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Generalised Bargaining Sets for Cooperative Games[1])

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Abstract: Although the M_1-bargaining set for games with side payments is known to exist, it frequently contains payoffs which are highly inequitable. For this reason the more restricted M_2-bargaining set is of interest. Since M_2 is not known to exist in general, this paper introduces an M_*-bargaining set, contained in M_1 and containing M_2, and presents an existence theorem. For the class of symmetric, simple games with decreasing returns, the M_2-bargaining set is shown to exist, and a fairly severe restriction on payoffs satisfying M_2-stability is obtained.

Introduction

A characteristic function game with transferable value is defined by a pair (N, v) where N is a set of players and v is a function which assigns to each coalition S in N, a real number $v(S)$ called the *value* of the coalition. This value may be divided among the coalition members. The obvious equilibrium set, the *core*, is typically empty for such games: for example if the game is constant sum then the core is empty [*Owen; Riker/Ordeshook*]. However the M_1 bargaining set, introduced by *Aumann/Maschler* [1964] and the kernel, due to *Davis/Maschler* [1963] are known to exist [*Peleg*, 1967; *Maschler/Peleg*].

For simple, constant sum games the M_1 bargaining set may contain payoff vectors which are counter intuitive. For example, consider a simple majority voting game with twenty five players where each winning coalition (of size at least thirteen) has value 1. A payoff distribution which gives 1/7 to the first seven members, and 0 to the other six members, of a winning thirteen person coalition, is in the bargaining set. In general, for simple games with many players the M_1-bargaining set appears to be too large. The kernel excludes such inequitable payoff distributions. However the kernel has not performed well as a predictor of actual payoffs in one experimental gaming study [*Michener/Sakurai*] and in an empirical analysis of European government coalitions [*Schofield*]. For these reasons the more restricted solution concepts, the M_2-bargaining set and the K_2-kernel are worth studying. The first part of the paper gives the definitions of these general solution notions M_2 and K_2. A new bargaining set M_*, lying between M_1 and M_2 is introduced, and an existence theorem obtained. However M_* does not forbid the inequitable payoff distribution in the simple game mentioned above. The second part of the paper shows that for a general class of simple symmetric games, which are called D-games, the M_2-bargaining set exists. More precisely it is shown that the equitable payoff vector (all payoffs are equal in the winning coalition) belongs to

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the M_2-bargaining set. Furthermore if the decision rule is at least two thirds then this equitable payoff vector satisfies the K_2-kernel symmetry constraints.

These results suggest that the strong stability properties of the M_2-bargaining set may be satisfied for a more general class of weighted majority games.

Generalised Bargaining Sets

Let $N = \{1, \ldots, n\}$ be a finite set of n players. A characteristic function game with transferable value for N is defined in terms of a real valued function $v : 2^N \to R$, where 2^N is the set of subsets of N. The *value* $v(S)$ of a coalition $S \subset N$ is regarded as the collective benefits that the coalition can guarantee, irrespective of the behavior of the players outside S. A *coalition structure* B is a disjoint partition of N. An individually rational payoff configuration (irpc) is a pair (x, B) where B is a partition $(M_1, \ldots, M_r)$ and $x \in R^n$ is a payoff vector, satisfying

(i) $\sum_{i \in M_j} x_i = v(M_j)$ for each $j = 1, \ldots, r$

(ii) $x_i \geqslant 0$ for all $i = 1, \ldots, n$.

Let (x, B) be an irpc and M a coalition in B, with L, K two disjoint *subsets* of M. Let T_{LK} be the class of coalitions which contain L and exclude K. An objection $y(C)$ by L against K, wrt (x, B) is a C-vector, where $C \in T_{LK}$, satisfying

(i) $\sum_{i \in C} y_i = v(C)$

(ii) $y_i > x_i$ for $i \in L$

(iii) $y_i \geqslant x_i$ for $i \in C$.

A ***rule*** ψ assigns to any such disjoint pair (K, L) of any coalition M in any structure B a class $T_{\psi(K,L)}$ of coalitions which *include* some specified subset of K and *exclude* some specified subset of L.

A ψ-counter objection $Z(D)$ to $y(C)$ is a D-vector, where $D \in T_{\psi(K,L)}$, which satisfies

(i) $\sum_{i \in D} z_i = v(D)$

(ii) $z_i \geqslant x_i$ for $i \in D \cap K$

(iii) $z_i \geqslant y_i$ for $i \in D \cap C$.

An objection $y(C)$ by L against K is said to be ψ-justified if there is no ψ-counter objection.

Definition 1: An irpc (x, B) is called ψ-stable iff for each coalition M in B, and any disjoint pair (L, K) in M, there is no ψ-justified objection by L against K. The M^{ψ}-bargaining set is the set of ψ-stable irpc's.

The M^{ψ}-bargaining set is said to exist iff, for every coalition structure B there is a non empty set of payoff vectors $M^{\psi}(B)$ such that for each $x \in M^{\psi}(B)$, (x, B) is an irpc belonging to M^{ψ}.

Example:

(i) *the M-bargaining set*: for each L, K let $T_{\psi(K,L)} = T_{KL}$. This definition means that the whole set K must be able to join the counter objecting coalition D and exclude all of L. We may also say that K has a *strong* counter objection, $z(D)$, to the objection $y(C)$ of L.

(ii) *the M_2-bargaining set*: for each L, K let $T_{\psi(K,L)} = T^2_{KL}$ be the class of subsets of N which include K and exclude some members of L. Thus the counter objecting coalition D must satisfy $K \subset D$ but $L \not\subset D$. We may call $z(D)$ a *weak* counter objection.

(iii) *the M_1-bargaining set*: for each L, K let $T_{\psi(K,L)} = T^1_{KL}$ be the class of subsets of N which include at least one member of K and exclude at least one member of L.

Since

$T_{KL} \subset T^2_{KL} \subset T^1_{KL}$ it is clear that

$M \subset M_2 \subset M_1$.

To indicate why we have used the notation M_2, suppose that L has an objection against K, and there is no weak counter by K. Obviously any subset L' of L also has an M_2-justified objection against K. Since this is true when L' is a single individual, M_2-stability is equivalent to the requirement that whenever a single individual i, say, objects to a group K of members of the same coalition that group has a (strong) counter.

Suppose now that a single player has an M_1-justified objection against K. This objection is also an objection against each individual member of K, and no member of K may counter. Consequently M_1-stability is equivalent to the requirement that for no pair of individuals i, j in a coalition M does i have a justified objection against j.

The classical proof [*Peleg*, 1967] of the existence of the M_1-bargaining set proceeds by way of the *excess* of a coalition.

Definition 2: Let x be any payoff vector and C any coalition. Define the *excess* of C with respect to x to be

$$e_x(C) = v(C) - \sum_{i \in C} x_i .$$

When x is fixed, we shall more briefly write $e(C)$ for the excess.

Lemma 1 [A generalisation of *Davis/Maschler*, 1967, lemma 3.1]: Let L, K be disjoint subsets of a coalition M in a structure B. If L has an objection y (C), $C \in T_{LK}$ against K, and there is some coalition D containing $K' \subset K$ which belongs to ψ (K, L) where ψ is some *rule*, such that $e(D) \geqslant e(C)$, then there is a ψ-counter objection z (D) for K.

Proof: Since y (C) is an objection,

$$e(C) = v(C) - \sum_C x_i$$

$$= v(C) - \sum_C y_i + \sum_C (y_i - x_i) > 0.$$

Define the counter z (D) by

$$z_i = x_i \qquad \text{for } i \in D - C - K'$$

$$= y_i \qquad \text{for } i \in D \cap C$$

and

$$\sum_{i \in K'} z_i = v(D) - \sum_{D\text{-}K'} z_j.$$

Thus

$$\sum_{j \in K'} (z_i - x_i) = v(D) - \sum_{D \cap C} y_i - \sum_{D\text{-}C\text{-}K'} x_i - \sum_{K'} x_i$$

$$\geqslant v(D) - v(C) + \sum_{C\text{-}D} x_i - \sum_{D\text{-}C} x_i$$

$$= e(D) - e(C).$$

Thus if $e(D) \geqslant e(C)$ then z (D) is a ψ-counter objection by K. Q.E.D.

The consequences of this lemma are

(i) if L has an M (resp. M_2) justified objection against K then there is some $C \in T_{LK}$ such that $e(C) > e(D)$ for all D in T_{KL} (resp. T^2_{KL}).

(ii) the excess can be used to define the generalised *kernels*.

Definition 3:

(i) The surplus s_{LK} of L over K is

$$s_{LK} = \max \{e(C) : C \in T_{LK}\}$$

(ii) under a rule ψ, the ψ-surplus of K over L is

$$s_{\psi(K,L)} = \max \{e(D) : D \in T_{\psi(K,L)}\}$$

(iii) L ψ-outweighs K iff
a) $s_{LK} > s_{\psi(K,L)}$ and
b) there is no $D \in T_{\psi(K,L)}$ such that $x_i = 0$ for all i in

$$K' = D \cap K.$$

(iv) An irpc (x, B) is K^{ψ}-stable iff for no disjoint groups L, K in each coalition M in B does L ψ-outweigh K.
(v) The set of K^{ψ}-stable irpc's is called the K^{ψ}-kernel.

Corollary 1: The K^{ψ}-kernel is a subset of the M^{ψ}-bargaining set.

Proof: Suppose first of all that condition (iiib) of Definition 3 is satisfied.

If there is a ψ-justified objection by L against K, then by lemma 1 there is some $C \in T_{LK}$ such that $e(C) > e(D)$ for all $D \in T_{\psi(K,L)}$. Consequently $s_{LK} \geqslant e(C) > s_{\psi(K,L)} \geqslant e(D)$ so L ψ-outweighs K. On the other hand if condition (iiib) fails, then L cannot ψ-outweigh K. But $K' = D \cap K$ has the counter objection $x(K')$ to any objection by L. Hence L has no ψ-justified objection to K. Q.E.D.

Definition 4: For $\{L, K\}$ a disjoint pair of the coalition M in the coalition structure B, and with respect to the irpc (x, B) write
a) $L\, P^{\psi}(x)\, K$ whenever L has a ψ-justified objection against K
b) $L\, Q^{\psi}(x)\, K$ whenever L ψ-outweighs K.

These relations have the interpretation that under the situation (x, B) and in the context of the bargaining rule ψ, the group L is "stronger", in the obvious sense, then group K. Corollary 1 also implies that

$$L\, P^{\psi}(x)\, K \Rightarrow L\, Q^{\psi}(x)\, K.$$

To fit in with our previous notation, we may define
a) the K-kernel through the rule given by $T_{\psi(K,L)} = T_{KL}$
b) the K_2-kernel through the rule $T_{\psi(K,L)} = T^2_{KL}$
c) the K_1-kernel through the rule $T_{\psi(K,L)} = T^1_{KL}$.

K_2 requires that for each coalition M, then $i \in M$, $K \subset M - \{i\}$ implies either $s_{iK} \leqslant s_{Ki}$ or $x_i = 0$ for all members of K.
K_1 requires either $s_{ij} \leqslant s_{ji}$ or $x_j = 0$ for each disjoint pair $\{i, j\}$ in a coalition M.
From Corollary 1 and the definitions we obtain.

Corollary 2:

$$\begin{array}{ccccc} K & \subset & K_2 & \subset & K_1 \\ \cap & & \cap & & \cap \\ M & \subset & M_2 & \subset & M_1 . \end{array}$$

The classical proof of the existence of M_1 [*Peleg*, 1967; *Billera/Peleg*, 1969], is to proceed as follows.

Definition 5: A relation $P \subset N \times N$ is
a) acyiclic iff $a_1 \; Pa_2 \; \ldots Pa_t \Rightarrow$ not $(a_t \, Pa_1)$ for $a_1, \ldots, a_t \in N$
b) asymmetric iff $aPb \Rightarrow$ not (bPa) for any $a, b \in N$.

Proposition: Suppose $P(x) \subset N \times N$ is an asymmetric, acyclic relation, parametised by $x \in X(B)$ where $X(B)$ is the set of payoffs st. (x, B) is an irpc.

For a coalition M, define

$$E_i^P = \{x \in X(B) : jP(x)\, i \qquad \text{for } no \; j \text{ in } M - \{i\}\}.$$

If E_i^P is closed in $X(B)$ and contains any $x \in X(B)$ satisfying $x_i = 0$, then the *bargaining set*

$$M^P = \bigcap_i E_i^P$$

is non empty.

Acyclicity of P is used to show that for each $x \in X(B)$, and M in B there is an $i \in M$ st. $x \in E_i^P$ [see for example *Peleg*, 1967, Corollary 2.3]. Then *Billera*'s theorem [1970] gives the proposition.

By *Davis/Maschler*, the relation $P_1\,(x)$ associated with the M_1-bargaining set, is acyclic and obviously asymmetric, and existence is thus obtained. However, *Billera*'s theorem cannot be used for M_2.

As in the introduction M_1 contains somewhat inequitable payoff distributions.

Example 1: Let

$$v(M) = 1 \qquad \text{for } |M| \geqslant 13, \text{ and } |N| = 25.$$

Consider a payoff vector:

$$x_i = 1/7 \qquad \text{for } i = 1, \ldots, 7 \in M$$

$$x_i = 0 \qquad \text{for } i = 8, \ldots, 13 \in M$$

$$x_i = 0 \qquad i \notin M.$$

Let 8 object to 1 by:

$$y_i = 0 \qquad i = 1, \ldots, 7 \notin C$$

$$y_i = 1/13 \qquad i = 8, \ldots, 20 \in C$$

$$y_i = 0 \qquad i = 21, \ldots, 25 \notin C.$$

For 1 to counter "efficiently" he needs seven members of C, and consequently must pay them 7/13 leaving $6/13 > 1/7$ for himself. Thus 8 has no justified objection against 1.

On the other hand 8 *has* an M_2-justified objection against $K = \{1, \ldots, 7\}$ since 8 may give each member of $N - K$ the payoff $y_i < 1/17$, to which K has no counter.

While there is no general existence theorem for M_2, it is possible to define a new non empty bargaining set M_* including M_2 and included in M_1. M_* also gives a means of comparing M_2 and M_1, and of generating a class of bargaining sets.

Definition 6: Let B be a coalition structure (c.s.) $x \in X(B)$, M in B, and j in M. For $K \subset M - \{j\}$ write $jP(x)K$ whenever j has a justified objection against K.

For $i \in M$ define $jP_*(x)\, i$ iff the following are satisfied:

a) $x_i > 0$
b) for some $K \subset M$, $K \in T_{ij}$, $jP(x)K$ *and* there is no L, $L \in T_{ji}$ st. $L \cap K \neq \emptyset$ with $iP(x)L$.

Define the M_*-bargaining set to the set of P_*-stable irpc's. The idea behind this notion is, if an individual j has a justified objection against some K containing i, then i may "block" this only by a justified objection against j and some of the members of K.

Lemma 2:

$$M_2 \subset M_* \subset M_1.$$

Proof:

a) Let B be a c.s. Suppose $M_*(B) \not\subset M_1(B)$. Then for some i, j in M in B, $jP(x)\, i$ although not $(jP_*(x)\, i)$ and $x_i > 0$. Since i must block j, there must be an $L \in T_{ji}$ satisfying $iP(x)L$ and $L \cap \{i\} \neq \emptyset$. But this is a contradiction. Consequently $jP(x)\, i$ implies $jP_*(x)\, i$ so $M_*(B) \subset M_1(B)$.
b) Obviously if $x \in M_2(B)$ then $jP(x)K$ for no $K \in T_{ij}$ so not $(jP_*(x)\, i)$. Consequently $M_2(B) \subset M_*(B)$.

Theorem 1: $M_*(B)$ is non empty for any c.s. B.

The theorem is proved by the following lemma.

Lemma 3: $P_*(x)$ is acyclic on $M \times M$ for each M in B, $x \in X(B)$.

Proof: Suppose $P_*(x)$ is *cyclic* on $M \times M$ for some $x \in X(B)$, M in B. Write P_* for $P_*(x)$ and take M to be $\{1, \ldots, t \ldots\}$ such that $1P_*2P_* \ldots P_*t$ and tP_*1.

To each pair $(r, r-1)$ let C_r be the coalition that maximises the surplus $\{s_{rK} : r+1 \in K\}$ and let C_s be the coalition that maximises $e(C_r)$, $r = 1, \ldots, t$.

Consider $(s-1)P_*sP_*(s+1)$ where the objecting coalitions of $(s-1)$ and s are C_{s-1}, C_s respectively. Suppose that $(s-1) \notin C_s$.

Now $(s-1)$ objects to $M - C_{s-1}$. If $M - C_{s-1} \subset C_s$ then since

$$e(C_s) \geqslant e(C_{s-1})$$

by lemma 1, s has a counter objection (via C_s) against $s - 1$. Consequently

$$M - C_{s-1} \cap M - C_s \neq \emptyset.$$

However $(s-1)\,P\,(M - C_{s-1})$ *and* $s\,P\,(M - C_s)$ with $M - C_s \cap M - C_{s-1} \neq \emptyset$. This contradicts $(s-1)\,P_*s$. Consequently $(s-1) \in C_s$.

By the same procedure as *Davis/Maschler* [1967, theorem 3.1], an induction argument shows that $(s-2), (s-3) \ldots 1, t, \ldots, (s+1)$ all belong to C_s. But C_s is the objecting coalition of s against $(s+1)$. This contradition shows that P_* must be *acyclic*.

Example 2: Since it may be the case not $(j\,P\,i)$ although $j\,P\,K$ for some K containing i, the M_*-bargaining set may well be a *proper* subset of M_1. However the M_*-solution does not exclude the inequitable distribution of Example 1. We know from Example 1 that 8 has no justified objection against 1 alone. In fact 8 has a justified objection against {1, 2, 3} say. However 1 has a justified objection against {2, 3, 4, 5, 8} and can thus "block" the objection of player 8. This example indicates that in the M_*-solution notion, a player may fairly easily block another objection. This implies obviously that in M_1, players may be balanced (there is no justified objection of one against another) although one may have what would appear on intuitive grounds to be a reasonable complaint against another. It is clear from definition 6 that one may modify the definition to make blocking more difficult, thus having the effect reducing the solution set. From this perspective it is interesting to note that the M_2 solution notion does not permit blocking. In the next section of the paper we shall show that M_2 exists in a class of games which includes the game of Example 1, and moreover forbids the kind of inequitable payoff distribution that we have been discussing.

The Kernel and Bargaining Set of D Games

The K_1-kernel has the virtue that if one player is "more desirable" than another, in the sense that the former contributes more to coalition value than the latter, then the former's payoff in the kernel must be at least as great as the latter's payoff. Unfortunately the M_1-bargaining set does not display this property.

Definition 7: Say a player i is *more desirable* than a player j iff, for any S in N which excludes both i and j,

$$v\,(S \cup i) \geqslant v\,(S \cup j).$$

Lemma 4: If i, j both belong to a coalition M, and i is more desirable than j, then for any irpc x (B),

(i) $x_j > x_i$ implies $s_{ij} > s_{ji}$

(ii) if $x_j \geqslant x_i$ then j has no (M_1)-justified objection against i.

Proof:

(i) See also *Peleg* [1968].
For any S in N, excluding i, j, let

$$U = S \cup \{i\},\ W = S \cup \{j\}.$$

Observe that

$$e(U) - e(W) = v(S \cup \{i\}) - v(S \cup \{j\}) + (x_j - x_i) > 0.$$

Hence

$$s_{ij} \geqslant e(U) > s_{ji} \geqslant e(W).$$

(ii) In precisely the same way, suppose j has an objection, via $S \cup \{j\}$ against i. Since

$$e(S \cup \{i\}) \geqslant e(S \cup \{j\})$$

i has a counter, by lemma 1.

Lemma 4 (i) immediately gives the K_1-kernel in symmetric games.

Definition 8: A game is symmetric iff, for any coalition M ofsize m,

$$v(M) = v(m).$$

In other words in a symmetric game if two coalitions are of the same size then their values are equal.

Corollary 3: In a symmetric game if $x(B)$ is an irpc belonging to K_1 (B), and i, j both belong to a coalition M in B, then $x_i = x_j$.

In the game of Example 1, the kernel payoff to the members of the 13-person coalitions is 1/13. On the other hand in Example 2, lemma 4 (ii) tells us only that player 1 (with payoff 1/7) has no justified objection against player 8 (with payoff 0).

Obviously the equitable payoff structure in a symmetric game belongs to K_1 and would appear to be the expected solution. We shall now show that in a typical class of symmetric games the M_2-bargaining set contains these equitable payoffs.

Definition 9: A D-game v is a characteristic function game with the following properties:

(i) it is *symmetric*

(ii) it is *proper simple*:
there is some *quota* q, satisfying $n/2 < q \leqslant n$ such that any coalition M of size m
a) if $m \geqslant q$ then M is winning, $v(m) > 0$
b) if $m < q$ then M is losing, $v(m) = 0$

(iii) v has *decreasing returns to scale*:
if S is *minimal* winning (i.e. of size $s = q$), and T is a coalition which contains S, of size $t > q$, then

$$\frac{v(s)}{s} > \frac{v(t)}{t}$$

For convenience we may assume that the value of a minimal winning coalition is 1, and that for any coalition T of size t greater than q, $1 \leqslant v(T) < t/q$.

Obviously the class of D-games is a slightly generalised form of the usual simple majority rule game illustrated in Example 1.

We shall show in Theorem 2 for D-games that the equitable payoff structures belong to M_2, and then in Theorem 3 obtain a constraint on payoff distribution sufficient for membership of M_2.

We make use of the fairly obvious lemma.

Lemma 5: Let v be a D-game with quota q, $n/2 < q < n$.

Let $M = \{1, \ldots, m\}$ be a winning coalition in the coalition structure B. Suppose that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m$ with some strict inequalities possible. Let $K \subset M - \{1\}$ be of size $k < q$.

For $C \in T_{1K}$, of size c, let

$$av(C) = \frac{1}{c-1}(v(C) - x_1)$$

and

$$av(K) = \frac{1}{q-k}(1 - \sum_K x_j)$$

(i) If there is some $C = N - K$ of size $c \geqslant q$ such that $av(C) > x_1$,

$$av(C) \geqslant \max_{C \cap M} \{x_i\}$$

and

$$av(C) > av(K)$$

then $\{1\}$ has a justified objection against K.

(ii) If $av(K) \geqslant av(C)$ for all $C \in T_{1k}$ then $\{1\}$ has no justified objection against K.

Proof:

(i) Construct an objection $y(C)$ for $\{1\}$ against K by:

$$x_1 < av(C) = y_1, \; av(C) \geqslant y_i > av(K) \text{ for } i \neq 1.$$

Since $y_i > av(K)$, for all $i \in N - K - \{1\}$, there is no counter objection $z(D)$, for D a minimal winning coalition. But by decreasing returns, if K had a counter objection $z'(D')$ for $|D'| > q$, then there would also be a counter objection $z(D)$, for $|D| = q$. Consequently K has no counter objection.

(ii) On the other hand if $av(K) \geqslant av(C)$, then K has a counter $z(D)$ to any objection $y(C)$, where

$$D = K \cup R, R \subset C \setminus \{1\},$$

and

$$|D| = q.$$ Q.E.D.

Theorem 2: Let v be a D-game with quota q, $n/2 < q < n$. Let M be a winning coalition in the coalition structure B.

(i) $K_1(B) \subset M_2(B)$

(ii) If furthermore $n > q \geqslant (2n)/3$, then

$$K_1(B) = K_2(B).$$

We prove this theorem by the following two lemmas. We shall take B to be a fixed coalition structure, and M a winning coalition in B of size $m \geqslant q$. By Corollary 3 we know that an irpc $x(B)$ belongs to $K_1(B)$ iff $x_i = x_j$ for all i, j in M. Consequently in these two lemmas we take $x(B)$ to be the *fixed* payoff vector which assigns $(v(M))/m$ to each member of M, and 0 to all members of $N - M$.

Lemma 6: If

$$2q + m \geqslant 2n \qquad \text{and } q < n$$

then

$$K_1(B) = K_2(B).$$

Proof: We seek to show that for any $i \in M$, $K \in M - \{i\}$, $s_{Ki} \geqslant s_{iK}$ for the equitable payoff vector $x(B)$. Note first of all that by assumption

$$n - q \leqslant m + q - n \leqslant m - 1.$$

Consider the two possibilities for $|K| = k$.

(i) $n - q + 1 \leqslant k \leqslant m - 1.$

Since $n - k \leqslant q - 1$, there is no winning coalition including i and excluding K. Thus $s_{iK} < 0$. But $n - 1 \geqslant q$, so there exists some coalition D of size q, which contains K. Hence

$$e(D) \geqslant 1 - \frac{q}{m} v(M).$$

By decreasing returns, $v(M) \leqslant m/q$.

Hence

$$s_{Ki} \geqslant e(D) \geqslant 0 > s_{iK}.$$

(ii) $1 \leqslant k \leqslant n - q \leqslant m + q - n$.

Consider a coalition $C = R' \cup (N \setminus M) \cup \{i\}$ where $R' \subset M \setminus K \setminus \{i\}$. Obviously

$$s_{iK} = v(C) - \frac{v(M)}{m}(c - n + m)$$

for some such C with $q \leqslant |C| \leqslant n - k$.

Now let $r'' = c - k + m - n$, and observe that $1 \leqslant r'' \leqslant m - k - 1$. Consequently there is a coalition $D = R'' \cup K \cup (N \setminus M)$ with $R'' \subset M \setminus K \setminus \{i\}$, such that $|R''| = r''$, and $|D| = |C|$.

Now

$$e(D) = \frac{v(M)}{m}(c - n + m),$$

so

$$s_{Ki} \geqslant e(D) \geqslant s_{iK} = e(C).$$

Corollary 4: If $n > q \geqslant (2n)/3$, then $K_1(B) = K_2(B)$.

Proof: Under the assumption, for a winning coalition M of size m, $2q + m \geqslant 2n$. By Lemma 6, the equitable payoffs belong to $K_2(B)$. By Corollary 3, this implies $K_1(B) \subset K_2(B)$. The result is obtained by applying Corollary 2. Q.E.D.

When $2q + m \geqslant 2n$, and $q < n$, Corollary 4 gives

$$K_1(B) = K_2(B) \subset M_2(B).$$

Thus to complete theorem 2 we need the following lemma.

Lemma 7: If $2q + m < 2n$, or $q = n$, then no individual in M has an (M_2)-justified objection against a subgroup $K \subset M \setminus \{i\}$.

Proof: Observe first of all that if $q = n$, then no i has an objection. So assume $q < n$. If $|K| > n - q$ then certainly no $i \in M \setminus K$ has an objection against K. So assume $|K| \leqslant n - q$.

Since $m - qv(M) \geqslant 0$, $q \leqslant k \leqslant 1$, then for any $c \geqslant q$,

$$1 - \left(\frac{q-k}{c-1}\right)\left(\frac{c}{q} - \frac{v(M)}{m}\right) \geqslant k\frac{v(M)}{m}.$$

Now

$$x_i = \frac{v(M)}{m}.$$

But for any $C \in T_{iK}$, $v(C) \leqslant c/q$.

Hence

$$av(K) = \frac{1}{q-k}\left(1 - k\,\frac{v(M)}{m}\right) \geqslant \frac{1}{c-1}\left(\frac{c}{q} - \frac{v(M)}{m}\right) \geqslant av(C).$$

By lemma 5 (ii), $\{i\}$ has no justified objection against K. Q.E.D.

Lemmas 6 and 7 together give Theorem 2. We have also shown the somewhat surprising result that with a two-thirds decision rule not only does the equitable payoff distribution satisfy M_2-stability, but the far more severe K_2-stability. It is easy to show furthermore that if $m < 2(n-q)$ for the winning coalition M in B, then $K_2(B) = \emptyset$. To illustrate this consider the thirteen person game with $q = 7$ and $M = \{1, 2, 3, 4, 5, 6, 7\}$ and payoffs $x_i = 1/7$ for $i \in M$. Let $K = M \setminus \{1\}$. Obviously $s_{1K} = 6/7$ although $s_{K1} = 1/7$, so this payoff vector does not belong to $K_2(B)$.

We now obtain a constraint which must be satisfied for an irpc (x, B) to belong to $M_2(B)$.

Theorem 3: Let v be a D-game with quota q, where $n/2 < q < n$. Let M be a winning coalition in the coalition structure B. If

a) there exist two individuals i, j in M such that $x_i < x_j$ in $x(B)$, *and*
b) there is a group K in M of size at most $(n-q)$ whose total payoffs in $x(B)$ sum to at least $(n-q)/q$ *then* $x(B)$ does not belong to $M_2(B)$.

Proof: Assume without loss of generality that M is of size m and that the payoffs in $x(B)$ satisfy $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_m$ with some strict inequalities.

Let $S = \{1, \ldots, s\}$ where $s = q - (n-m) > 0$ and let C be the minimal winning coalition $(N \setminus M) \cup S$. We now construct an objection by 1 against $M \setminus S = K$.

(i) If $x_s \leqslant 1/q$, then since $x_1 < 1/q$ by assumption,

$$av(C) > \frac{1}{q-1}\left(1 - \frac{1}{q}\right) = \frac{1}{q}.$$

On the other hand,

$$\sum_K x_j \geqslant \frac{n-q}{q}$$

and

$$|K| \leqslant (n-q)$$

so that

$$av\,(K) \leqslant \frac{1}{q}.$$

By lemma 5 (i), {1} has a justified objection against K.

(ii) If $x_s > 1/q$, construct the objection $y\,(C)$:

$$x_1 < y_1 < 1/q,\ x_i < y_i \leqslant x_{s+1} \quad \text{for } i \neq 1, \sum_C y_i = 1.$$

Suppose $z\,(D)$ is a minimal winning counter objection to $y\,(C)$.

Now

$$C = (C \setminus D) \cup (C \cap D), D = K \cup (D \setminus K)$$

and

$$(D \setminus K) = C \cap D.$$

Since

$$|C| = |D| = q,$$

we find

$$|K| = |C \setminus D|.$$

By construction

$$\sum_{C \setminus D} y_i < \sum_K z_i$$

and

$$\sum_{C \setminus D} y_i + \sum_{C \cap D} y_i = \sum_K z_i + \sum_{D \setminus K} z_i = 1.$$

Thus

$$\sum_{C \cap D} y_i > \sum_{C \cap D} z_i$$

which is contrary to the assumption that $z\,(D)$ is a counter objection.

Since K has no counter objection $z\,(D)$, for D minimal winning, there is no counter objection $z'\,(D')$ against $y\,(C)$. Consequently {1} has a justified objection against K.

Q.E.D.

Corollary 5: Let v be a D-game with quota q where $n/2 < q < n$, and let M be a minimal winning coalition, in the c.s.B. If there exist two individuals i, j in M such that $x_i < x_j$ in $x\,(B)$ then $x\,(B)$ does not belong to $M_2\,(B)$.

Proof: As before suppose $x_1 \leqslant x_2 \leqslant x_q$, and let $S = \{1, \ldots, 2q-n\}, K = M - S$.
Now

$$\sum_S x_i < \frac{2q-n}{q}, \sum_K x_i > \frac{n-q}{q}.$$

By theorem 3, $\{1\}$ has a justified objection against K. Q.E.D.

Theorem 3 is a generalisation of the result by *McKelvey/Smith* [1974] that the inequitable payoff structure in a *minimal* winning coalition does not belong to M_2. However inequitable payoffs may belong to M_2, as we can illustrate by the following example.

Example 3: Consider the D-game with $n = 13, q = 7$. For convenience take $v(Q) = 7$, for Q minimal winning. Let x be the payoff vector which assigns 0.9 each to the first six players of a ten person coalition, and 1.0 to the remaining four. Here $v(M) = 9.4$, consistent with decreasing returns.

(i) Consider an objection by $\{1\}$ against $K = \{5, 6, 7, 8, 9, 10\}$.
Observe that $\frac{1}{6}(7-(0.9)) < (7-5.8)$
so by lemma 5 (i), player $\{1\}$ has no justified objection against K.

(ii) On the other hand suppose $v(R) = 8.95$ for any nine person coalition R. Let $K = \{7, 8, 9, 10\}$, and observe that

$$\frac{1}{8}(8.95 - 0.9) > \frac{1}{3}(7-4).$$

Since $av(N \setminus K) > 0.9$, $\{1\}$ has a justified objection against K.

Note also that if $v(R) \leqslant 8.9$, for $|R| = 9$, then $\{1\}$ has *no* justified objection against any subgroup of $M - \{1\}$. Obviously the inequality permitted in M_2 depends very much on the actual values assigned to larger than minimal winning coalitions.

Concluding Remarks

The results obtained in the previous section shed some light on the debate between *Butterworth* [1971] and *Riker* [1971] on the status of the minimal winning size principle of *Riker* [1962]. Although *Butterworth* based his argument on a five person symmetric zero sum game, the point he made was that players outside a minimal winning coalition could be expected to bribe their way into the winning coalition. This is obviously true for a D-game. *Shepsle* [1974] joined this debate by arguing that the resultant payoff associated with a larger than minimal winning coalition was unstable.

Suppose we let Q be the original minimal winning coalition and x be the payoff vector which say, gives $1/q$ to each member of Q, and zero to the other players. Let R be some subset of $N - Q$. *Shepsle* argued that any payoff vector for $R \cup Q$ which gave Q less than 1 did not belong to the Von Neumann-Morgenstern V-solution set. On the other hand if Q receives 1, then some player *outside* $R \cup Q$ appeared to have a justified objection against some player in R. *Shepsle* concluded that only payoffs associated with a minimal winning coalition could be stable. His argument is somewhat complicated since it invokes both the V-solution set and an unusual bargaining set notion.

In fact *McKelvey/Smith* [1974] justified the minimal size principle for D-games by requiring that each subgroup of an extant coalition receive its value. Their result that inequitable payoffs for a minimal winning coalition did not belong to M_2 was interpreted as an additional argument in support of the minimal size principle.

However we have shown here that for *any* coalition there are M_2-stable payoffs. Theorem 3 does indicate however that the kind of bribery envisaged by *Butterworth* cannot lead to M_2-stable payoffs for large coalitions. For this bribery to be effective, the original minimal winning group Q must receive its value. But since $q > n - q$, by Theorem 3, the resultant payoff vector cannot be M_2-stable. On the other hand, if the payoff vector associated with a larger than minimal winning coalition S is M_2-stable, then there must be some minimal winning subset Q of S which does not attain its value. Consequently Q may reject the other members of S.

The minimal size principle can, of course, be deduced from the assumption that each player in a coalition receives a payoff proportional to its weight [see *Gamson*, for example]. This assumption had however little justification [*Browne/Franklin; Schofield*].

We have seen that M_2-bargaining theory severely restricts the variation in payoffs among members of winning coalitions, and that this restriction lends some credence to the minimal size principle. It remains to be seen whether the same argument with respect to M_2, or perhaps M_*, is valid for a general class of weighted majority games.

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