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An Impossibility Result Concerning n-Person Bargaining Games[1])

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Abstract: In this note we show that a solution proposed by *Raiffa* for two-person bargaining games, which has recently been axiomatized by *Kalai/Smorodinsky*, does not generalize in a straightforward manner to general n-person bargaining games. Specifically, the solution is not Pareto optimal on the class of all n-person bargaining games, and no solution which is can possess the other properties which characterize *Raiffa*'s solution in the two-person case.

An n-person bargaining game consists of a set of players $N = \{1, \ldots, n\}$, a vector of utilities $d = (d_1, \ldots, d_n)$, and a compact convex subset $S \subset R^n$ of utility vectors which contains the vector d and at least one element x such that $x > d$. The interpretation is that the set S is the set of feasible expected utility payoffs to the players, any one of which can be achieved by unanimous agreement. In the event that no unanimous agreement is reached, the disagreement vector d is the result.[3])

A *solution* is a function defined on the class B of such bargaining games, which selects a feasible outcome for every game: i.e., $f: B \to R^n$ such that for any game (S, d) in B, $f(S, d) = x$ is an element of S. *Nash* [1950] was the first to study solutions to the bargaining game, which he interpreted as modelling the anticipated outcome of the game when it is played by rational players. *Raiffa* [1953] also studied solutions, which he interpreted as indicating the manner in which games might be arbitrated by an impartial arbiter.

Nash approached the problem axiomatically, and specified a list of properties which collectively characterize a unique solution. Although *Nash* explicitly studied only the case of two-person bargaining his treatment generalizes without any difficulty to the general n-person case.[4]) *Nash* required that a solution possess the following properties.[5])

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[3]) That is, if the players unanimously agree on some x in S, then each player i receives the utility x_i. If no agreement is reached, player i receives the utility d_i.

[4]) The two-person case is the one which has been studied the most, both because of its simplicity, as well as because most two-person cooperative games can be thought of as bargaining games, whereas bargaining games are only a subclass of games with more than two players.

[5]) These properties are stated here in a slightly different form than in *Nash*'s original paper.

Property 1. Pareto optimality: If $f(S, d) = x$, then there exists no y in S, distinct from x, such that $y \geqslant x$.

Property 2. Symmetry: If (S, d) is a symmetric game,[6]) then $f_i(S, d) = f_j(S, d)$ for all i, j in N.

Property 3. Independence of equivalent utility representations: Let $T : R^n \to R^n$ be an order-preserving linear transformation of the players' utility functions (i.e., if $y = T(x)$, then $y_i = a_i x_i + b_i$, where $a_i > 0$). Then $f(T(S), T(d)) = T(f(S,d))$.

Property 4. Independence of irrelevant alternatives: If T contains S, and $f(T, d) \in S$, then $f(S, d) = f(T, d)$.

These properties have been extensively discussed in the literature [cf. *Luce/Raiffa; Harsanyi; Kalai*, 1977a; *Roth*, 1977b, 1978, 1979b], and so we will not review that discussion here.

Nash proved that there is a unique solution which possesses properties 1–4. It is the solution $f = F$ defined by $F(S, d) = x$ such that $x > d$ and $\Pi (x_i - d_i) \geqslant \Pi (y_i - d_i)$ for all y in S such that $y > d$.
That is, *Nash*'s solution selects the strongly individually rational outcome which maximizes the geometric average of the gains available to the players. Formally, this result can be stated as follows.

Theorem 1: F is the unique solution which is Pareto optimal, symmetric, independent of equivalent utility representations, and independent of irrelevant alternatives.

It has recently been shown [*Roth*, 1977a] that the property of strong individual rationality (i.e., $f(S, d) > d$) can replace Pareto optimality in the characterization of Nash's solution.

Since Nash's solution is independent of irrelevant alternatives, it is not sensitive to the range of outcomes contained in the feasible set, as reflected, for instance, by the *ideal point* $\bar{x}$ defined by $\bar{x}_i = \max\{x_i \mid x \in S \text{ and } x \geqslant d\}$ for $i \in N$. The point $\bar{x}$ can be thought of as reflecting the potential aspirations of the players, and *Raiffa* [1953] proposed a solution for two-player games which is sensitive to changes in $\bar{x}$.[7]) Specifically, *Raiffa* proposed the solution G for two-player games such that $G(S, d) = x$ is the Pareto optimal point at which $(x_1 - d_1)/(x_2 - d_2) = (\bar{x}_1 - d_1)/(\bar{x}_2 - d_2)$. That is, the solution G selects the maximal point on the line joining d to $\bar{x}$, yielding each player the largest reward consistent with the constraint that the players' actual gains should be in proportion to their potential gains, as measured by the ideal point $\bar{x}$.

Kalai/Smorodinsky [1975] give an elegant axiomatization of the solution G, and show that it is the unique solution defined on the class of two-person games which

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[6]) A game (S, d) is symmetric if $d_i = d_j$ for all i, j in N, and if for every element x in S, every permutation $y = \pi x$ is also contained in S.

[7]) The same solution has been proposed by *Crott* [1971], on the basis of some experimental results, and it has also been studied by *Butrim* [1976], who attributes it to the Russian game-theorist Germeier.

possesses properties 1–3 as well as the following property.[8])

Property 5: Restricted monotonicity: Let (T, d) and (S, d) be games which share a common ideal point, and such that T contains S. Then $f(T, d) \geqslant f(S, d)$.

That is, *Kalai/Smorodinsky* proved the following.

Theorem 2: The solution G is the unique solution defined on the class of two-person games which possesses properties 1, 2, 3, and 5.

In view of the fact that *Nash*'s solution is so easily generalized to bargaining games with more than two players, we might expect that this would also be the case for the solution G. Instead, it turns out that for games with more than two players, the solution G no longer possesses all the properties named in the theorem, and there exists no solution which does. We will see that the solution G fails to select a Pareto optimal outcome for all n person games with $n \geqslant 3$, although it continues to obey the other properties named in Theorem 2. In fact, for games with more than two players, there exists no solution defined on the class B which possesses properties 1, 2, and 5. Formally, we will prove the following.

Theorem 3: For bargaining games with three or more players, no solution exists which possesses the properties of Pareto optimality, symmetry, and restricted monotonicity.

Proof: We will assume that f is a solution which possesses the properties named in the theorem, and show that this leads to a contradiction. Consider the n-person game ($n \geqslant 3$) whose disagreement point is equal to the origin (which we will denote by $\bar{O}$), and whose feasible set S is equal to the convex hull of $\bar{O}$ and the points p and q such that $p_1 = 0$ and $p_i = 1$ for $i \neq 1$, and $q_2 = 0$ and $q_j = 1$ for $j \neq 2$. Then the set of Pareto optimal points in S is the line segment joining p to q, so $f(S, \bar{O}) = x$ is a convex combination of p and q. In particular, $x_3 = 1$.

Now consider the game $(T, \bar{O})$, where $T = \{x \geqslant \bar{O} \mid \Sigma x_i \leqslant (n-1) \text{ and } x_i \leqslant 1 \text{ for } i = 1, \ldots, n\}$. Then $(T, \bar{O})$ is a symmetric game, so $f_1(T, \bar{O}) = f_2(T, \bar{O}) = \ldots = f_n(T, \bar{O})$, and the Pareto optimality of f implies that

$$f(T, \bar{O}) = z = \left(\frac{(n-1)}{n}, \frac{(n-1)}{n}, \ldots, \frac{(n-1)}{n} \right).$$

But the point $(1, 1, \ldots, 1)$ is the ideal point both of the game $(S, \bar{O})$ and of the game $(T, \bar{O})$, so the fact that T contains S and the restricted monotonicity of f imply that $z \geqslant x$. Since we have shown that $z_3 = (n-1)/n$ and $x_3 = 1$, this gives the contradiction needed to complete the proof.

[8]) Actually, *Kalai/Smorodinsky* consider a stronger property which they call individual monotonicity, and which implies the property of restricted monotonicity stated here. However their treatment makes it clear that only this weaker condition is necessary to characterize the solution G for two-person games.

[9]) Note that the solution G is not even weakly Pareto optimal, as it would be if we confined our attention to games with disposable utility. (For a discussion of some matters related to this point, see *Kalai* [1977b], *Myerson* [1977], and *Roth* [1979a]).

Note that the solution G does not fail to be well-defined for $n \geqslant 3$, since for any game (S, d) there is still a unique maximal point on the line joining d to the ideal point $\bar{x}$. Furthermore, it is straightforward to confirm that the solution G possesses properties 2, 3, and 5 for any n. Theorem 3 therefore implies that G must not possess property 1, and in fact, in the game $(S, \bar{O})$ considered in the proof of the theorem, $G(S, \bar{O}) = \bar{O}$, which is not Pareto optimal.[9])

These results suggest that there may be some fundamental differences between the general case and the two-person case, and that other solution concepts will need to be explored in any attempt to incorporate players' aspirations in a consistent way into multi-player games.

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