm{SV}(G, k) = \{S \mid S \in \theta_k \text{ and } \varphi^k(S) \geqslant \varphi^k(T) \text{ for all } T \in \theta_k\}. \tag{6.1}$$

We now state our hypothesis:

(SV) *Let* $G = (N, W)$ *be a dominated simple game and let* $i = h(G)$. *Under Assumption* (A) *only coalitions in* SV (G, i) (see (6.1)) *are formed.*

Our first duty is to check whether (SV) is compatible with (H) and (D) (see Sections 4 and 5 respectively). We start by proving two lemmata which are very helpful in computing the sets SV (G, k). (These lemmata are, indeed, essential for the computation of the results in Table 8.1.) First a definition.

Definition 6.1: Let $G = (N, W)$ be a monotonic simple game. A player $k \in N$ is a *dummy* (in the game G) if there exists no $S \in W^m$ (see Remark 2.13), such that $k \in S$.
The following Remark is obvious.

Remark 6.2: Let $G = (N, W)$ be a monotonic simple game. A player $k \in N$ is a dummy if and only if $\varphi^k(G) = 0$.
We now state and prove:

Lemma 6.3: Let $G = (N, W)$ be a monotonic simple game, let S be a coalition and let $k \in S$. If k is not a dummy in G and if $S - \{k\} \in W$, then $S \notin$ SV (G, k).

Proof: If k is a dummy in $G \mid S$ then $S \notin$ SV (G, k) by Remark 6.2. Hence we may assume that k is not a dummy in $G \mid S$. Also, since k is not a dummy in G, $\emptyset \notin W$. Hence, since $S - \{k\} \in W$, $s = |S| \geqslant 2$. Denote now by L_S the set of all linear orders of S. For $j \in S$ let

$$\lambda_j = \{R \mid R \in L_S \text{ and } m\,R\,j \quad \text{for all } m \in S\},$$

i.e., λ_j is the set of all linear orders of S in which j is the last player. Let further

$$\pi_k(S) = \{R \mid R \in L_S \text{ and } k \text{ is a pivot with respect to } R\},$$

and $\pi_{k,j} = \pi_k(S) \cap \lambda_j$. Then, if $j \in S - \{k\}$,

$$\varphi^k(S - \{j\}) = |\pi_{k,j}| / (s-1)!$$

Since $S - \{k\} \in W$, $\pi_{k,k} = \pi_k(S) \cap \lambda_k = \emptyset$. Hence

$$\varphi^k(S) = \Sigma\,\{\varphi^k(S - \{j\}) \mid j \in S - \{k\}\} / s.$$

Since k is not a dummy in $G \mid S$ there exists $j \in S - \{k\}$ such that $\varphi^k(S - \{j\}) > 0$. Thus, as $s \geqslant 2$, $\varphi^k(S) < \max\,\{\varphi^k(S - \{j\}) \mid j \in S - \{k\}\}$, which proves the lemma.

Lemma 6.4: Let $G = (N, W)$ be a monotonic simple game, let S be a coalition and let $k \in S$. If there exists $j \in N - S$ such that $(S \cup \{j\}) - \{k\} \notin W$, then $\varphi^k(S) \leqslant \varphi^k(S \cup \{j\})$. If, furthermore, there exists $T \subset S$ such $k \notin T$, $T \cup \{k\} \notin W$, but $T \cup \{k\} \cup \{j\} \in W$, then $\varphi^k(S) < \varphi^k(S \cup \{j\})$.

Proof: Once more let L_S be the set of all linear orders of S, and

$$\pi_k(S) = \{R \mid R \in L_S \text{ and } k \text{ is a pivot with respect to } R\}.$$

$L_{S \cup \{j\}}$ and $\pi_k (S \cup \{j\})$ are defined similarly. Let $Q \in L_{S \cup \{j\}}$. We denote by $t_u (Q)$ the u-th player in the order Q, $u = 1, \ldots, s + 1$, where $s = | S |$. Thus, $t_1 (Q)$ is the first player in the order Q, $t_2 (Q)$ is the second, and so on. Let now $R \in L_S$. We define $s + 1$ linear orders of $S \cup \{j\}$, $R^{(1)}, \ldots, R^{(s+1)}$, in the following way. Let $1 \leqslant u \leqslant s + 1$. If $a, b \in S$ then $[a R^{(u)} b \Leftrightarrow a R b]$, and $t_u (R^{(u)}) = j$. If $R \in \pi_k (S)$, then, since $(S \cup \{j\}) - \{k\} \notin W$, $R^{(u)} \in \pi_k (S \cup \{j\})$ for $u = 1, \ldots, s + 1$. Hence $| \pi_k (S \cup \{j\}) | \geqslant | \pi_k (S) | (s + 1)$.
Thus,

$$\varphi^k (S \cup \{j\}) = | \pi_k (S \cup \{j\}) | / (s + 1) ! \geqslant | \pi_k (S) | / s ! = \varphi^k (S).$$

In addition, if T has the above properties, we may choose $R \in L_S$ such that $T = \{ i \mid i \neq k \text{ and } i R k \}$. Then, since $T \cup \{k\} \notin W$, $R \notin \pi_k (S)$. However, since $T \cup \{j\} \cup \{k\} \in W$, $R^{(1)} \in \pi_k (S \cup \{j\})$ (see the previous paragraph). Combining the last observation with the previous argument, we conclude that $\varphi^k (S) < \varphi^k (S \cup \{j\})$.

Example 6.5: Consider the proper and strong homogeneous weighted majority game $G = [5; 3,2,2,1,1]$. G is dominated and $1 = h (G)$ is almost winning (see Definitions 2.21 and 5.9). By Lemmata 6.3 and 6.4 SV $(G, 1) \subset \{\{1,2,3\}, \{1,2,4,5\}, \{1,3,4,5\}\}$. Now, by direct computation, $\varphi^1 (\{1,2,3\}) = 2/3$ while $\varphi^1 (\{1,2,4,5\}) = 7/12$. Thus, SV $(G, 1) = \{\{1,2,3\}\}$. However, as the reader can easily verify, $\{1,2,3\} \notin H (G)$ (see (4.1)), and $\{1,2,3\} \notin D (G)$ (see Definition 5.1).

The following Remark shows that in dominated weak games all our hypotheses are compatible.

Remark 6.6: If $G = (N, W)$ is a dominated weak game and $\mathrm{v} = h (G)$, then $N \in H (G) \cap D (G) \cap \mathrm{SV} (G, \mathrm{v})$.

Proof: By the proof of Remark 5.15 $N \in H (G) \cap D (G)$. Also, since v is a vetoer it follows from Lemma 6.4 that $N \in \mathrm{SV} (G, \mathrm{v})$.

For the sake of easy reference we now restate some of the implications of the hypothesis (SV).

Remark 6.7: Let $G = (N, W)$ be a dominated simple game and let $i = h (G)$. (SV) implies that i erects S with maximum $\varphi^i (S)$. Thus, i considers only his "power" within the coalition which he chooses. In particular, no information on the bargaining process which leads to the erection of a coalition is explicit in (SV). Furthermore, only the situation *after* the formation of a coalition S is valued by i, and it is assumed, implicitly, that the opposition, $N - S$, has no influence on the process of decision making by S. Finally, no direct reference to payoff considerations is made. However, it is quite obvious from Lemma 6.4 that for weighted majority games (SV) implies, in most cases, the rejection of the Riker-Gamson minimum size principle, which *is* based on payoff considerations [see *De Swaan*].

7. The Nucleolus Approach

In this section we consider simple games as cooperative games with side payments. This enables us to use the theories of the kernel and the nucleolus in order to investigate coalition formation in dominated games. At the end of this section we shall try to elucidate, under certain assumptions, the relationship between coalition formation in committees (i.e., simple games), and bargaining processes over the final distribution of payoffs in cooperative simple games with side payments (see Remark 7.11).

Let $G = (N, W)$ be a simple game. We assume that G is non-null, i.e., $N \in W$. The *characteristic function* c of G is the function $c : 2^N \to \{0,1\}$ which satisfies $[c(S) = 1 \Leftrightarrow S \in W]$. A *payoff vector* is an n-tuple $x = (x^1, \ldots, x^n)$ (recall that $N = \{1, \ldots, n\}$) of real numbers which satisfies $x^i \geqslant 0, i = 1, \ldots, n$ and $x(N) = 1$ (see Remark 2.5). We denote by X the set of all payoff vectors. For $x \in X$, let $\theta(x)$ be a 2^n-tuple whose components are the numbers $c(S) - x(S), S \subset N$, arranged according to their magnitude, i.e., $\theta_i(x) \geqslant \theta_j(x)$ for $1 \leqslant i < j \leqslant 2^n$. The *nucleolus* of G, $\nu = \nu(G)$, is the payoff vector which is "closest" to c in the sense that $\theta(\nu)$ is the minimum, in the lexicographic order, of the set $\{\theta(x) \mid x \in X\}$ [see *Schmeidler*]. Let now $S \in W$. The *reduced game* (with respect to S) is the game RG $(S) = (S, W_S)$, where $T \in W_S \Leftrightarrow [T \subset S$ and there exists $B \subset N - S$ such that $T \cup B \in W]$.

If G is *proper* then the *nucleolus of* S is $\nu(S) = \nu(\text{RG}(S))$ [see *Aumann/Dreze*, p. 222].

Remark 7.1: Let $G = (N, W)$ be a simple game and let $S \in W$. In the subgame $G \mid S$ (see Section 6) the effect of the opposition $N - S$ is completely ignored. Thus, $G \mid S$ describes the situation after the erection of S (see Remark 6.7). In the reduced game RG (S) the situation is completely different: the members of $N - S$ are ready to cooperate with sub-coalitions of S for any positive payoff. Thus, RG (S) is more likely to reflect the "power" of the various sub-coalitions of S *during the process of establishing* S (i.e., during the negotiations about the details of an agreement on which the erection of S will be based).

Let $G = (N, W)$ be a dominated game (see Definition 2.21) and let $i = h(G)$. If the distribution of payoffs among the members of a coalition which forms is determined by the nucleolus (of that coalition), then, under Assumption (A) (see Section 4), i will choose S to maximize $\nu^i(S)$ (the component of $\nu(S)$ which corresponds to player i). This leads us to the following hypothesis. First we need the following notation. Let

$$\theta = \{S \mid S \in W \text{ and } i \in S\},$$

and

$$\text{NUC}(G) = \{S \mid S \in \theta \text{ and } \nu^i(S) \geqslant \nu^i(T) \text{ for all } T \in \theta\}. \tag{7.1}$$

We are now able to state our hypothesis:

(NUC) *Let $G = (N, W)$ be a dominated simple game and let $i = h(G)$. Under Assumption* (A) *only coalitions in* NUC (G) (see (7.1)) *are formed.*

Again, we have to check whether (NUC) is compatible with (H), (D) and (SV) (see Sections 4, 5 and 6 respectively). It is convenient first to introduce the kernel, a solution concept which is closely related to the nucleolus. Let $G = (N, W)$ be a non-null *monotonic* simple game, and let c be the characteristic function of G. Let $x \in X$ and let $k, m \in N, k \neq m$. We denote:

$$s_{k,m}(x) = \max \{c(S) - x(S) \mid S \subset N, k \in S \text{ and } m \notin S\}.$$

The *kernel* of G, $K(G)$, is defined by

$$K(G) = \{x \mid x \in X \text{ and } s_{k,m}(x) = s_{m,k}(x) \text{ for all } k, m \in N, k \neq m\}, \qquad (7.2)$$

[see *Maschler/Peleg/Shapley*, 76–77]. It is well-known that $\nu(G) \in K(G)$ [see *Schmeidler*]. One advantage of the kernel (of a monotonic game) over the nucleolus, is that the kernel is determined by an explicit system of equations (see (7.2)), and is therefore easier to compute.

Remark 7.2: The kernel is, of course, defined also for non-monotonic games. However, its definition in that case is somewhat more complicated than (7.2).

Let now $G = (N, W)$ be a monotonic simple game and let $S \in W$. If G is *proper* then the *kernel of* S is $K(S) = K(\mathrm{RG}(S))$, where $\mathrm{RG}(S)$ is the reduced game with respect to S [see *Maschler/Peleg*, Theorem 2.9]. The following remarks are very helpful for computing kernels of simple games. (Indeed, they are essential for the computation of Table 8.1.)

Remark 7.3: Let $G = (N, W)$ be a (non-null) proper and monotonic simple game and let $S \in W$. If $i \in S$ is not a vetoer (see Definition 2.9), and $x \in K(S)$, then $x^i \leqslant 1/2$.

Remark 7.3 is a direct consequence of (7.2).

Remark 7.4: Let $G = (N, W)$ be a proper, monotonic and strong simple game. If $S \in W^m$ (see Remark 2.13) and $x \in K(S)$ then $x^i = 1/|S|$ for all $i \in S$.

Remark 7.4 follows from the simple observation that in the reduced game $\mathrm{RG}(S)$ every player $i \in S$ is winning. (Indeed, if $i \in S$ then, since $S \in W^m$, $S - \{i\} \notin W$. Hence, since G is strong, $\{i\} \cup (N - S) \in W$, which implies that $\{i\} \in W_S$.)

Remark 7.5: Let $G = [q; w^1, \ldots, w^n]$ be a weighted majority game (see Definition 2.6). If $S = \{i_1, \ldots, i_k\} \in W$ then the reduced game $\mathrm{RG}(S) = [q - w(N - S); w^{i_1}, \ldots, w^{i_k}]$.

We now check the compatibility of (NUC) with (H), (D) and (SV).

Example 7.6: Let $G = [5; 3,1,1,1,1,1,1]$. G is a strong and homogeneous weighted majority game and $1 = h(G)$. Using Remark 7.5, Table 1 in *Aumann/Peleg/Rabinowitz* [1965] and *Kopelowitz* [1967] we find that

$$\mathrm{NUC}(G) = \{S \mid 1 \in S \text{ and } |S \cap \{2,3,4,5,6,7\}| = 4\} \quad (\text{see } (7.1)).$$

Hence $\mathrm{NUC}(G) \cap H(G) = \emptyset$ and $\mathrm{NUC}(G) \cap D(G) = \emptyset$ (see (4.1) and Definition 5.1). Note that 1 is *not* almost winning (see Definition 5.9).

Remark 7.7: If $G = (N, W)$ is a dominated weak game then $N \in \mathrm{NUC}(G)$.

Proof: Let $i = h(G)$. Then i is a vetoer. Let $x \in K(G)$. By Theorem 4.1 in *Maschler/Peleg* [1967] $x^i = 1$. Hence, since the nucleolus of G $\nu \in K(G)$, $\nu^i = 1$. Thus, since $\nu(G) = \nu(N)$, $N \in \mathrm{NUC}(G)$.

Remark 7.8: It follows from Remarks 6.6 and 7.7 that, in dominated weak games, all the four assumptions (H), (D), (SV) and (NUC) are simultaneously compatible.

Theorem 7.9: Let $G = (N, W)$ be a dominated weighted majority game and let $i = h(G)$. If i is almost winning (see Definition 5.9) *then* NUC $(G) \cap H(G) \neq \emptyset$.

Proof: By Remark 7.8 we have to prove the theorem only when i is not a vetoer. Since i is almost winning there exists $j \in N, j \neq i$, such that $S = \{i, j\} \in W$. Since i is not a vetoer, $N - \{i\} = \{j\} \cup (N - S) \in W$. Hence $\{j\} \in W_S$ (where RG $(S) = (S, W_S)$). Clearly, $\{i\} \in W_S$. Hence, by symmetry, $\nu^i(S) = 1/2$. By Remark 7.3 $S \in$ NUC (G). By Lemma 2.4, $i \leftarrow j$. Hence, $S \in H(G)$.
Thus, $S \in H(G) \cap$ NUC (G).

Example 7.10: Let $G = [5; 3,2,1,1,1,1]$. Then G is a dominated, strong and homogeneous weighted majority game, $1 = h(G)$ and 1 is almost winning. Denote $T = \{3,4,5,6\}$. Then, as the reader can easily verify

$$D(G) = \{S \mid 1 \in S \text{ and } |S \cap T| \geqslant 3\} \cup \{\{2,3,4,5,6\}\}. \tag{7.3}$$

Since $\{1,2\} \in W$ it follows from Remarks 7.3 and 7.4 that

$$\max \{\nu^1(S) \mid 1 \in S \text{ and } S \in W\} = 1/2. \tag{7.4}$$

Using Remark 7.5, Table 1 in *Aumann/Peleg/Rabinowitz* [1965] and *Kopelowitz* [1967], one shows that $D(G) \cap$ NUC $(G) = \emptyset$. (Indeed, if $S = \{1,3,4,5\}$ then RG $(S) = [2; 2,1,1,1]$ and $\nu^1(S) = 2/5$; if $S = \{1,3,4,5,6\}$then RG $(S) = [3; 3,1,1,1,1]$ and $\nu^1(S) = 3/7$; if $S = \{1,2,3,4,5\}$ then RG $(S) = [4; 3,2,1,1,1]$ and $\nu^1(S) = 3/8$; finally, if $S = N$ then $\nu^1(S) = 1/3$. Hence, by (7.3) and (7.4), $D(G) \cap$ NUC $(G) = \emptyset$.) By Lemmata 6.3 and 6.4 $\varphi^1(S)$ is maximized either at $S_1 = \{1,2,3,4\}$ or at $S_2 = \{1,3,4,5,6\}$. By direct computation $\varphi^1(S_1) = 7/12$ and $\varphi^1(S_2) = 3/5$. Thus SV $(G, 1) = \{S_2\}$ (see (6.1)). As we have already shown $S_2 \notin$ NUC (G) (see (7.1)). Hence SV $(G, 1) \cap$ NUC $(G) = \emptyset$.
We conclude this section with the following remarks.

Remark 7.11: Let $G = (N, W)$ be a non-null simple game. We address ourselves to the following problem: under what assumptions can G be considered as a cooperative game with side payments. Since we apply the theoretical results of this paper mainly to parliaments, we shall assume further that G represents a parliament. Thus, the players of G (i.e., the members of N) are parties, and if a (winning) coalition S forms then the parties in S have to divide among themselves a certain number k of portfolios. (The number k my not be fixed in advance; e.g., it may be part of the outcome of the bargaining between the members of S. However, this does not affect the following discussion.) Let $p_1, \ldots, p_k$ be the different portfolios and let the budget allocated to portfolio p_j be y_j, $j = 1, \ldots, k$. We assume that $y_1, \ldots, y_k$ (and k itself), are determined during the "first phase" of negotiations among the parties in S. Now, for each portfolio p_j, $j = 1, \ldots, k$, there is a real number f_j, where $0 \leqslant f_j \leqslant y_j$, which equals that part of y_j which is not a-priori "tied", i.e., which is completely controlled by the minister holding the office p_j. We assume further that in the second (and final) phase of negotiations the parties in S have both the time required and the legal possibility to reach, by binding agreements, every distribution of the sum $\sum_{j=1}^{k} f_j$ (which is independent of the coalition S) between themselves. If we add the usual assumption that the utilities (for money) of the players of G are linear and increasing in money, then the theories of the kernel and the nucleolus can be applied to G.

Remark 7.12: The question whether the assumptions made in Remark 7.11 in order to describe parliaments as games with side payments are realistic, remains open. However, our feeling is that *during the period of the formation of a coalition* our assumptions do apply (at least approximately), to parliaments. Indeed, this is reflected in the results of Tables 8.2 and 8.3.

8. Classification of Ordinary Coalitions

In this section we do in full detail the classification of ordinary coalitions for 5 out of the 9 nations investigated in *De Swaan* [1973] (see Table 8.1). We also summarize the results of such a classification for the 41 ordinary coalitions in Table 8.1 of *Peleg* [1980] (see Table 8.3). Throughout this section we use the following notation. Let $G = (N, W)$ be a dominated simple game (see Definition 2.21) and let $i = h(G)$. We denote by W^m the set of minimal winning coalitions (see Remark 2.13). $H = H(G)$ is defined by (4.1). $D = D(G)$ is the set of determining coalitions in G (see Definition 5.1). SV = SV (G, i) is determined by (6.1). NUC = NUC (G) is defined by (7.1). Finally, if R is a policy order (see Definition 3.2), then we denote by CLMR = CLMR (G, R) the set of *minimal* elements of the set of all connected (see Definition 4.2) *and* winning coalitions in G. (The reader is referred to *De Swaan* [1973] for a discussion of the set CLMR (G, R).)

The data of the following table are taken from *De Swaan* [1973].

Tab. 8.1: Classification of Ordinary Coalitions in 5 Democracies

Denmark

No.	Date	Assembly	Coalition	Type
1	1920 a)	[71; 42,17,*49*,28]	{3,4}	W^m, H, NUC, CLMR
2	1920 b)	[71; 42,16,*52*,26]	{3,4}	W^m, H, NUC, CLMR
3	1924	[75; *55*, 20,45,28]	{1,2}	W^m, H, NUC, CLMR
4	1929	[75; *61*,16,44,24]	{1,2}	W^m, H, NUC, CLMR
5	1932	[75; *62*,14,39,27]	{1,2}	W^m, H, NUC, CLMR
6	1935	[75; *68*,14,29,26]	{1,2}	W^m, H, NUC, CLMR
7	1939 a)	[75; *64*,14,31,26]	{1,2}	W^m, H, NUC, CLMR
8	1939 b)	"	{1,2,3,4}	NUC, SV
9	1953	[89; 8,*75*,14,43,30]	{2,3}	W^m, H, NUC, CLMR
10	1957	[88; *70*,14,46,30,9]	{1,2,5}	W^m, H
11	1960	[89; 11,*78*,11,39,32]	{2,3}	W^m, H, NUC, CLMR
12	1966	[89; 20,*70*,13,35,34]	{1,2}	W^m, H, NUC, CLMR
13	1971	[90; 17,*73*,27,30,31]	{1,2}	W^m, H, NUC, CLMR

Israel

No.	Date	Assembly	Coalition	Type
1	3/49	[61; 4,19,*46*,4,16,5,7,14]	{3,4,5,6}	H, NUC
2	11/50	〃	〃	〃
3	12/52	[61; 5,15,*45*,10,4,5,20,8]	{3,4,5,7}	H, NUC
4	1/54	〃	〃	〃
5	1/55	[61; 6,9,10,*40*,11,5,6,13,15]	{2,3,4,5,6}	H, D
6	1/58	〃	〃	〃
7	7/59	[61; 9,7,*47*,12,6,6,8,17]	{1,2,3,4,5}	H, D
8	11/61	[61; 5,9,8,*42*,12,6,17,17]	{3,4,5}	W^m, H, CLMR
9	6/63	〃	〃	〃
10	12/64	〃	〃	〃
11	1/66	[61; 8,*45*,10,11,6,5,26]	{1,2,4,6}	H
12	5/67	〃	{1,2,3,4,6,7}	SV, NUC(?)
13	3/69	[61; 4,*56*,4,12,6,4,26]	{2,4,6,7}	H, D, NUC
14	8/70	〃	{2,4,6}	H, NUC

Italy

No.	Date	Assembly and Senate	Coalition	Type
1	6/46	[279; 104,115,23,*207*,41,16,30] no Senate	{1,2,3,4,5}	D, NUC
2	2/47	〃	{1,2,4}	SV, NUC
3	12/47	〃	{2,4,5}	H, NUC
4	5/48	[288; 183,33,*305*,19,14] [173; 119,24,*148*,27,8]	{2,3,4}	H, NUC
5	10/54	[295; 143,75,19,*263*,13,40,29] [119; 55,31,4,*114*,3,15,9]	{3,4,5}	W^m, H, CLMR
6	7/55	〃	〃	〃
7	2/59	[299; 140,84,23,*273*,16,23,25] [124; 60,37,5,*123*,4,7,8]	{4,5,6,7}	H, D, NUC
8	7/60	〃	{3,4,5}	W^m, H, NUC, CLMR
9	12/63	[315; 166,87,33,*260*,39,27] [158; 85,44,14,*133*,19,15]	{2,3,4}	H, NUC, CLMR
10	7/64	〃	〃	〃
11	2/66	〃	〃	〃
12	12/66	[315; 166,24,95,*260*,38,26] [158; 85,12,46,*133*,19,15] [316; 177,23,91,*266*,31,24]	{3,4}	W^m, H, NUC, CLMR

No.	Date	Assembly and Senate	Coalition	Type
13	12/68		{3,4}	W^m, H, NUC, CLMR
		[158; 87,14,46,*135*,16,11]		
		[316; 177,23,62,29,*266*,31,24]		
14	8/69		{3,4,5}	H, NUC, CLMR
		[158; 87,14,34,12,*135*,16,11]		
15	4/70	"	"	"
16	8/70	"	"	"

The Netherlands

No.	Date	Assembly	Coalition	Type
1	9/22	[51; 20,5,*32*,16,11,10]	{3,4,5}	W^m, H, CLMR
2	8/25	[51; 24,7,*31*,13,11,9]	{3,4,5}	W^m, H, CLMR
3	3/26	"	"	"
4	8/29	[51; 24,7,*30*,12,11,8,3]	{3,4,5}	W^m, H, CLMR
5	5/33	[51; 4,24,7,*28*,14,10,7,3]	{3,4,5,6,7}	H, D
6	7/35	"	"	"
7	6/37	[51; 3,23,6,*31*,17,8,4,4]	{4,5,6}	W^m, H, CLMR
8	8/39	"	{2,3,4,6}	NUC
9	7/46	[51; 10,29,*32*,13,8,6]	{2,3}	W^m, H, NUC, CLMR
10	8/48	[51; 8,27,*32*,13,9,8]	{2,3,5,6}	NUC
11	3/51	"	"	"
12	10/56	[76; 7,*50*,49,15,13,13]	{2,3,4,5}	D
13	7/63	[76; 4,4,43,*50*,13,13,16]	{4,5,6,7}	H, D
14	4/65	"	{3,4,5}	NUC, CLMR
15	4/67	[76; 5,4,37,7,*42*,15,12,17,7]	{5,6,7,8}	W^m, H, CLMR
16	5/73	[76; 7,7,*43*,6,27,14,7,6,22]	{2,3,4,5,6}	NUC(?)

Sweden

No.	Date	Assembly and Senate	Coalition	Type
		[116; *105*,23,33,65]		
1	1924		{1,3}	W^m, H, NUC, CLMR
		[191; *157*,41,68,109]		
		[116; 8,*104*,36,24,58]		
2	1932		{2,3}	W^m, H, NUC, CLMR
		[191; 9,*162*, 54,47,108]		
		[116; 6,*112*,36,27,44]		
3	1936		{2,3}	W^m, H, NUC, CLMR
		[191; 7,*178*,58,43,89]		
4	1939	"	{2,3,4,5}	SV, NUC
		[116; 8,*112*,30,57,23]		
5	1948		{2,3}	W^m, H, NUC, CLMR
		[191; 11,*196*,51,75,47]		
		[116; *110*,26,58,31]		

No.	Date	Assembly	Coalition	Type
6	1952		{1,2}	W^m, H, NUC, CLMR
		[191; *189*,51,80,51]		
		[116; 6,*106*,19,58,42]		
7	1956		{2,3}	W^m, H, NUC, CLMR
		[191; 9,*185*,44,88,55]		
		[176; 17,*163*,71,58,41]		
8	1970		{1,2}	W^m, H, NUC, CLMR
		no Senate		

The following remarks seem to be necessary for a complete understanding of Table 8.1.

Remark 8.1: The intersection symbol is omitted in our notation for the type of a coalition. For example, Coalition No. 1 of Denmark, $S = \{3,4\}$, is classified as "W^m, H, NUC, CLMR". This notation means that

$$S \in W^m \cap H \cap \text{NUC} \cap \text{CLMR}, \qquad \text{(see the next remark).}$$

Remark 8.2: If $[q; w^1, \ldots, w^n]$ is a representation (see Definition 2.6) of an assembly, or a senate (which appears in the third column of Table 8.1), then the order in which the weights are written is De Swaan's policy order for that assembly (or senate). This remark explains how to determine whether a coalition belongs to CLMR or not. Also, the weight of the dominant player is in italics.

Remark 8.3: The representations of assemblies (and senates) in Table 8.1 are not complete: Parties with less than 2.5 percent of the seats are omitted. This has not interrupted with the computations made in order obtain the desired classification, except at one point: we could not determine whether the coalition that formed in the Netherlands in 5/73 (Coalition No. 16) belongs to NUC.

Remark 8.4: Coalition No. 12 in Israel contains six parties. The corresponding reduced game (see Section 7) is a six-person *non*superadditive game, and therefore it does not appear on the list of *Kopelowitz* [1967]. Hence we had to leave open the question whether that coalition is in NUC.

Finally, the following remark is quite obvious.

Remark 8.5: We have chosen to include Denmark, Israel, Italy, the Netherlands and Sweden in Table 8.1, since in these countries the formation of ordinary coalitions seems to be the general rule. The situation in the other 4 countries is clearly different (see Table 3.1).

We now turn to a discussion of the results of Table 8.1. It is very convenient to introduce Table 8.2 at this point.

Type / Nation	W^m	H	D	SV	NUC	CLMR	Total
Denmark	12	12	0	1	12	11	13
Israel	3	13	4	1	6	3	14
Italy	5	14	2	1	14	11	16
The Netherlands	7	10	4	0	5	8	16
Sweden	7	7	0	1	8	7	8
Total	34	56	10	4	45	40	67
Percentage	51	84	15	6	67	60	100

Tab. 8.2: Summary of the Results of Table 8.1

Remark 8.6: For a discussion of the occurrence of minimal winning and minimal connected coalitions the reader is referred to *De Swaan* [1973]. We restrict ourselves to an examination of the hypotheses (H), (D), (SV) and (NUC) (see Sections 4, 5, 6 and 7 respectively).

Remark 8.7: The (relative) success of (H) is not surprising. (H) is a very simple (and, therefore, easily understandable) assumption, and is also very reasonable (see Section 4).

Remark 8.8: The following explanation is suggested for the failure of (D). Although (H) and (D) may be incompatible for a dominated strong weighted majority game (see Example 5.6), as far as applications are concerned there is a wide range wherein these assumptions are compatible (see Theorems 5.7, 5.10 and Corollary 5.12). (See also Table 8.3 and Remark 8.11.) However, it seems to us, that the achievement of the combination "D, ČLMR" is practically impossible. (Indeed, no coalition in Table 8.1 belongs to both D and CLMR.) Thus, the results of Table 8.1 seem to imply that, when there exists a well defined policy order, the desire to establish a (minimal) connected coalition results in the rejection of (D). Indeed, the possibility of the incompatibility of (CLMR) and (D) was conjectured in Remark 8.4 of *Peleg* [1980].

Remark 8.9: The complete failure of (SV) can be explained very easily. It follows from Lemma 6.4 that (H) and (SV) are almost always incompatible (indeed, there exists no coalition in Table 8.1 which belongs to both SV and H). Thus, the acceptance of (H) implies, practically, the rejection of (SV). For further criticism of (SV) the reader is referred to Remark 6.7.

Remark 8.10: The (relative) success of (NUC) is very surprising; especially, if we recall that the nucleolus is defined only for cooperative games with side payments, while parliaments are, formally, games without side payments. The explanation we offer is based on our remarks in Section 7. First, it seems that *during the process of coalition formation* parliaments, in many cases, can be approximated by games with side payments (see Remark 7.11). Secondly, during the same time the reduced game which corresponds to the coalition which forms, seems to reflect quite faithfully the bargaining possibilities of the members of that coalition (see Remark 7.1). The fact that the nucleolus of a coalition consists of a good (additive) approximation of the *reduced game* which corresponds to the coalition, explains its predictive capability.

We now comment briefly on coalition formations in town councils in Israel.

Type	W^m	H	D	SV	NUC	Total
Total	10	33	26	9	35	41
Percentage	24.4	80.5	63.4	22.0	85.4	100

Tab. 8.3: Classification of Ordinary Coalitions in Towns in Israel

Remark 8.11: We conclude from Table 8.3 that in the absence of a well defined policy order, determining coalitions are quite likely to form in a dominated game. Also, (NUC) is the most successful hypothesis in such situations.

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