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Core and Competitive Equilibria with Indivisibilities[1])

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Abstract: The paper presents a model of an exchange economy with indivisible goods and money. There are a finite number of agents, each one initially endowed with a certain amount of money and at most one indivisible good. Each agent is assumed to have no use for more than one indivisible good. It is proved that the core of the economy is nonempty. If utility functions are increasing in money, and if the initial resources in money are in some sense "sufficient" the core allocations coincide with the competitive equilibrium allocations.

With restrictions on the set of feasible allocations, the same model is used to prove the existence of stable solutions in the generalized "marriage problem". However it is shown that, even if money enters the model, these solutions cannot generally be obtained as competitive equilibria.

1. Introduction

At the end of the paper "On Cores and Indivisibility" *Shapley/Scarf* [1974] review a series of models involving indivisible goods which have been studied in the literature from the point of view of the core. They conclude: "It would be interesting if a general framework could be found that would unify some or all these scattered results".

The purpose of this paper is both to present a general framework and to generalize some of the existing results.

We consider a model of an exchange economy with n agents and two goods. The first good is perfectly divisible and will be called money. The other is a good present in the economy in indivisible units subject to quality differentiation. The main restriction on the model is that each agent does not initially own more than one indivisible item and has no use for more than one of these items. Under these conditions, we prove that, whatever the preferences of the agents, the economy is balanced and so has a nonempty core.

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[1]) Part of this work was done in the winter 1982 when I was visiting the Cowles Foundation for Research in Economics.

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To see how this model allows a unified study of the models reviewed in *Shapley/Scarf* [1974], it is convenient to classify them in two categories: the "exchange models" and the "pairing models". The first category includes the model of exchange of houses[3]) without money studied by *Shapley/Scarf* [1974] and the model of assignment between buyers and sellers presented by *Shapley/Shubik* [1972], and generalized by *Kaneko* [1982, 1983]. In the second category, we find the "college admission" model of *Gale/Shapley* [1962], with the particular case of "marriage", interpreted after introduction of money as a "job matching model" by *Crawford/Knoer* [1981]. The "roommate problem", for which *Gale/Shapley* [1962] show that there may exist no stable solution, also belongs to this category.

The exchange models are special cases of the general model presented above. Their characteristic features can be captured by making restrictive assumptions on the distribution of initial resources and on the preferences of the agents. These restrictions do not alter the result of existence of the core. What we prove is that a market for one kind of indivisible goods has always a nonempty core. No assumption of complete symmetry as in *Shapley/Scarf* [1974], or of complete assymetry as in *Shapley/Shubik* [1972], or of transferable utility is needed for this result.

Nevertheless, to adapt our model to the pairing models, we have to impose restrictions on the allocations which are feasible. Roughly speaking, we have to translate the fact that if A is married to B, then B of course is married to A. Unfortunately, the proof of balancedness no longer works with this restriction. However we can prove that the core still exists in models involving two types of agents, men and women for a marriage model, firms and workers for a job market, colleges and students for a college admission.

The study of the relation between the core, when it exists, and competitive equilibria is the subject of the last section of the paper. The main result of this section is that the core coincides with the competitive equilibrium allocations in an exchange economy of the type presented above, as soon as money really enters the model. In fact we need for this result two assumptions. The first one ensures that money really enters in the preferences of the agents. The other implies that the initial resources in money are in some sense "sufficient".

The other conclusion of this section is that the "pairing models", even with two types of agents, have completely different behaviour with respect to price decentralization. To prove this, we give an example of a pairing model with money which has no competitive equilibrium.

2. The Model of Exchange

Let us consider an exchange economy with n agents and two goods. The first good is a perfectly divisible good which will be called money. The second good exists only in indivisible units called "items". These units can be different in quality but have all the same function for the consumers. In consequence, we assume that each agent

[3]) All the models quoted in the introduction will be described in the main body of the paper.

has no use for more than one item. A typical example of such a good is a house. We assume that initially an agent does not own more than one item. The number of items in the economy is therefore at most equal to the number of agents.

Let ω_i be the initial endowment of agent i in money. If agent i owns one item before the exchange, this item is denoted e^i. Let us rank the agents in such a way that each of the first q initially owns an item, but the others have none. The initial resources of agent i are (ω_i, e^i) if i belongs to $[1, \ldots, q]$, and $(\omega_i, 0)$ if i belongs to $[q+1, \ldots, n]$. In the following, it will be convenient to use the notation e^i for the initial resources of all the agents. Then, it will be understood that if $q+1 \leqslant i \leqslant n$, then $e^i = 0$.

The preferences of agent i are represented by a utility function u_i defined on $\mathbf{R}_+ \times \{e^1, \ldots, e^q, 0\}$. For every $e^j \in \{e^1, \ldots, e^q, 0\}$, it is assumed that $u_i(., e^j)$ is continuous and non decreasing with respect to the quantity of money.

An allocation for this economy is a vector $(m_i, e_i^j)_{i=1,\ldots,n}$ in $\mathbf{R}_+^n \times \{e^1, \ldots, e^q, 0\}^n$. e_i^j means that agent i has been allocated the item (or possibly the absence of item) which was owned initially by agent j. (If $j \in [q+1, \ldots, n]$, then $e_1^j = 0$).

An allocation is feasible if there exists a permutation σ of $N = \{1, \ldots, n\}$ such that the allocation is of the form $(m_i, e_i^{\sigma(i)})_{i=1,\ldots,n}$ with $\sum_{i=1}^{n} m_i \leqslant \sum_{i=1}^{n} \omega_i$. The existence of a permutation σ implies that each item has been attributed to one and only one agent.

Since we want to study the core of the economy we must describe the allocations feasible for a coalition S, $S \subset N$. An allocation $(m_i, e_i^j)_{i=1,\ldots,n}$ is feasible for a coalition S if $\sum_{i \in S} m_i \leqslant \sum_{i \in S} \omega_i$ and if there exists a permutation σ_S of S such that: $\forall\, i \in S \;\; e_i^j = e_i^{\sigma_S(i)}$.

Let us denote $A(S)$ the feasible allocations of the coalition S and Σ_S the set of permutations of S.

$$A(S) = \{(m_i, e_i^j)_{1 \leqslant i \leqslant n} \;\big|\; \exists\, \sigma_S \in \Sigma_S, j = \sigma_S(i)\ \forall\, i \in S \text{ and } \sum_{i \in S} m_i \leqslant \sum_{i \in S} \omega_i\}.$$

Let us note that this definition of feasible allocations implies that there is no free disposition of the indivisible goods. So this model covers the cases of markets where some agents want to get rid of their items (for example a used car or an aging house) but cannot do it economically. In order that the results below stay valid with a free disposal assumption, we have to add the following assumption of desirability:

$$u_i(m, e^j) \geqslant u_i(m, 0) \quad \forall\, m \geqslant 0 \quad \forall\, i \in [1, \ldots, n] \quad \forall\, j \in [1, \ldots, q].$$

The set of feasible allocations is then larger than $A(S)$ but only the allocations in $A(S)$ are relevant for our problem.

The core of the economy consists of those allocations which are feasible for N and such that no coalition S can find an allocation in $A(S)$ which is strictly preferred by all its members.

We associate with the economy a game without sidepayments whose characteristic function V is defined by

$$V(S) = \{v = (v_i)_{1 \leqslant i \leqslant n} \in \mathbf{R}^n \mid \exists\, (m_i, e_i^j) \in A(S), v_i \leqslant u_i(m_i, e_i^j)\ \forall\, i \in S\}.$$

The game is well defined in the sense that it has the following properties

a) $V(S)$ is closed in $\mathbf{R}^n$.

b) If $v \in V(S)$ and $v'_i \leqslant v_i\ \forall\, i \in S$, then $v' \in V(S)$.

c) $\operatorname{Proj}_{\mathbf{R}^S} [V(S) - \bigcup_{i \in S} \operatorname{Int} V(\{i\})]$ is nonempty and bounded.

Theorem 1: The exchange economy has a nonempty core.

That the core of the economy is nonempty if and only if the game V has a nonempty core is of course immediate. To establish the existence of the core for V we prove that V is balanced and so, by *Scarf*'s theorem [1967], has a core.

A family B of coalitions $S \subset N$ is balanced if there exist positive weights $(\delta_S)_{S \in B}$ such that $\sum_{S \in B / i \in S} \delta_S = 1$ for all $i \in N$. A game V is balanced if for every balanced family B of coalitions, $v \in \bigcap_{S \in B} V(S)$ implies $v \in V(N)$.

The proof that V is balanced depends on the following lemma.

Lemma 1: Let $M = (m_{ij})$ be an $n \times n$ matrix with coefficients in $\mathbf{R} \cup \{+\infty\}$. Let B be a balanced family of coalitions of $N = [1, \ldots, n]$, with weights $(\delta_S)_{S \in B}$. Then for any family $(\sigma_S)_{S \in B}$ of permutations of the sets S, we have the following inequality

$$\min_{\sigma \in \Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)} \leqslant \sum_{S \in B} \delta_S \sum_{i \in S} m_{i,\sigma_S(i)}.$$

Before proving lemma 1, we show how it implies Theorem 1.

Proof of Theorem 1: Let B be a balanced family of coalitions of N with weights $(\delta_S)_{S \in B}$ and let v be a vector in $\bigcap_{S \in B} V(S)$. Associate to v the matrix $M(v) = (m_{ij}(v))$ defined as follows

$$m_{ij}(v) = \inf \{m_i \in \mathbf{R}_+ \mid u_i(m_i, e^j) \geqslant v_i\}$$

with the convention that if the set $\{m_i \in \mathbf{R}_+ \mid u_i(m_i, e^j) \geqslant v_i\}$ is empty then $m_{ij}(v) = +\infty$.

The interpretation of the matrix $M(v)$ is clear: $m_{ij}(v)$ is the minimum amount of money that must be given to agent i in conjunction with the item e^j, initially owned by agent j, to guarantee the utility level v_i to agent i.

Given a coalition S, it follows from the definitions of $M(v)$ and $V(S)$ that v belongs to $V(S)$ if and only if there exists a permutation σ_S of S such that

$$\sum_{i\in S} m_{i,\sigma_S(i)}(v) \leqslant \sum_{i\in S} \omega_i .$$

Hence, since $v \in \bigcap_{S\in B} V(S)$, we know that there exists a family $(\sigma_S)_{S\in B}$ of permutations of the sets S such that

$$\forall S \in B \quad \sum_{i\in S} m_{i,\sigma_S(i)}(v) \leqslant \sum_{i\in S} \omega_i .$$

From Lemma 1 this implies that

$$\min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)}(v) \leqslant \sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)}(v) \leqslant \sum_{S\in B} \delta_S \sum_{i\in S} \omega_i$$

$$= \sum_{i=1}^{n} \omega_i \,(\sum_{S|i\in S} \delta_S) = \sum_{i=1}^{n} \omega_i .$$

Therefore, there exists at least one permutation σ of N such that

$$\sum_{i=1}^{n} m_{i,\sigma(i)}(v) \leqslant \sum_{i=1}^{n} \omega_i$$

and thus v belongs to $V(N)$. □

Before proving Lemma 1, we need several definitions. For $S \subset N$, a S-permutation matrix is an n by n zero-one matrix containing one 1 in each row and each column indexed by a member of S, and zeros in rows and columns indexed by members of $N \setminus S$. If σ_S is a permutation of the set S, the S-permutation matrix $A_{\sigma_S} = (a_{ij})$ associated with σ_S is such that $a_{ij} = 1$ if and only if $i \in S$, $j \in S$, and $j = \sigma_S(i)$.

An N-permutation matrix will be simply called a permutation matrix.

A matrix is said to be doubly stochastic if all its components are nonnegative real numbers, and if each of its rows and columns sums to 1.

Proof of Lemma 1: Let $M = (m_{ij})$ be an $n \times n$ matrix with coefficients in $\mathbf{R} \cup \{+\infty\}$ and B a balanced family of coalitions of $N = [1, \ldots, n]$ with weights $(\delta_S)_{S\in B}$.

For each $S \in B$, let $B_S = (b_{ij}^S)$ be the S-permutation matrix associated with σ_S. Consider the matrix $A = (a_{ij})$ defined by: $A = \sum_{S\in B} \delta_S B_S$.

The matrix A has the following properties.

1°) $\sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} m_{ij}.$

Proof:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} m_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} (\sum_{S\in B} \delta_S b_{ij}^S) m_{ij}$$

$$= \sum_{S\in B} \delta_S \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^S m_{ij}$$

$$= \sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)}.$$

2°) A is a doubly stochastic matrix.

Proof:

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} (\sum_{S\in B} \delta_S b_{ij}^S) = \sum_{S\in B} \delta_S \sum_{j=1}^{n} b_{ij}^S = \sum_{S\in B/i\in S} \delta_S = 1.$$

The same reasoning gives the proof for the sums of the columns.

By the Birkhoff-Von Neumann Theorem, A is a convex combination of permutation matrices. There exist $\sigma_1, \ldots, \sigma_K$, permutations of N and non-negative coefficients $\lambda_1, \ldots, \lambda_K$ with $\sum_{k=1}^{K} \lambda_k = 1$, such that

$$A = \sum_{k=1}^{K} \lambda_k A_{\sigma_k}$$

From 1°)

$$\sum_{S\in B} \delta_S \sum_{i\in S} m_{i,\sigma_S(i)} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} m_{ij} = \sum_{k=1}^{K} \lambda_k \sum_{i=1}^{n} m_{i,\sigma_k(i)}$$

$$\geqslant (\sum_{k=1}^{k} \lambda_k) \min_{\sigma\in\Sigma_N} \sum_{i=1}^{n} m_{i,\sigma(i)}. \qquad \square$$

Let us now turn to the exchange models studied in the literature. *Shapley/Scarf* [1974] consider a model with n traders, each with initially one item (for example a house). Each trader has a preference ordering on the n items available in the economy and has no use for more than one. The problem is to find a redistribution of the items in accordance with the preferences of the traders. It is proved that the problem has a solution since the model has a nonempty core.

This model is a special case of our general model with $m = n$ and $\omega_i = 0$ for all i. Theorem 1 proves that money can be introduced in the model without altering the result of existence of the core.

In constrast to the symmetric model of Shapley-Scarf, we find the completely asymmetric model of *Shapley/Shubik* [1972]. Here there are two kinds of traders: m sellers, each with initially one item, and p buyers which initially have the money. Sellers and buyers have asymmetric preferences. A seller values only his own item. A buyer has no use for more than one item but his preferences hold on all available items. The problem is to find a redistribution of the items with compensations in money which cannot be improved by any coalition of buyers and sellers. Shapley and Shubik make the assumption of transferable utility and prove the nonemptiness of the core using linear programming. *Kaneko* [1982] generalized the model and the result to the case of non-transferable utility.

The model just described corresponds to the following specification of our model:

- for all i and j, $u_i(m_i, e^j)$ is increasing in m_i
- for $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant q$ and $m_i \geqslant 0$
  $u_i(m_i, e^j) < u_i(m_i, 0) < u_i(m_i, e^i)$.

The asymmetry in the preferences of buyers and sellers implies that a seller will never buy the item of another seller. An allocation in the core of the economy such that buyer i $(q + 1 \leqslant i \leqslant n)$ gets the item of seller j must be of the form $(\omega_j + c, 0)$ for j, and $(\omega_i - c, e_i^j)$ for i, with $c \geqslant 0$. If not, it would be blocked either by the coalition $\{i, j\}$ or by the coalition $N - \{i, j\}$. Thus the only coalitions relevant for the problem are singletons or pairs of agents of different types and the problem can be solved as a "pairing problem" of assignment between buyers and sellers.

Theorem 1 proves that the asymmetry in the preferences of buyers and sellers is not necessary for the existence of a core. We may allow a seller to sell his item and buy another that he prefers and still have a core. The other important conclusion of *Shapley/Shubik* [1972] in the transferable utility case and of *Kaneko* [1982] in the non-transferable utility case is that the core allocations coincide with the competitive equilibrium allocations. We will generalize this result in Section 4.

3. The Pairing Model

The "pairing" models referred to in the introduction fit the framework of the model presented in the preceding section only if we make restrictions on the feasible allocations. Consider, for example, the simple marriage model of *Gale/Shapley* [1962]. "A certain community consists of n women and n men. Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner. The problem is to find a satisfactory way of marrying all members of the community".

We may try to describe this problem with our model. Take an exchange economy with $2n$ agents, each agent endowed initially with no money and one, namely himself, indivisible item. We can translate the fact that each person has preferences only for persons of the opposite sex by the following assumption:

$$1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n, i \neq j, n+1 \leqslant k \leqslant 2n, n+1 \leqslant k' \leqslant 2n, k \neq k'$$

$$u_i(0, e^j) < u_i(0, e^i) < u_i(0, e^k)$$

$$u_k(0, e^{k'}) < u_k(0, e^k) < u_k(0, e^i).$$

If we study the core of such an economy, we will find allocations of the form $(0, e_i^{\sigma(i)})_{1 \leqslant i \leqslant n}$ where σ is a permutation of $N = \{1, \ldots, 2n\}$. The assumption made on preferences will imply that if $i \in [1, \ldots, n]$ then $\sigma(i) \in [n+1, \ldots, 2n]$. But nothing in the model can ensure that the agents are paired. We have to impose the additional condition: $\sigma(i) = j \Rightarrow \sigma(j) = i$[4]).

Hence the following definition:

Definition: A model as described in Section 2 is called a "pairing" model if the set of feasible allocations for the coalitions $S \subseteq N$ is restricted by the condition $\sigma_S \circ \sigma_S = Id_S$ where Id_S is the identity mapping of the set S.

The set of feasible allocations for a coalition S is then:

$$\bar{A}(S) = \{(m_i, e_i^j)_{i=1,\ldots,n} \mid \exists\, \sigma_S \in \Sigma_S, \sigma_S \circ \sigma_S = Id_S, j = \sigma_S(i) \;\forall\, i \in S \text{ and } \sum_{i \in S} m_i \leqslant \sum_{i \in S} \omega_i\}.$$

The associated game will be denoted $\bar{V}(S)$.

In the literature we find some pairing models which have an empty core. For example, the "roommate problem" presented by *Gale/Shapley* [1962]. "An even number of boys wish to divide up into pairs of roommates. A set of pairing is called stable if under it there are no two boys who are not roommates and who prefer each other to their actual roommates. An easy example shows that there can be situations in which there exists no stable pairing. Namely consider boys α, β, γ and δ, where α ranks β first, β ranks γ first and α, β and γ all rank δ last. Then regardless of δ's preferences there can be no stable pairing, for whoever has to room with δ will want to move out, and one of the other two will be willing to take him in."

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[4]) In spite of the apparent similarity between the model of marriage and the model of assignment between buyers and sellers, this difficulty does not appear when we translate the model of Shapley and Shubik in our exchange model. In this case, if j is a "seller" ($j \in [1, \ldots, q]$) and if i and i' are two "buyers" ($i \in [q+1, \ldots, n]$ and $i' \in [q+1, \ldots, n]$), the allocations $((m_j, e_j^{i'}), (m_{i'}, e_{i'}^j))$ and $((m_j, e_j^i), (m_{i'}, e_{i'}^j))$ are the same since $e^i = e^{i'} = 0$. In other words, a "seller" is indifferent between the "no item" of i or the "no item" of i'. The assumptions on the preferences and initial resources then imply that an allocation in the core can always be represented by a permutation σ such that $j = \sigma(i) \Rightarrow i = \sigma(j)$. On the other hand, in the marriage model, the "man" j typically will not be indifferent between the "woman" i and the "woman" i' and the allocation $((m_j, e_j^{i'}), (m_{i'}, e_{i'}^j))$ is really different from $((m_j, e_j^i), (m_{i'}, e_{i'}^j))$.

The reason for emptiness of the core is the following. If we try to prove that the game $\bar{V}$ is balanced by constructing the matrix $M(v)$[5]) as in the proof of Theorem 1 we would have to prove:

$$\min_{\substack{\sigma \in \Sigma_N \\ \sigma \circ \sigma = Id_N}} \sum_{i=1}^{n} m_{i,\sigma(i)}(v) \leqslant \sum_{S \in B} \delta_S \sum_{i \in S} m_{i,\sigma_S(i)}(v).$$

But restriction on the admissible permutations σ can change the result. In fact to prove the above inequality we would have to prove that the matrix A (defined in the same way and with the same properties as in Theorem 1) can be decomposed into a convex combination of *symmetric* permutation matrices. And this is not always possible.

However, in the literature, the pairing models involving two different types of agents (marriage problem, college admission problem, job matching) have a nonempty core. We can prove that this result is general.

Definition: We will say that the pairing model involves two types of agents if there exists an integer number p < n such that the utility functions have the following property

for

$$1 \leqslant i \leqslant p, \quad 1 \leqslant j \leqslant p \quad i \neq j$$

$$p+1 \leqslant h \leqslant n, \quad p+1 \leqslant k \leqslant n \quad k \neq h$$

and for $m \geqslant 0$

$$u_i(m, e^j) < u_i(m, e^h)$$

$$u_h(m, e^k) < u_h(m, e^i).$$

This assumption means that agents of one type strictly prefer rather to be paired with any agent of the other type than to be paired with an agent of his own type.

Theorem 2: The pairing model with two types of agents has a nonempty core.

Remark: This theorem is another form of *Kaneko's* result [1982] that the "Central Assignment Game" has a nonempty core. Nevertheless, a proof coherent with the logic of this paper seems to be of interest.

Proof: We have to prove that if a vector v belongs to $\bigcap_{S \in B} V(S)$ for a balanced family B of coalitions of N, then v belongs to $V(N)$. It suffices to prove this property for a minimal balanced family of coalitions, since every balanced family B contains at least one minimal balanced family B' and $\bigcap_{S \in B} V(S) \subset \bigcap_{S \in B'} V(S)$.

[5]) The interpretation of $m_{ij}(v)$ is now: $m_{ij}(v)$ is the minimum amount of money needed by agent i to reach the utility level v_i when paired with agent j.

Consider the matrix $M(\nu)$ defined as in the proof of Theorem 1. ν belongs to $V(N)$ if and only if there exists a *symmetric* permutation σ of Σ_N such that

$$\sum_{i=1}^{n} m_{i,\sigma(i)}(\nu) \leqslant \sum_{i=1}^{n} \omega_i .$$

As $\nu \in \bigcap_{S \in B} V(S)$, there exists for each $S \in B$, a symmetric permutation σ_S such that $\sum_{i \in S} m_{i,\sigma_S(i)}(\nu) \leqslant \sum_{i \in S} \omega_i$. The assumption made on the preferences implies that σ_S can always be chosen such that

$$1 \leqslant i \leqslant p \qquad \sigma_S(i) = i \text{ or } \sigma_S(i) \in [p+1, \ldots, n]$$

$$p+1 \leqslant i \leqslant n \quad \sigma_S(i) = i \text{ or } \sigma_S(i) \in [1, \ldots, p].$$

The permutation matrix B_S associated to σ_S is then of the "form"

$$\left[\begin{array}{c|c} D_S & {}^tM_S \\ \hline M_S & D'_S \end{array}\right]$$

where D_S is a $p \times p$ diagonal matrix and D'_S an $(n-p) \times (n-p)$ diagonal matrix.

On the other hand, B is a minimal balanced family of coalitions. It is easy to deduce from the proof of uniqueness of the associated weights $(\delta_S)_{S \in B}$ [*Shapley*] that they are rational numbers. Let d be a common denominator so that $\delta_S = c_S / d$ with c_S a positive integer.

The matrix $A = \sum_{S \in B} \delta_S B_S$ has the following properties:

(1) $\sum_{S \in B} \delta_S \sum_{i \in S} m_{i,\sigma_S(i)}(\nu) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} m_{ij}(\nu)$ (same proof as in Lemma 1).

(2) A is doubly stochastic (same proof as in Lemma 1).

(3) A is of the "form":

$$\left[\begin{array}{c|c} D & {}^tM \\ \hline M & D' \end{array}\right]$$

where D is a $p \times p$ diagonal matrix and D' an $(n-p) \times (n-p)$ diagonal matrix.

(4) The coefficients of A are rational numbers with common denominator d.

Hence, the matrix $\widetilde{A} = dA$ has integer coefficients, rows and columns which sum to d, and is of the same "form" as A.

The proof of Theorem 2 depends on the following lemma.

Lemma 2: Suppose that an $n \times n$ *matrix* $\widetilde{A} = (\widetilde{a}_{ij})$ *has non negative integer coefficients whose rows and columns each sum to an integer* d *and that:*

$$\widetilde{A} = \left[\begin{array}{c|c} D & {}^tM \\ \hline M & D' \end{array}\right]$$

where D *is a* $p \times p$ *diagonal matrix and* D' *an* $(n-p) \times (n-p)$ *diagonal matrix. Then there exist symmetric permutations* $\sigma_1, \ldots, \sigma_d$ *of* N *with matrices* $A_{\sigma_1}, \ldots, A_{\sigma_d}$ *such that:*

$$\widetilde{A} = A_{\sigma_1} + \ldots + A_{\sigma_d}.$$

Proof of Lemma 2: The proof is by induction on d. If $d = 1$, $\widetilde{A}$ is itself a symmetric permutation matrix. Let us suppose that the property holds for d and let $\widetilde{A}$ be a matrix with the properties of the Lemma and whose columns and rows sum to $d + 1$. $(1/(d+1))\,\widetilde{A}$ is a doubly stochastic matrix and can be written as a convex combination of permutation matrices. The integer coefficients of $\widetilde{A}$ are each greater than or equal to the coefficients of any permutation matrix which enters the decomposition of $(1/(d+1))\,\widetilde{A}$. So there must be at least one permutation σ of the decomposition for which $\widetilde{A} \geqslant A_\sigma$. If σ is symmetric our job is essentially done. If σ is not symmetric, define σ' by

$$1 \leqslant i \leqslant p \qquad \sigma'(i) = \sigma(i)$$

$$p+1 \leqslant i \leqslant n \qquad \sigma'(i) = \sigma^{-1}(i).$$

We first prove that $\sigma' \circ \sigma' = Id_N$.

(i) Let $i \in [1, \ldots, p]$ $\sigma' \circ \sigma'(i) = \sigma'(\sigma(i))$.
Since A_σ is inferior to $\widetilde{A}$ which has the form indicated in the Lemma, $i \in [1, \ldots, p] \Rightarrow \sigma(i) = i$ or $\sigma(i) \in [p+1, \ldots, n]$. If $\sigma(i) = i$, $\sigma'(\sigma(i)) = \sigma'(i) = \sigma(i) = i$. If $\sigma(i) \in [p+1, \ldots, n]$, $\sigma'(\sigma(i)) = \sigma^{-1}(\sigma(i)) = i$.

(ii) Let $i \in [p+1, \ldots, n]$, $\sigma' \circ \sigma'(i) = \sigma'(\sigma^{-1}(i))$.
For the same reason, $i \in [p+1, \ldots, n] \Rightarrow [\sigma^{-1}(i) = i$ or $\sigma^{-1}(i) \in [1, \ldots, p]]$. If $\sigma^{-1}(i) = i$, $\sigma'(\sigma^{-1}(i)) = \sigma'(i) = \sigma^{-1}(i) = i$. If $\sigma^{-1}(i) \in [1, \ldots, p]$, $\sigma'(\sigma^{-1}(i)) = \sigma(\sigma^{-1}(i)) = i$.

To prove that $A_{\sigma'} \leqslant \widetilde{A}$, notice that, from the construction of σ', the non zero coefficients of $A_{\sigma'}$ correspond to positive coefficients of A_σ or $A_{\sigma^{-1}}$. But $A_{\sigma^{-1}}$ is the transposed matrix of A_σ, so since $\widetilde{A}$ is symmetric, $A_\sigma \leqslant \widetilde{A}$ implies $A_{\sigma^{-1}} \leqslant \widetilde{A}$ and therefore $A_{\sigma'} \leqslant \widetilde{A}$.

$\widetilde{A} - A_{\sigma'}$ (which is equal to $\widetilde{A} - A_\sigma$ if σ is symmetric) has all the properties of $\widetilde{A}$ with rows and columns which sum to d, so the result obtains by induction. □

Proof of Theorem 2 (completed): We can deduce from Lemma 2 that $A = \widetilde{A} / d$ is a convex combination of *symmetric* permutation matrices. This implies (from property (1)) that

$$\sum_{S \in B} \delta_S \sum_{i \in S} m_{i,\sigma_S(i)}(v) \geqslant \min_{\substack{\sigma \in \Sigma_N \\ \sigma \circ \sigma = Id_N}} \sum_{i=1}^{n} m_{i,\sigma(i)}(v),$$

which in turn implies (since $\sum_{i \in S} m_{i,\sigma_S(i)}(v) \leqslant \sum_{i \in S} \omega_i$ for every S in B)that

$$\min_{\substack{\sigma \in \Sigma_N \\ \sigma \circ \sigma = Id_N}} \sum_{i=1}^{n} m_{i,\sigma(i)}(v) \leqslant \sum_{i=1}^{n} \omega_i$$

and thus that $v \in V(N)$. □

Theorem 2 gives an alternative proof of the result of *Gale/Shapley* [1962] that the marriage problem has a stable solution.

The college admission problem does not directly enter the framework of the pairing model since more than one student may go to the same college. The problem is the following: "A set of n applicants is to be assigned among m colleges where q_i is the quota of the i-th college. Each applicant ranks the colleges in order of his preferences. Each college similarly ranks the students who have applied to it in order of preference. An assignment of applicants to colleges will be called unstable if there are two applicants α and β who are assigned to colleges A and B, respectively, although β prefers A to B and A prefers β to α."

The existence of a stable assignment can nevertheless be deduced from Theorem 2, by considering a pairing model with no endowments in money, where the m students are the agents of one type, and where there are $q_1 + q_2 + \ldots + q_n$ agents of the other type, college i being replicated q_i times. Of course, the rank of two replica of the same college is the same in the preferences of the students, and two replicated colleges have the same ranking of the students. A core allocation of this model gives a stable assignment of students to colleges.

However, the constructive proof given by *Gale/Shapley* [1962] of the existence of stable assignments for the college admission problem and the marriage problem is more interesting that the proof by balancedness since it gives a procedure to find them. The main interest here is more the comparison of the structures of the exchange models and the pairing models than the result of Theorem 2 itself.

A version of the college admission model with money is studied by *Crawford/Knoer* [1981] and *Kelso/Crawford* [1982] with an interpretation in terms of job matching. One type of agents is composed of firms and the other of workers. Money enters the model in the form of salaries given by firms to workers. If it is assumed that each firm hires at most one worker, the existence of a stable assignment can be deduced from Theorem 2, without any assumption on the utility functions. But when firms can hire more than one worker, it becomes difficult to use the pairing model to get results. Replicating firms is not possible if the preferences of firms for workers are not separable and if the firms have budget constraints (a difficulty not taken into account in the two papers quoted above). Here again, the constructive proof based on a generalization of the Gale-Shapley algorithm is interesting.

These constructive proofs will be seen to be even more appealing at the end of Section 4 where it will be shown that, in contrast to the exchange model, the core allocations of a pairing model cannot in general be decentralized by means of competitive prices.

4. Competititive Prices

In the models of exchange with indivisibilities introduced by *Shapley/Scarf* [1974] and *Shapley/Shubik* [1972] the relation between the core allocations and the competitive equilibrium allocations was studied. The results were different from one paper to the other.

For the model without money of Shapley-Scarf, it was proved that at least one core allocation can be decentralized as a competitive equilibrium allocation (and this implies the existence of a competitive equilibrium for the model). But Shapley and Scarf give an example of a core allocation that cannot be decentralized by means of prices. As noted by the authors, the main reason for this comes from the definition of core allocations. Core allocations are defined as allocations that cannot be improved strictly by all members of a coalition. In a model with only indivisible items, this implies that some core allocations are weak Pareto optima but not Pareto optima. This fact introduces complexities in the question of decentralization by prices of core allocations, a problem which has been studied more completely by *Roth/Postlewaite* [1977].

On the other hand, in the model of "assignment" between buyers and sellers with compensation in money presented by Shapley and Shubik, it was proved that all core allocations could be decentralized as competitive equilibrium allocations. This was done in the case of linear utility of money in the original paper [*Shapley/Shubik*], and in the case of nontransferable utility by *Kaneko* [1982] under assumptions which ensure that money really enters the model.

We will prove in Theorem 3 that this result can be generalized with the same assumptions (assumption A.1 and A.2 below) to the general model of exchange. Before stating this Theorem, we give a precise definition of competitive equilibria in our model.

Definition: Let $E = \{(\omega_i, e^i)_{1 \leq i \leq n}, (u_i)_{1 \leq i \leq n}, q\}$ *be an exchange economy, where* q *is an integer belonging to* $[1, \dots, n]$ *such that:* $i > q \Rightarrow e^i = 0$. *A price vector for this economy is a vector* $p = (p_1, \dots, p_n) \in \mathbf{R}^n$ *such that:* $i > q \Rightarrow p_i = 0$.

A competitive equilibrium is a pair consisting of a price vector $\bar{p}$ *and a feasible allocation* $(\bar{m}_i, e_i^{\sigma(i)})_{1 \leq i \leq n}$ *such that*

$$\bar{m}_i + \bar{p}_{\sigma(i)} \leq \omega_i + \bar{p}_i \qquad \forall\, i \in [1, \dots, n]$$

$$[u_i(m_i, e_i^j) > u_i(\bar{m}_i, e_i^{\sigma(i)})] \Rightarrow [m_i + \bar{p}_j > \omega_i + \bar{p}_i] \qquad \forall\, i \in [1, \dots, n]$$

A competitive equilibrium allocation is an allocation $(\bar{m}_i, e_i^{\sigma(i)})_{i=1,\dots,n}$ *for which one can find a price vector* $\bar{p}$ *such that* $(\bar{p}, (\bar{m}_i, e_i^{\sigma(i)})_{1 \leq i \leq n})$ *is a competitive equilibrium.*

Theorem 3: Let $E = \{(\omega_i, e^i)_{1 \leq i \leq n}, (u_i)_{1 \leq i \leq n}, q\}$ *be an exchange economy. Given the following assumptions:*

A.1 *the functions* u_i *are increasing with respect to the money and* $\lim u_i(m, e^j) = +\infty$ *when* $m \to +\infty$ *for all* i *and* j *in* $[1, \dots, n]$

A.2 $u_i(\omega_i, e^i) \geq u^i(0, e^j)$ *for every* i *and* j *in* $[1, \dots, n]$

the set of core allocations and the set of competitive equilibrium allocations of E *coincide.*

Assumption A.2 can be justified as in *Kaneko* [1982]. It is argued there that a model with two goods must be considered as a partial analysis model where money is a composite good of all other commodities which are not considered explicitly in the model. Then it is not "normal" for an agent to enjoy an indivisible item (a house for example) but to consume nothing else.

Proof of Theorem 3 [6]. The usual reasoning can be applied to prove that a competitive equilibrium allocation is in the core. We are interested in the proof of the inverse implication.

Let

$$(\bar{m}_i, e_i^{\sigma(i)})_{1 \leq i \leq n}$$

be a core allocation.

[6]) This proof is due in large part to David Gale who introduced me to the notion of shortest path and made several suggestions that led to a considerable simplification of my original proof.

Define for all i and j in $[1, \ldots, n]$ the quantity m_{ij} by

$$m_{ij} = \min \{m \in \mathbf{R}_+ \mid u_i (m, e_i^j) \geq u_i (\bar{m}_i, e_i^{\sigma(i)})\}.$$

In the notation of Section 2, $m_{ij} = m_{ij} (\bar{v})$, where $\bar{v}$ is defined by

$$\bar{v}_i = u_i (\bar{m}_i, e_i^{\sigma(i)}).$$

If $j = \sigma(i)$, $m_{i,\sigma(i)} = \bar{m}_i$.

Assumption A.1 implies that $m_{ij} < +\infty$ and Assumption A.2 that

$$u_i (m_{ij}, e_i^j) = \bar{v}_i.$$

To prove that $p = (p_1, \ldots, p_n)$ is an equilibrium price associated with the allocation $(\bar{m}_i, e_i^{\sigma(i)})$, it is enough, from Assumption A.1, to prove that

1) $\bar{m}_i + p_{\sigma(i)} = \omega_i + p_i \quad \forall i \in [1, \ldots, n]$
2) $m_{ij} + p_j \geq \omega_i + p_i \quad \forall i \in [1, \ldots, n] \ \forall j \in [1, \ldots, n]$
3) $p_i = 0$ if $i > q$.

To find a price vector which satisfies these conditions, let us consider the directed graph with nodes $1, \ldots, n$ and such that the "transportation cost" or "length" from i to j is $m_{ji} - \omega_j$ (see Figure 1). Let l_{ij} denote this length.

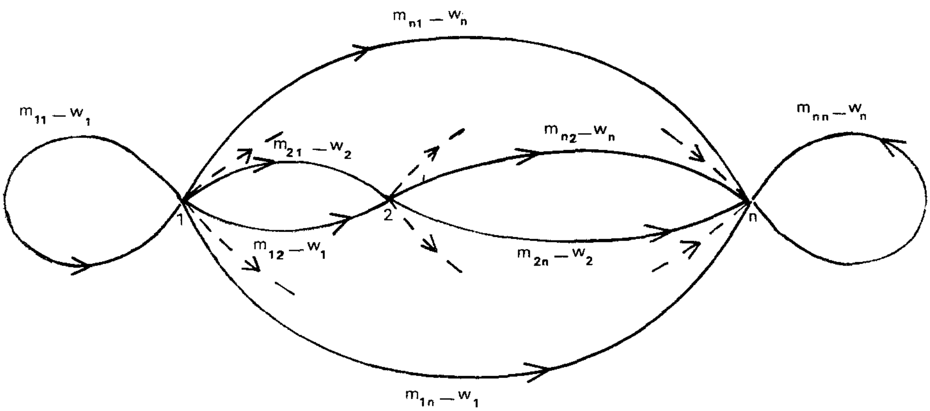

Fig. 1

A path from i to j is a sequence $(i, i_1, i_2, \ldots, i_m, j)$ and its length is $l_{ii_1} + l_{i_1 i_2} + \ldots + l_{i_m j}$.

Let Π_{ij} be the set of all paths from i to j.

Let p_n be arbitrary and take $p_i - p_n$ to be the length of the shortest path from n to i.

It is well known in graph theory that the minimization problem $\min_{\Pi_{ij}} [l_{ii_1} + l_{i_1 i_2} + \ldots + l_{i_m j}]$ has a solution if and only if there is no cycle of negative length. We prove that this is the case here.

Suppose that there exists a cycle $(i, i_1, \ldots, i_m, i)$ such that

$l_{ii_1} + \ldots + l_{i_m i} = m_{i_1 i} - \omega_{i_1} + \ldots + m_{ii_m} - \omega_i < 0$, or

$\omega_{i_1} + \ldots + \omega_i > m_{i_1 i} + \ldots + m_{ii_m}$. This means that the coalitions $\{i_1, \ldots, i_m, i\}$ can ensure the utility level $\bar{v}$ to its members by giving to i_1 the item of i and the amount of money $m_{i_1 i}$, to i_2 the item of i_1 and the amount of money $m_{i_2 i_1}$, etc. This coalition will, moreover, have a positive surplus of money, which can be used to strictly increase the utility of each of its members. But then, this coalition blocks the allocation $(\bar{m}_i, e_i^{\sigma(i)})_{1 \leq i \leq n}$, which is impossible since the allocation is in the core.

Therefore $p_i - p_n = \min_{\Pi_{ni}} l_{ni_1} + \ldots + l_{i_m i}$ defines the prices p_i without ambiguity.

We show that the prices p_i so defined satisfy (1) and (2).

Suppose that there exist i and j in $[1, \ldots, n]$ such that

$$m_{ij} + p_j < \omega_i + p_i.$$

This is equivalent to: $p_i > p_j + m_{ij} - \omega_i$, or to $p_i - p_n > p_j - p_n + l_{ji}$, which contradicts the defintion of p_i. Thus (2) holds. To prove (1) consider the following inequalities

$$\bar{m}_i + p_{\sigma(i)} \geq \omega_i + p_i$$

$$\bar{m}_{\sigma(i)} + p_{\sigma^2(i)} \geq \omega_{\sigma(i)} + p_{\sigma(i)}$$

$$m_{\sigma^2(i)} + p_{\sigma^3(i)} \geq \omega_{\sigma^2(i)} + p_{\sigma^2(i)}$$

$$\ldots$$

with $\sigma^2 = \sigma \circ \sigma, \sigma^3 = \sigma \circ \sigma^2, \ldots$

Since σ is a permutation of the finite set $[1, \ldots, n]$ there exists, for each i, a minimum number $\lambda_i \geq 0$ such that $\sigma^{\lambda_i}(i) = i$. The coalition $S^i = \{i, \sigma(i), \ldots, \sigma^{\lambda_i - 1}(i)\}$ is a "trading cycle" for the core allocation that we consider in the sense that the exchange of indivisible items takes place inside this coalition S^i. The item of $\sigma^{\lambda_i - 1}(i)$ goes to $\sigma^{\lambda_i - 2}(i), \ldots$, the item of $\sigma(i)$ goes to i, and the item of i to $\sigma^{\lambda_i - 1}(i)$. Then the core allocation must be such that $\sum_{j \in S^i} m_j = \sum_{j \in S^i} \omega_j$ since

$\sum_{j\in S^i} \bar{m}_j < \sum_{j\in S^i} \omega_j$, S^i would block the allocation, and if $\sum_{j\in S^i} \bar{m}_j > \sum_{j\in S^i} \omega_j$, $N\setminus S^i$ would block the allocation.

Therefore, adding the inequalities

$$\bar{m}_i + p_{\sigma(i)} \geqslant \omega_i + p_i$$

$$\bar{m}_{\sigma(i)} + p_{\sigma^2(i)} \geqslant \omega_{\sigma(i)} + p_{\sigma(i)}$$

$$\cdots$$

$$\bar{m}_{\sigma^{\lambda_i - 1}(i)} + p_i \geqslant \omega_{\sigma^{\lambda_i-1}(i)} + p_{\sigma^{\lambda_i-1}(i)}$$

we must obtain an equality, which is possible only if all inequalities are equalities. This proves (1).

If $q = n$, whatever the choice of p_n, the price p defined above is an equilibrium price.

If $q < n$, we have to deal with condition (3). This condition imposes the choice $p_n = 0$ and we must prove that $p_{q+1}, \ldots, p_{n-1}$ are then equal to 0.

Let i be an index such that $i \in [q+1, \ldots, n-1]$. There exists $k \in [1, \ldots, n]$ such that $\sigma(k) = i$. We must have, from (1)

$$\bar{m}_k + p_i = \omega_k + p_k \qquad (\text{since } \bar{m}_k = m_{ki}).$$

Since both i and n have no house as initial resources, $m_{kn} = m_{ki} = \bar{m}_k$ and we have, from (2)

$$\bar{m}_k + p_n \geqslant \omega_k + p_k$$

which implies, with the above equality, that $p_i \leqslant p_n$. The same reasoning applied to the agent k' such that $\sigma(k') = n$ implies that $p_n \leqslant p_i$, and thus $p_i = p_n = 0$. □

Remark 1: The proof of Theorem 3 shows, and this can be seen directly, that, if $q = n$ and p is an equilibrium price associated with an allocation $(m_i, e_i^{\sigma(i)})_{1\leqslant i\leqslant n}$, then for every $a \in \mathbf{R}$, $p + a$ is also an equilibrium price. (Only the differences $p_i - p_n$ are significant.) In this case, the equilibrium prices can be chosen to be positive.

However, if $q < n$, the condition $p_i = 0$ if $i > q$ imposes a normalization. Then, so ensure that equilibrium prices are non negative, we should have a "desirability" condition on the indivisible items, for example the assumption that we mentioned earlier:

(A.3) $\forall\, m \geqslant 0 \quad \forall\, i \in [1, \ldots, n] \quad \forall\, j \in [1, \ldots, q]$

$$u_i(m, e^j) \geqslant u_i(m, 0).$$

Remark 2: The reason for which Assumptions A.1 and A.2 allows to overcome the problem encountered by Shaley-Scarf in the model of exchange without money is that these assumptions ensure that the core and the strong core of E coincide.

The property of the exchange model that all core allocations can be decentralized by means of prices if we introduce money is not true for the pairing model with two types, (for which the core is nonempty). To illustrate this, the following example of a pairing model with money is one that has no competitive equilibrium at all.

There are two "men" α_1 and α_2 and two "women" A_1 and A_2. Each person initially owns one unit of money. The first number of each pair in the following matrix gives the ranking of women by the men, the second number the ranking of the men by the women.

$$\begin{array}{ccc} & A_1 & A_2 \\ \alpha_1 & 1,2 & 2,1 \\ \alpha_2 & 2,1 & 1,2 \end{array}$$

The utility functions are defined by:

$$u_{A_i}(m, \alpha_j) = \frac{1}{c_{A_i\alpha_j}} m$$

where $c_{A_i\alpha_j}$ is the rank of α_j in the ordering of A_i given by the matrix

$$u_{A_i}(m, A_j) = \frac{m}{4} .$$

Similarly

$$u_{\alpha_i}(m, A_j) = \frac{1}{c_{\alpha_i A_j}} m$$

$$u_{\alpha_i}(m, \alpha_j) = \frac{m}{4} .$$

These utility functions are such that:

- for a given amount of money, the ranking of one person on the possible partners of the other sex is the one given in the matrix;
- for a given amount of money, a person always prefers to be paired with a person of the other sex than either to stay alone or to be paired with a person of the same sex;
- assumptions A.1 and A.2 are fulfilled.

It is clear from these properties that core allocations must be associated with one of the pairings $(\alpha_1, A_1)(\alpha_2, A_2)$ or $(\alpha_1, A_2)(\alpha_2, A_1)$.

Let $(p_{A_1}, p_{A_2}, p_{\alpha_1}, p_{\alpha_2})$ be prices attached to each person. As a competitive equilibrium allocation is in the core, these prices, if they are competitive prices, must decentralize an allocation of form

$$(m_{\alpha_1}, A_1)\,(m_{A_1}, \alpha_1) \quad \text{for the pair } (\alpha_1, A_1)$$

$$(m_{\alpha_2}, A_2)\,(m_{A_2}, \alpha_2) \quad \text{for the pair } (\alpha_2, A_2)$$

or

$$(m_{\alpha_1}, A_2)\,(m_{A_2}, \alpha_1) \quad \text{for the pair } (\alpha_1, A_2)$$

$$(m_{\alpha_2}, A_1)\,(m_{A_1}, \alpha_2) \quad \text{for the pair } (\alpha_2, A_1).$$

In the first case we must have

$$m_{\alpha_1} + p_{A_1} = 1 + p_{\alpha_1}$$

$$m_{A_1} + p_{\alpha_1} = 1 + p_{A_1}$$

$$m_{\alpha_2} + p_{A_2} = 1 + p_{\alpha_2}$$

$$m_{A_2} + p_{\alpha_2} = 1 + p_{A_2}.$$

If $p_{\alpha_1} \leqslant p_{\alpha_2}$, then (m_{A_2}, α_1) satisfies the budget constraint of A_2 and is preferred by A_2 to (m_{A_2}, α_2). If $p_{\alpha_2} \leqslant p_{\alpha_1}$, (m_{A_1}, α_2) satisfies the budget constraint of A_1 and is preferred by A_1 to (m_{A_1}, α_1).

Therefore $(p_{\alpha_1}, p_{\alpha_2}, p_{A_1}, p_{A_2})$ cannot decentralize any allocation associated with the pairs $(\alpha_1, A_1)\,(\alpha_2, A_2)$. Changing the role of men and women the same reasoning proves that prices cannot decentralize neither an allocation associated with the pairs $(\alpha_1, A_2)\,(\alpha_2, A_1)$. Thus there is no competitive equilibrium.

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