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# On the Convexity of Communication Games

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*Abstract:* A communication situation consists of a game and a communication graph. By introducing two different types of corresponding communication games, point games and arc games, the Myerson value and the position value of a communication situation were introduced.

This paper investigates relations between convexity of the underlying game and the two communication games. In particular, assuming the underlying game to be convex, necessary and sufficient conditions on the communication graph are provided such that the communication games are convex. Moreover, under the same conditions, it is shown that the Myerson value and the position value are in the core of the point game. Some remarks are made on superadditivity and balancedness.

## 1 Introduction

In this paper we consider cooperative games with communication restrictions. We assume that the communication possibilities are modelled by means of a *communication graph* in which the points are the players and the arcs correspond to pairs of players who can communicate directly.

These so-called *communication situations* were first studied by Myerson (1977). He introduced corresponding point games and provided and axiomatic characterization of the *Shapley value* of these games. Alternatively, Borm, Owen and Tijs (1990) introduced arc games and the *position value*. This value could be characterized axiomatically in case the communication graph contains no cycles.

The present paper investigates under what conditions on the communication graphs nice properties of the underlying game are inherited by the point game and the arc game. The main result of this paper can be found in section 3: if a communication graph is cycle-complete (cycle-free) and the underlying game is convex, then the corresponding point game (arc game) is convex and the Myerson value (position value) is in the core of the point game. The paper concludes with some remarks on the inheritance of superadditivity and balancedness in section 4. First we recall the main definitions concerning communication situations in section 2.

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## 2 The Model

A *communication situation* is a triple $(N, v, A)$, where $N := \{1,\ldots,n\}$ is the set of players, $(N, v)$ is a *coalitional game* having player set $N$ and characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, and $(N,A)$ is an undirected *communication graph*. For convenience we assume throughout this paper that the game $(N, v)$ is *zero-normalized*, i.e. $v(\{i\}) = 0$ for all $i \in N$.

Let $(N, v, A)$ be a communication situation. The players in a *coalition* $S \subseteq N$ can effect communication through all communication links of $A(S) := \{\{i,j\} \in A \mid \{i,j\} \subseteq S\}$. Hence, a coalition $S$ splits up into *(communication) components* in the following way: $T \subseteq S$ is a component within $S$ if and only if the graph $(T, A(T))$ is connected and there is no set $\overline{T}$ such that $T \subsetneq \overline{T} \subseteq S$ and $(\overline{T}, A(\overline{T}))$ is connected. We denote the resulting partition of $S$ by $S/A$. Correspondingly, the reward of a coalition $S \subseteq N$ having available the communication links in $A(S)$ can be defined as

$$r^v(S,A) := \sum_{C \in S/A} v(C).$$

Note that the fact that $(N, v)$ is zero-normalized implies that for all $S \subseteq N$

$$r^v(S,A) = \sum_{C \in S/A} v(C) = \sum_{C \in S/A(S)} v(C) = \sum_{C \in N/A(S)} v(C) = r^v(N, A(S)),$$

because the components of $(N, A(S))$ are the components of $(S, A(S))$ and all singletons $\{i\}$ with $i \in N \backslash S$.

*Definition:* Let $(N, v, A)$ be a communication situation. *The point game* $(N, r_A^v)$ is defined by

$$r_A^v(S) := r^v(A,S) \text{ for all } S \subseteq N.$$

The *arc game* $(A, r_N^v)$ assigns to every subset $L$ of communication links the corresponding reward of the grand coalition $N$, i.e.

$$r_N^v(L) := r^v(N,L) \text{ for all } L \subseteq A.$$

These two communication games give rise to two allocation rules for communication situations, the Myerson value and the position value. Both values are based on the *Shapley value*, given by (cf. Shapley (1953))

$$\Phi_i(N, v) := \sum_{S \subseteq N: i \notin S} \frac{|S|!(n-1-|S|)!}{n!} (v(S \cup \{i\}) - v(S))$$

for all coalitional games $(N, v)$ and all $i \in N$.

*Definition:* The *Myerson value* $\mu(N,v,A) \in \mathbb{R}^N$ (cf. Myerson (1977)) is defined as the Shapley value of the corresponding point game, i.e.

$$\mu(N,v,A) := \Phi(N,r_A^v).$$

The *position value* $\pi(N,v,A) \in \mathbb{R}^N$(cf. Borm, Owen and Tijs (1990)) is obtained from the Shapley value of the corresponding arc game in the following way:

$$\pi_i(N,v,A) := \sum_{a \in A_i} \frac{1}{2} \Phi_a(A,r_N^v)$$

for all $i \in N$, where $A_i := \{\{i,j\} \in A \mid j \in N\}$, the set of all communication links of which player $i$ is an end point.

## 3 Convexity

In this section we provide classes of communication graphs for which the convexity of the original game is inherited by the point game and the arc game, respectively.

*Definition:* A coalitional game $(N,v)$ is called *convex* if it is more advantageous to join larger coalitions, i.e.

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$. It is obvious that a convex game $(N,v)$ is *superadditive*, i.e.

$$v(S \cup T) \geq v(S) + v(T)$$

for all $S,T \in 2^N$ with $S \cap T = \emptyset$.

*Definition:* A graph $(N,A)$ is called *cycle-complete* if the following holds: if there is a cycle $(x_1,x_2,...,x_t,x_1)$ in $(N,A)$ where $x_1,...,x_t$ are all distinct elements of $N$, then the complete graph on $\{x_1,...,x_t\}$ is a subgraph of $(N,A)$. Note that both graphs without cycles and complete graphs are cycle-complete.

*Example 1:* Consider the communication situations $(N,v,A)$ and $(N,v,B)$, where $N = \{1,...,5\}$, $v(S) = |S| - 1$ for all $S \neq \emptyset$ and $(N,A)$ and $(N,B)$ are the graphs represented in figure 1.

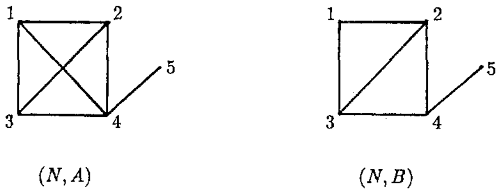


**Fig. 1.**

The graph $(N, A)$ is cycle-complete, whereas $(N, B)$ is not. Note that $(N, v)$ is convex.
Further, since $r_A^v(S) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = \{5\} \\ |S| - 2 & \text{if } 5 \in S \text{ and } 4 \notin S \text{ and } S \neq \{5\}, \\ |S| - 1 & \text{else} \end{cases}$
$(N, r_A^v)$ is convex too. However, $(N, r_B^v)$ is not convex, because

$$r_B^v(\{1,4\} \cup \{3\}) - r_B^v(\{1,4\}) = 2 > 1 = r_B^v(\{1,2,4\} \cup \{3\}) - r_B^v(\{1,2,4\}).$$

Theorem 1 shows that it is no coincidence that the game $r_A^v$ of example 1 is convex.

*Theorem 1:* Let $(N, v, A)$ be a communication situation where the underlying game $(N, v)$ is convex and the communication graph $(N, A)$ is cycle-complete. Then the corresponding point game $(N, r_A^v)$ is convex.

*Proof:* Let $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$.
We have to prove that $r_A^v(S \cup \{i\}) - r_A^v(S) \leq r_A^v(T \cup \{i\}) - r_A^v(T)$, i.e.

$$\sum_{C \in S \cup \{i\}/A} v(C) - \sum_{C \in S/A} v(C) \leq \sum_{D \in T \cup \{i\}/A} v(D) - \sum_{D \in T/A} v(D). \tag{1}$$

Clearly, with $C_i := \cup \{C \in S/A \mid \exists j \in C : \{i,j\} \in A\} \cup \{i\}$, $C_i \in S \cup \{i\}/A$ and if $C \in S \cup \{i\}/A$, $C \neq C_i$, then $C \in S/A$.
So, with $\mathfrak{C} := \{C \in S/A \mid \exists j \in C : \{i,j\} \in A\}$, we have

$$\sum_{C \in S \cup \{i\}/A} v(C) - \sum_{C \in S/A} v(C) = v(\{i\} \cup \bigcup_{C \in \mathfrak{C}} C) - \sum_{C \in \mathfrak{C}} v(C). \tag{2}$$

Analogously we obtain

$$\sum_{D \in T \cup \{i\}/A} v(D) - \sum_{D \in T/A} v(D) = v(\{i\} \cup \bigcup_{D \in \mathfrak{D}} D) - \sum_{D \in \mathfrak{D}} v(D), \tag{3}$$

where $\mathfrak{D} := \{D \in T/A \mid \exists j \in D : \{i,j\} \in A\}$.

Hence, it remains to prove that

$$v(\{i\} \cup \bigcup_{C \in \mathcal{C}} C) - \sum_{C \in \mathcal{C}} v(C) \leq v(\{i\} \cup \bigcup_{D \in \mathcal{D}} D) - \sum_{D \in \mathcal{D}} v(D). \tag{4}$$

We can number the elements of $\mathcal{C}$ and $\mathcal{D}$ in such a way that $\mathcal{C} = \{C_1,\dots,C_s\}$ and $\mathcal{D} = \{D_1,\dots,D_t\}$, where $t \geqslant s$ and $C_r \subseteq D_r$ for all $r \in \{1,\dots,s\}$. This can be seen as follows:
It easily follows that for all $C \in \mathcal{C}$ there exists precisely one $D \in \mathcal{D}$ such that $C \subseteq D$, because $S \subseteq T$. Now suppose there are $E^1, E^2 \in \mathcal{C}$, $E^1 \neq E^2$, and $F \in \mathcal{D}$ such that $E^1 \subseteq F$ and $E^2 \subseteq F$. Let $j_1 \in E^1$ and $j_2 \in E^2$ be such that $\{i,j_1\} \in A$ and $\{i,j_2\} \in A$. Note that $\{j_1,j_2\} \notin A$. Since $\{j_1,j_2\} \subseteq F \in T/A$, there is a path in $(T,A)$ from $j_1$ to $j_2$. So, since $i \notin T$, there is a cycle from $i$ to $i$ over $j_1$ and $j_2$ in the graph $(N,A)$. However, since $(N,A)$ is cycle-complete this should imply that $\{j_1,j_2\} \in A$.
Superadditivity of the game $(N,v)$ implies

$$v(\{i\} \cup \bigcup_{D \in \mathcal{D}} D) \geq v(\{i\} \cup \bigcup_{r=1}^{s} D_r) + \sum_{r=s+1}^{t} v(D_r) \tag{5}$$

and, by convexity,

$$v(\{i\} \cup \bigcup_{r=1}^{s} D_r) - v(D_1) \geq v(\{i\} \cup \bigcup_{r=2}^{s} D_r \cup C_1) - v(C_1),$$

$$v(\{i\} \cup \bigcup_{r=2}^{s} D_r \cup C_1) - v(D_2) \geq v(\{i\} \cup \bigcup_{r=3}^{s} D_r \cup \bigcup_{r=1}^{2} C_r) - v(C_2),$$

$$v(\{i\} \cup D_s \cup \bigcup_{r=1}^{s-1} C_r) - v(D_s) \geq v(\{i\} \cup \bigcup_{r=1}^{s} C_r) - v(C_s).$$

Adding these inequalities we obtain

$$v(\{i\} \cup \bigcup_{r=1}^{s} D_r) - \sum_{r=1}^{s} v(D_r) \geq v(\{i\} \cup \bigcup_{C \in \mathcal{C}} C) - \sum_{C \in \mathcal{C}} v(C). \tag{6}$$

Now (5) and (6) readily imply (4). □

The following example shows that the condition of cycle-completeness in theorem 1 is necessary in the sense that for each communication graph that is not cycle-complete there exists a convex game such that the corresponding point game is not convex.

*Example 2:* Let $(N,A)$ be a communication graph that is not cycle-complete. Then, clearly, there is a cycle $(x_1,...,x_t,x_1)$ in $(N,A)$ with $x_1,...,x_t$ all distinct and there are $i,j \in \{1,...,t\}$ such that $i < j-1$, $\{x_i,x_j\} \notin A$ and for all $k \in \{i+1,...,j-1\}$ $\{x_k,x_j\} \in A$. Consider the convex game $(N,v)$ where $v(S) = |S| - 1$ if $S \neq \emptyset$, and define $S := \{x_i,x_j\}$ and $T := \{x_1,...,x_t\} \setminus \{x_{i+1}\}$. It follows that

$$r_A^v(S \cup \{x_{i+1}\}) - r_A^v(S) = 2 > 1 = r_A^v(T \cup \{x_{i+1}\}) - r_A^v(T).$$

Hence, $(N,r_A^v)$ is not convex.

For the analogue of theorem 1 with respect to arc games one needs a strengthening of cycle-completeness towards cycle-freeness. This follows from

*Theorem 2:* Let $(N,v,A)$ be a communication situation where the communication graph $(N,A)$ is cycle-free and the point game $(N,r_A^v)$ is convex. Then the arc game $(A,r_N^v)$ is convex.

*Proof:* Let $a = \{i,j\} \in A$ and $K \subseteq L \subseteq A \setminus \{a\}$.
We have to prove that $r_N^v(L \cup \{a\}) - r_N^v(L) \geq r_N^v(K \cup \{a\}) - r_N^v(K)$ or, equivalently, that

$$\sum_{T \in N/L \cup \{a\}} v(T) - \sum_{T \in N/L} v(T) \geq \sum_{S \in N/K \cup \{a\}} v(S) - \sum_{S \in N/K} v(S). \quad (7)$$

Define $T_k(S_k)$ as the component in $N/L$ $(N/K)$ that contains player $k \in N$. Clearly,

$$N/(L \cup \{a\}) = \{T \in N/L \mid i \notin T, j \notin T\} \cup \{T_i \cup T_j\} \text{ and}$$
$$N/(K \cup \{a\}) = \{S \in N/K \mid i \notin S, j \notin S\} \cup \{S_i \cup S_j\}.$$

Since $(N,A)$ is cycle-free, we know $S_i \cap S_j = \emptyset$ and $T_i \cap T_j = \emptyset$. Hence, (7) is equivalent to

$$v(T_i \cup T_j) - v(T_i) - v(T_j) \geq v(S_i \cup S_j) - v(S_i) - v(S_j). \quad (8)$$

Because $(N,v)$ is zero-normalized, (8) is equivalent to

$$r_A^v(T_i \cup T_j) - r_A^v(T_i) - r_A^v(T_j) \geq r_A^v(S_i \cup S_j) - r_A^v(S_i) - r_A^v(S_j). \quad (9)$$

Since $K \subseteq L$ we have that $S_i \subseteq T_i$ and $S_j \subseteq T_j$. So, convexity of the game $(N,r_A^v)$ implies

$$r_A^v(T_i \cup T_j) - r_A^v(T_i) \geq r_A^v(S_i \cup T_j) - r_A^v(S_i) \text{ and}$$
$$r_A^v(S_i \cup T_j) - r_A^v(T_j) \geq r_A^v(S_i \cup S_j) - r_A^v(S_j).$$

Adding these inequalities we obtain (9). □

*Corollary 1: Let* $(N, v, A)$ be a communication situation where the original game $(N, v)$ is convex and the communication graph $(N, A)$ is cycle-free. Then the corresponding arc game $(A, r_N^v)$ is convex.

The following example shows that for each communication graph that is not cyle-free we can find a convex game such that the corresponding arc game is not convex.

*Example 3:* Let $(N, A)$ be a communication graph and let $(x_1, \ldots, x_t, x_{t+1} = x_1)$ be a cycle in the graph $(N, A)$. Consider the convex game $(N, v)$, with $v(S) = |S| - 1$ for all $S \neq \emptyset$. Defining $a_k := \{x_k, x_{k+1}\} \in A$ for $k \in \{1, \ldots, t\}$, $K := \{a_1\}$ and $L := \{a_1, \ldots, a_{t-1}\}$, it follows that

$$r_N^v(K \cup \{a_t\}) - r_N^v(K) = 1 > 0 = r_N^v(L \cup \{a_t\}) - r_N^v(L).$$

Hence, $(A, r_N^v)$ is not convex.

With respect to the converse of theorem 2, we have that convexity of a non-negative arc game immediately implies convexity of the point game. So, in particular, for this result we do not have to restrict to cycle-free communication graphs.

*Theorem 3:* Let $(N, v, A)$ be a communication situation where the arc game $(A, r_N^v)$ is non-negative and convex. Then the point game $(N, r_A^v)$ is convex.

*Proof:* Let $i \in N$ and $S \subseteq T \subseteq N \backslash \{i\}$. We have to prove that

$$r_A^v(S \cup \{i\}) - r_A^v(S) \leq r_A^v(T \cup \{i\}) - r_A^v(T). \tag{10}$$

Because $(N, v)$ is zero-normalized, (10) is equivalent to

$$r_N^v(A(S \cup \{i\})) - r_N^v(A(S)) \leq r_N^v(A(T \cup \{i\})) - r_N^v(A(T)). \tag{11}$$

Since $(A, r_N^v)$ is non-negative, the superadditivity of $(A, r_N^v)$ implies

$$r_N^v(A(T \cup \{i\})) \geq r_N^v(A(S \cup \{i\}) \cup A(T)). \tag{12}$$

Further, by the convexity of the game $(A, r_N^v)$, we have

$$\begin{aligned} & r_N^v(A(S \cup \{i\}) \cup A(T)) - r_N^v(A(T)) \geq \\ & r_N^v(A(S \cup \{i\})) - r_N^v(A(S \cup \{i\}) \cap A(T)) = \\ & r_N^v(A(S \cup \{i\})) - r_N^v(A(S)). \end{aligned} \tag{13}$$

Now, inequality (11) is a direct consequence of (12) and (13). □

Combining theorems 2 and 3 we obtain

*Corollary 2:* Let $(N, v, A)$ be a communication situation where the game $(N, v)$ is non-negative and the communication graph $(N, A)$ is cycle-free. Then the point game $(N, r_A^v)$ is convex if and only if the arc game $(A, r_N^v)$ is convex.

Finally, we show that under the conditions of theorem 1 and corollary 1, respectively, the Myerson value and the position value are core-elements of the point game.

*Definition:* The *core* $C(N, v)$ of a coalitional game $(N, v)$ is the set of all division rules of the amount $v(N)$ against which no subcoalition can protest effectively, i.e.

$$C(N,v) := \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}.$$

Shapley (1971) proved that the Shapley value of a convex game is a core-element.

*Theorem 4:* Let $(N, v, A)$ be a communication situation where the underlying game $(N, v)$ is convex. Then the following two assertions hold:

(i) If the communication graph is cycle-complete, then $\mu(N, v, A) \in C(N, r_A^v)$.
(ii) If the communication graph is cycle-free, then $\pi(N, v, A) \in C(N, r_A^v)$.

*Proof:* Part (i) is a direct consequence of theorem 1.
Part (ii). Suppose $(N, A)$ is cycle-free. According to corollary 1 the arc game $(A, r_N^v)$ is convex. Hence,

$$\Phi(A, r_N^v) \in C(A, r_N^v). \tag{14}$$

Next we show that

$$\Phi(A, r_N^v) \geq 0. \tag{15}$$

Clearly, it suffices to prove that $r_N^v(L \cup \{a\}) - r_N^v(L) \geq 0$ for all $a \in A$ and all $L \subseteq A \setminus \{a\}$. Let $a = \{i, j\} \in A$ and $L \subseteq A \setminus \{a\}$. Then,

$$r_N^v(L \cup \{a\}) - r_N^v(L) \geq r_N^v(\{a\}) - r_N^v(\emptyset) = v(\{i, j\}) \geq v(\{i\}) + v(\{j\}) = 0,$$

where the first inequality follows from the convexity of $(A, r_N^v)$ and the second one from the superadditivity of $(N, v)$.

Now let $S \subseteq N$. Then, using (14) and (15),

$$\sum_{i \in S} \pi_i(N,v,A) = \sum_{i \in S} \sum_{a \in A_i} \tfrac{1}{2}\Phi_a(A,r_N^v) \geq$$

$$\sum_{i \in S} \sum_{a \in A_i \cap A(S)} \tfrac{1}{2}\Phi_a(A,r_N^v) = \sum_{a \in A(S)} \Phi_a(A,r_N^v) \geq$$

$$r_N^v(A(S)) = r_A^v(S)$$

and

$$\sum_{i \in N} \pi_i(N,v,A) = \sum_{a \in A} \Phi_a(A,r_N^v) = r_N^v(A) = r_A^v(N).$$

Hence, $\pi(N,v,A) \in C(N,r_A^v)$. □

## 4 Remarks

Owen (1986) proved that, without imposing any restrictions on the communication graphs, superadditivity of the underlying game is inherited by the point game. With respect to arc games, however, one cannot hope to find a non-trivial class of communication graphs for which the superadditivity of the underlying game is inherited by the corresponding arc games. This follows from

*Example 4:* Let $(N,v,A)$ be the communication situation where $N = \{1,2,3\}$, $A = \{\{1,2\},\{2,3\}\}$ and $v(S) = \begin{cases} 0 & \text{if } |S| \leq 1 \\ |S| & \text{if } |S| \geq 2 \end{cases}$. The game $(N,v)$ is superadditive, but, with $a := \{1,2\}$ and $b := \{2,3\}$, we have $r_N^v(\{a\}) + r_N^v(\{b\}) = 4 < 3 = r_N^v(\{a,b\})$. Hence, the game $(A,r_N^v)$ is not superadditive.

Further, it may be noted that, similarly to theorem 3, superadditivity of a non-negative arc game immediately implies superadditivity of the point game.

*Definition:* A coalitional game $(N,v)$ is called *balanced* if it has a non-empty core and it is called *totally balanced* if for every $S \subseteq N$ the subgame $(S, v|_S)$ is balanced. Here, $v|_S(T) := v(T)$ for all $T \subseteq S \subseteq N$.

For connected communication graphs, it is easy to see that balancedness of the underlying game is inherited by the corresponding point game. Moreover, if the underlying game is totally balanced, then for each communication graph the point game is also totally balanced.

Example 4 shows that the fact that the underlying game is totally balanced does not even imply balancedness of the arc game. Finally, it can be proved that balancedness of a non-negative arc game immediately implies balancedness of the corresponding point game.

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