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On Milnor's Classes "L" and "D"

By K. Kikuta[1] and L. S. Shapley[2]

Abstract: A twenty-one player counterexample is presented which disposes of two questions raised by J. W. Milnor in 1952 concerning the existence of certain pre-solutions, based on plausible lower and upper bounds to what a coalition should expect to receive in a cooperative game in characteristic function form. In the counterexample, the lower-bound set, known as "L", is empty, and the upper-bound set, known as "D", contains no efficient outcomes.

1 Background

In 1952, John Milnor [4][3] introduced three criteria for "reasonable" outcomes to cooperative games in characteristic function form. They amount to what have since been termed "pre-solutions" – that is, classes of outcomes, of which it is asserted (with respect to some particular view of the cooperative process) *not* that those within the class are necessarily plausible, but only that those outside the class are implausible[4]. The best-known and most successful of Milnor's classes is the so-called *reasonable set* "R", consisting of those payoff vectors which give no player more than his maximum marginal worth. This concept has been widely applied. The set "R" is always non-empty, and it has been shown to contain most of the standard solutions of cooperative game theory [1, 2, 3, 4, 6].

Less is known about the other two pre-solutions, known as "L" and "D", which put lower and upper bounds on the payoff to any *coalition*. Milnor gave examples in [4] to show that they do not necessarily contain the von Neumann-Morgenstern solutions or the Shapley value, and proved that "L" and the efficient part of "D" are non-empty for certain classes of games. Nineteen years later however, one of the present authors found a 21-person game for which "L" is empty (see [5]), and subsequently the other author, using the same game, disposed of "D" as well. The purpose of this note is to document these results.

1 Kensaku Kikuta, Toyama University, Dept. of Economics, Gofuku, Toyama City 930, Japan.
2 Lloyd S. Shapley, University of California, Dept. of Mathematics, Los Angeles, CA 90024, USA.
3 An extended discussion of [4] will be found in Luce and Raiffa [2], pp. 237–245.
4 The Pareto set and the individually rational set (and their intersection, the imputation space) are familiar examples of pre-solutions. Another example is the set of payoff vectors that exhibit all the symmetries of the game.

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2 The Sets L and D

A *game* (N, v) consists of a finite *player set* N and a superadditive *characteristic function* v, mapping the subsets of N to the real numbers $\mathbb{R}$ with $v(\phi) = 0$. The space of *payoff vectors* (or simply, *payoffs*), with components x_i indexed by $i \in N$, is denoted $\mathbb{R}^N$. The subset $\mathbb{F}^N$ of *feasible payoffs* is defined by $x(N) \leqslant v(N)$, and the subset $\mathbb{E}^N$ of *efficient payoffs* is defined by $x(N) = v(N)$. (Here, $x(\cdot)$ is a short notation for $\sum_{i \in \cdot} x_i$.)

Following [3], we define

$$l(S) = \min_{R \subseteq S} [v(R) + v(S \setminus R)]$$

for each $S \subseteq N$, and

$$L = L(x) = \{x \in \mathbb{F}^N : x(S) \geqslant l(S), \text{ all } S \subseteq N\}.$$

We see that $l(S) \leqslant v(S)$ for all S, with equality if $|S| = 1$. Intuitively, the difference $v(S) - l(S)$ measures the degree of vulnerability of a "shaky" coalition S to factionalism or internal dissension. We may therefore think of $l(S)$ as a lower bound to what the members of S could reasonably expect to salvage if their coalition should break in two.[5]

Continuing, we define

$$d(S) = \min_{z \in \mathbb{F}^{N \setminus S}} \max_{R \subseteq N \setminus S} [v(S \cup R) - z(R)]$$

for each $S \subseteq N$, and

$$D = D(v) = \{x \in \mathbb{F}^N : x(S) \leqslant d(S), \text{ all } S \subseteq N\}.$$

By taking $R = N \setminus S$, we see that $d(S) \geqslant v(N) - v(N \setminus S) \geqslant v(S)$ for all S.[6] Intuitively, $d(S)$ is an upper bound to what an aggressively-expanding coalition S could expect to get by persuading the players R to defect from the opposing coalition $N \setminus S$, by offering them more than they would get under the optimum "campaign promise" that $N \setminus S$ can make, namely the minimizing vector $z \in \mathbb{F}^{N \setminus S}$.[7]

Milnor [4] proved that $L \neq \phi$ for all games that can be expressed as positive linear combinations of games fully symmetric in their non-dummy players, and also that $\bar{D} \neq \phi$ for at least the fully symmetric games, where $\bar{D} = D \cap \mathbb{E}^N$ is the efficient part

[5] The defining inequalities for L thus represent an orderly retreat from the better-known "core" inequalities: $x(S) \geqslant v(S)$, which often do not admit a feasible solution. In fact (since l can be shown to be superadditive), (N, l) is a game in its own right, and L is its core.

[6] Note also that $d(\phi) = 0$ if and only if the core of (N, v) is nonempty.

[7] Somewhat different rationales for L and D are given in [4] and [2].

of D. (Note that D itself is trivially nonempty.) Interestingly, our counterexample is also symmetric in the players, but only in the weaker sense of being invariant under a transitive subgroup of the group of all permutations of the players.

3 The 21-Player Example

Let $N = \{P_1, P_2, \ldots, P_{21}\}$, and let $C_1, C_2, \ldots, C_7$ be seven special subsets of players, with the property that the columns of their incidence matrix include all $\binom{7}{5} = 21$ possible arrangements of five 1's and two 0's:

	P_1	P_2	…	…	P_{20}	P_{21}
C_1	1	1			0	0
C_2	1	1			1	0
C_3	1	1			0	1
C_4	1	1	…	…	1	1
C_5	1	0			1	1
C_6	0	1			1	1
C_7	0	0			1	1

Thus, each C_k has fifteen members. Using these special sets, we can now define the characteristic function:

$$v(\phi) = 0$$

$$v(S) = \begin{cases} -1 & \text{if } S \subseteq C_k \text{ for some } k, \text{ and } S \neq \phi \\ -2 & \text{if } S \nsubseteq C_k \text{ for all } k, \text{ and } S \neq N \end{cases}$$

$$v(N) = -3.$$

To see that v is in fact superadditive, observe that any possible superadditivity violation

$$v(S) + v(T) > v(S \cup T)$$

must have numerical form

$$(-1) + (-1) > (-3).$$

Hence $S \subseteq C_i$ and $T \subseteq C_j$, for some i and j not necessarily distinct, and $S \cup T = N$. It follows that $C_i \cup C_j = N$. But this is impossible, since for each i and j there is some player *not* belonging to $C_i \cup C_j$, by definition. Hence v is superadditive.

4 Proof that L is Empty

The set L is convex and has the symmetry of the game, so if it is not empty it must contain a point of the form $y = (\eta, \eta, \dots, \eta)$. Feasibility of L requires that $\eta \leqslant -1/7$. Hence

$$y(C_1) \leqslant -15/7.$$

To calculate $l(C_1)$, we note that

$$v(R) + v(C_1 \backslash R) = \begin{cases} -2 & \text{if } \phi \subset R \subset C_1, \quad \text{and} \\ -1 & \text{if } R = \phi \text{ or } R = C_1. \end{cases}$$

From this we see that

$$l(C_1) = -2,$$

and hence that $y(C_1) < l(C_1)$, showing that L is empty.

5 Proof that $\bar{D}$ is Empty

The set $\bar{D}$, like L, is convex and has the symmetry of the game, so if it is not empty it contains a point of the form $y = (\eta, \eta, \dots, \eta)$. In this case, however, efficiency of $\bar{D}$ requires that η be exactly $-1/7$. Hence

$$y(N \backslash C_1) = -6/7.$$

By definition,

$$d(N \backslash C_1) = \min_{z \in \mathbb{F}^{C_1}} \max_{R \subseteq C_1} [v((N \backslash C_1) \cup R) - z(R)].$$

Note that $(N \backslash C_1) \cup R \subseteq C_k$ implies $C_1 \cup C_k = N$, which is impossible, as we have already pointed out. So $v((N \backslash C_1) \cup R) = -2$ for all $R \subset C_1$ and -3 for $R = C_1$, giving us

$$d(N \backslash C_1) = \min_{z \in \mathbb{F}^{C_1}} \max \{ \max_{R \subset C_1} [-2 - z(R)], -3 - z(C_1)\}$$

$$= -2 + \min_{w : w(C_1) = 1} \max \{ \max_{R \subset C_1} w(R), 0\}$$

(replacing $-z$ by w for convenience). Write $M(w)$ for $\max \{ \max_{R \subset C_1} w(R), 0\}$ and let $\omega = \min \{w_i : i \in C_1\}$. Then $\omega \leqslant 1/15$ and, taking $|R| = 14$, we see that $M(w) \geqslant 1 - \omega \geqslant 14/15$. So the value of the min max is at least $14/15$. On the other hand, taking

$w = w^* = (1/15, \ldots, 1/15)$ yields $M(w^*) = 14/15$, so the value of the min max is exactly 14/15, and we obtain

$$d(N \backslash C_1) = -2 + 14/15 = -16/15.$$

Hence $y(N \backslash C_1) > d(N \backslash C_1)$, showing that $\bar{D}$ is empty.

6 References

1. Gerard-Varet LA, Zamir S (1984) Remarks on the reasonable set of outcomes in a general coalition function form game. Working Paper 8406, GREQE, Universités d'Aix-Marseille II et III, Marseilles
2. Luce RD, Raiffa H (1957) Games and decisions. John Wiley and Sons, New York
3. Maschler M, Peleg B, Shapley LS (1979) Geometric properties of the kernel, nucleolus and related solution concepts. Mathematics of Operations Research 4:303–338
4. Milnor JW (1952) Reasonable outcomes for n-person games. RM-916, Rand Corporation, Santa Monica
5. Shapley LS (1971) On Milnor's class "L". R-795, Rand Corporation, Santa Monica
6. Wesley E (1971) An application of nonstandard analysis to game theory. Journal of Symbolic Logic 36:385–394

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