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Characterization of Cores of Assignment Games[1,2]

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Abstract: We consider the assignment game of Shapley and Shubik (1972). We prove that the class of possible cores of such games (expressed in terms of payoffs for players on one side of the market) is exactly the same as a special class of polytopes, called "45°-lattices". These results parallel similar work done by Conway (in Knuth, 1976) and Blair (1984) for marriage markets.

1 Introduction

The so-called "assignment game" of Shapley and Shubik (1972) has been the subject of much research. One particular thrust has been in the area of the structure of the core when expressed as a set of utility vectors for players on one side of the market. Shapley and Shubik themselves prove this set is a lattice. Other papers (Thompson (1980), Balinski and Gale (1988)) take more advantage of the special characteristics of these games in order to prove their results.

Certainly it is easy to see that the core will always be a special type of lattice in these games, here called a "45°-lattice". Geometrically, such a lattice is formed by starting with a cube and cutting away triangular cylinders where the "triangles" are 45°–45°–90°. In this paper, we show that the converse is true, i.e., that every such 45°-lattice is the core of an appropriately defined assignment game. Hence, our results parallel the lattice characterization theorems of Conway (in Knuth, 1976) and Blair (1984) concerning another type of two-sided matching market – the stable matching problem (Gale, Shapley 1962).

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2 Background

Let us briefly review the assignment game of Shapley and Shubik (1972). This is a TU-game in which there is a set $I = \{1,...\ell\}$ of *sellers* and a set $J = \{1,...,m\}$ of *buyers*. Let $N = I \cup J$ be the *player set*. The data for the model is an $\ell \times m$ nonnegative matrix C, whose entries c_{ij} represent the worth of a coalition containing only seller i and buyer j.

Define an *assignment*, or *matching*, as a sequence of seller-buyer pairs $\mu = \{(i_1 j_1),...,(i_k j_k)\}$ in which no player appears more than once. If pair $(ij) \in \mu$, we write $\mu(i) = j$ and $\mu^{-1}(j) = i$. On the other hand, if a seller i does not appear in any pair in μ, we say i is *unmatched* by μ and write $\mu(i) = \varnothing$. Similarly, j unmatched is written $\mu^{-1}(j) = \varnothing$.

Next, define a *maximal matching* μ^* as a matching for which

$$\sum_{\substack{i \in I \\ \mu^*(i) \neq \varnothing}} c_{i\mu^*(i)} \geq \sum_{\substack{i \in I \\ \mu(i) \neq \varnothing}} c_{i\mu(i)}$$

for all other assignments μ. Note that a *full* maximal matching, i.e., a maximal matching μ^* where $\mu^*(i) \neq \varnothing \; \forall i$ or $\mu^{*-1}(j) \neq \varnothing \; \forall j$, must exist.

The game's characteristic function V is given by

$$V(S) = \max_{\mu} \sum_{i,\mu(i) \in S} c_{i\mu(i)}.$$

The interpretation here is that, for any coalition of players $S \subseteq N$, the best that S can do is "split into (one-seller-one-buyer) pairs and pool the profit". In game theoretic terms, these pairs are the essential coalitions of the game. Finally, note that $V(N) = \Sigma_i c_{i\mu^*(i)}$ for any maximal matching μ^*.

The core of this game is never empty, and turns out to be equivalent to the set of utility vectors (u,v) for which

$$u_i + v_{\mu^*(i)} = c_{i\mu^*(i)} \quad \forall i : \mu^*(i) \neq \varnothing \tag{2.1}$$

$$u_i + v_j \geq c_{ij} \quad \forall i,j \tag{2.2}$$

$$u_i , v_j \geq 0 \quad \forall i,j \tag{2.3}$$

$$u_i = 0 \quad \forall i : \mu^*(i) = \varnothing \tag{2.4}$$

$$v_j = 0 \quad \forall j : \mu^{*-1}(j) = \varnothing, \tag{2.5}$$

for any full maximal matching μ^*. We can think of (2.1), (2.4), and (2.5) as feasibility constraints, (2.2) as stability constraints, and (2.3) as individual rationality.

Using (2.1) and (2.5), we can substitute for the v's and write the core as a vector u of utilities for the sellers:

$$u_i + c_{\mu^{*-1}(j)j} - u_{\mu^{*-1}(j)} \geq c_{ij} \quad \forall i,j : \mu^{*-1}(j) \neq \emptyset \tag{2.6}$$

$$u_i \geq c_{ij} \quad \forall i,j : \mu^{*-1}(j) = \emptyset \tag{2.7}$$

$$u_i \geq 0 \quad \forall i \tag{2.8}$$

$$u_{\mu^{*-1}(j)} \leq c_{\mu^{*-1}(j)j} \quad \forall j : \mu^{*-1}(j) \neq \emptyset \tag{2.9}$$

$$u_i = 0 \quad \forall i : \mu^*(i) = \emptyset \tag{2.10}$$

Here (2.6) and (2.7) come from (2.2), and (2.8) and (2.9) from (2.3). We call the set of u's which satisfy (2.6)–(2.10) the *u-space core of assignment game C.*

Next, denote by P^ℓ the set of nonempty polytopes $P \subseteq \mathbb{R}^\ell$ which can be written as $(u_1,\ldots,u_\ell)$ satisfying:

$$u_i - u_k \geq d_{ik} \quad \forall i,k \in 1,\ldots,\ell : i \neq k \tag{2.11}$$

$$b_i \leq u_i \leq e_i \quad \forall i \in 1,\ldots,\ell \tag{2.12}$$

for some constants $\{d_{ik}\}_{i,k=1}^{\ell}$, and some nonegative constants $\{b_i,e_i\}_{i=1}^{\ell}$. We call P^ℓ the set of $45°$*-lattices* in $\mathbb{R}^\ell$. The reason for this is that, geometrically, we are starting with a cube (2.12), and "lopping off" the 45°–45°–90° triangular cylinders indicated by (2.11).[4]

We can see that the u-space core of any assignment game C with ℓ sellers is an element of P^ℓ. Indeed, this can be seen by just setting

$$d_{ik} = \begin{cases} c_{i\mu^*(k)} - c_{k\mu^*(k)} & \text{if } \mu^*(k) \neq \emptyset \\ -M & \text{otherwise,} \end{cases}$$

$$b_i = \begin{cases} \max\limits_{j:\mu^{*-1}(j)=\emptyset} c_{ij} & \text{if } \exists j : \mu^{*-1}(j) = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$e_i = \begin{cases} c_{i\mu^*(i)} & \text{if } \mu^*(i) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

M is a large positive number.

4 Of course, while maintaining the lattice property.

Next, consider the special case in which $\ell \geq m$. In this case, there are no buyers j for which $\mu^{*-1}(j) = \emptyset$. Thus, the (u-space) core will be an element of $P^{\ell z}$, where $P^{\ell z}$ is defined as the subset of P^{ℓ} consisting of P's in which $b_i = 0\ \forall\, i$. We call $P^{\ell z}$ the set of *zero-possible* 45°-lattices because they are the only ones that could possibly contain the point 0.

3 The Characterization Theorem

We now state and prove the converse of the results of the previous section.

Theorem: Let $\hat{P} \in P^{\ell}$, (thus $\hat{P} \neq \emptyset$), say $\hat{P}$ can be written as $(u_1,\dots,u_\ell)$ satisfying:

$$u_i - u_j \geq \hat{d}_{ij} \quad \forall\, i,j \in 1,\dots,\ell : i \neq j \tag{3.1}$$

$$\hat{b}_i \leq u_i \leq \hat{e}_i \quad \forall\, i \in 1,\dots,\ell \tag{3.2}$$

for some constants $\{\hat{d}_{ij}\}$, $\{\hat{b}_i\} \geq 0$, and $\{\hat{e}_i\} \geq 0$. Then we can define an assignment game $C(\hat{P})$ with ℓ sellers in which the u-space core is exactly P. If $\hat{P} \in P^{\ell z}$, then $C(\hat{P})$ will have $\ell \geq m$ (the number of sellers will be at least the number of buyers).

Proof: Define the $\ell \times (\ell + 1)$ assignment game C by:

$$c_{ij} = \hat{e}_i \quad \forall\, i \in 1,\dots,\ell \tag{3.3}$$

$$c_{ij} = [\hat{e}_j + \hat{d}_{ij}]^+ \quad \forall\, i,j \in 1,\dots,\ell,\ i \neq j \tag{3.4}$$

$$c_{i,\ell+1} = \hat{b}_i \quad \forall\, i \in 1,\dots,\ell \tag{3.5}$$

Here the notation $[x]^+$ means max $(x,0)$. We will prove that C has the necessary properties for the first part of the Theorem.

Lemma: μ^* defined by $\mu^*(i) = i\ \forall\, i$ is a maximal matching for C.

Proof: It sufficies to prove $\Sigma_{i=1}^{\ell}\ c_{i\,\mu(i)} \leq \Sigma_{i=1}^{\ell}\ c_{i\mu^*(i)}$ for any matching μ. To show this, we need a preliminary result:

Definition: Given assignment μ, a *cycle D* of μ is a sequence of distinct numbers $i_1,\dots,i_r$ for which $\mu(i_{k-1}) = i_k$ for all $k \in 2,\dots,r$ and $\mu(i_r) = i_1$.

Proposition: Let D be a cycle of μ, in which $r \geq 2$. Then $\Sigma_{i \in D}\ \hat{d}_{i\mu(i)} \leq 0$.

Proof: Let $u \in \hat{P}$. Then

$$\begin{aligned}
u_{i_1} - u_{i_2} &\geq \hat{d}_{i_1 i_2} \\
&\vdots \\
u_{i_{r-1}} - u_{i_r} &\geq \hat{d}_{i_{r-1} i_r} \\
u_{i_r} - u_{i_1} &\geq \hat{d}_{i_r i_1}
\end{aligned}$$

Adding the above inequalities gives the Proposition.

Now, given μ, let $T = \{i: c_{i\mu(i)} > 0\}$. For all $i \in T$, if $\mu(i) \notin \{i, \ell+1\}$, $c_{i\mu(i)} = [\hat{e}_{\mu(i)} + \hat{d}_{i\mu(i)}]^+ = \hat{e}_{\mu(i)} + \hat{d}_{i\mu(i)}$. [Otherwise, if $\mu(i) = i$, $c_{i\mu(i)} = \hat{e}_i$ and, $f\mu(i) = \ell+1$, $c_{i\mu(i)} = \hat{b}_i$.] Define the directed graph G with nodes $\{1,\ldots,\ell + 1\}$ and where arc $\overrightarrow{ij}$ exists iff $\mu(i) = j$. Let G_T be the subgraph of G consisting of arcs $\overrightarrow{ij}$ where $i \in T$, together with the incident nodes for these arcs.

It is easy to see that $\sum_{i=1}^{\ell} c_{i\mu(i)} = \sum_{i \in T} c_{i\mu(i)} = \sum_{\overrightarrow{ij} \in G_T} c_{ij}$.

Furthermore, the set of arcs of G_T can be partitioned into paths $\{\rho_1,\ldots,\rho_h\}$, where ρ_x and ρ_y contain no common incident nodes if $x \neq y$. Call this partition $\mathscr{P}$.

Finally, define the notation $i \in \rho$ to mean that node i is incident to at least one of the arcs of path ρ. In this sense, $\mathscr{P}$ is a partition of R, where R is the set of nodes of G_T.

Claim: $\sum_{\overrightarrow{ij} \in \rho_x} c_{ij} \leq \sum_{i \in \rho_x : i \neq \ell+1} \hat{e}_i \;\forall \rho_x \in \mathscr{P}$.

Proof: There are two cases:

Case 1: ρ_x is a cycle. If it is also a loop (i.e., it contains only one arc), the Claim holds trivially with equality. Otherwise, $\mu(i) \neq i$ and $i \neq \ell + 1 \;\forall i \in \rho_x$. We have:

$$\sum_{\overrightarrow{ij} \in \rho_x} c_{ij} = \sum_{i \in \rho_x} \hat{e}_{\mu(i)} + \hat{d}_{i\mu(i)} \leq \sum_{i \in \rho_x} \hat{e}_{\mu(i)} = \sum_{i \in \rho_x} \hat{e}_i \,.$$

The inequality above of course follows from the Proposition.

Case 2: ρ_x is not a cycle. In this case, let $\overrightarrow{i_1 i_2}, \overrightarrow{i_2 i_3},\ldots,\overrightarrow{i_{K-1} i_K}$ be the arcs of ρ_x. From the construction, $i_k \in T$ and $\mu(i_k) = i_{k+1}$ for $k \in 1,\ldots,K-1$, while $i_K \notin T$. Again, we have $\mu(i) \neq i \;\forall i \in \rho_x$. Let $u \in \hat{P}$.

Case 2A: $i_K = \ell + 1$. Then

$$\sum_{\vec{ij} \in \rho_x} c_{ij} = \sum_{k=1}^{K-2} (\hat{e}_{\mu(i_k)} + \hat{d}_{i_k\,\mu(i_k)}) + \hat{b}_{i_{K-1}}$$

$$\leq \sum_{k=1}^{K-2} (\hat{e}_{\mu(i_k)} + u_{i_k} - u_{i_{k+1}}) + \hat{b}_{i_{K-1}}$$

$$= (\sum_{k=2}^{K-1} \hat{e}_{i_k}) + (u_{i_1}) + (\hat{b}_{i_{K-1}} - u_{i_{K-1}}) \leq \sum_{k=1}^{K-1} \hat{e}_{i_k}.$$

Both of the inequalities in the above chain follow from the fact that $u \in \hat{P}$.

Q.E.D.

Case 2B: $i_K \neq \ell + 1$.

$$\sum_{\vec{ij} \in \rho_x} c_{ij} = \sum_{k=1}^{K-1} (\hat{e}_{\mu(i_k)} + \hat{d}_{i_k\,\mu(i_k)})$$

$$\leq \sum_{k=1}^{K-1} (\hat{e}_{\mu(i_k)} + u_{i_k} - u_{i_{k+1}})$$

$$= (\sum_{k=2}^{K} \hat{e}_{i_k}) + (u_{i_1}) - (u_{i_K}) \leq \sum_{k=1}^{K} \hat{e}_{i_k}.$$

Now, to prove the Lemma, we need only to take the Claim and sum over all the paths ρ_x:

$$\sum_{\rho_x \in \mathscr{P}} \sum_{\vec{ij} \in \rho_x} c_{ij} \leq \sum_{\rho_x \in \mathscr{P}} \sum_{i \in \rho_x : i \neq \ell+1} \hat{e}_i$$

$$\Rightarrow \sum_{i \in T} c_{i\mu(i)} \leq \sum_{i \in R: i \neq \ell+1} \hat{e}_i \Rightarrow \sum_{i=1}^{\ell} c_{i\mu(i)} \leq \sum_{i=1}^{\ell} c_{i\mu^*(i)} .$$

Having finally proven the Lemma, we can thus write the following equations for the core of C:

$$u_i + v_i = c_{i\mu^*(i)} = \hat{e}_i \qquad \forall i \in 1,\ldots,\ell \tag{3.7}$$

$$u_i + v_j \geq c_{ij} = [\hat{e}_j + \hat{d}_{ij}]^+ \qquad \forall i,j : i \neq j,\, j \neq \ell+1 \tag{3.8}$$

$$u_i + 0 \geq c_{i(\ell+1)} = \hat{b}_i \qquad \forall i \in 1,\ldots,\ell \tag{3.9}$$

$$u_i \geq 0,\; v_j \geq 0 \qquad \forall i,j \in 1,\ldots,\ell \tag{3.10}$$

Substituting using (3.7), we have that the u-space core of C is the set of u's which satisfy

$$u_i + \hat{e}_j - u_j \geq [\hat{e}_j + \hat{d}_{ij}]^+ \qquad \forall i,j : i \neq j, j \neq \ell+1 \tag{3.11}$$

$$\hat{b}_i \leq u_i \leq \hat{e}_i \qquad \forall i \in 1,\ldots,\ell \tag{3.12}$$

So, in order to prove (the first part of) the Theorem, we need to show that u satisfies (3.11) and (3.12) if and only if it satisfies (3.1) and (3.2).

First, we prove the "if" statement above. If $\hat{e}_j + \hat{d}_{ij} \geq 0$, then $u_i - u_j \geq \hat{d}_{ij}$ implies $u_i + \hat{e}_j - u_j \geq \hat{e}_j + \hat{d}_{ij} = [\hat{e}_j + \hat{d}_{ij}]^+$. And, if $\hat{e}_j + \hat{d}_{ij} < 0$, then $u_i + (\hat{e}_j - u_j) \geq u_i \geq 0 = [\hat{e}_j + \hat{d}_{ij}]^+$. Hence, (3.1) and (3.2) together imply (3.11). Also, trivially, (3.2) implies (3.12).

For the "only if" part of the statement, again first suppose $\hat{e}_j + d_{ij} \geq 0$. Then $u_i + \hat{e}_j - u_j \geq [\hat{e}_i + \hat{d}_{ij}]^+$, which implies $u_i + \hat{e}_j - u_j \geq \hat{e}_j + \hat{d}_{ij}$, which in turn implies $u_i - u_j \geq \hat{d}_{ij}$. And, if $\hat{e}_j + \hat{d}_{ij} < 0$, then, since $u_i \geq 0$ and $u_j \leq \hat{e}_j$, we have $u_i - u_j \geq -\hat{e}_j \geq \hat{d}_{ij}$. Hence, (3.11) and (3.12) together imply (3.1). Trivially, (3.12) implies (3.2).

For the second part of the Theorem, note that if $\hat{P} \in P^{\ell z}$, i.e., all the $\hat{b}_i$'s are equal to zero, then our assignment game $C(\hat{P})$ has its $\ell + 1^{\text{st}}$ column all zeroes. Thus, we can just use as "$C(\hat{P})$" the $\ell \times \ell$ matrix consisting of the first ℓ columns of $C(\hat{P})$.

4 Discussion

A natural question to ask is to what extent $C(\hat{P})$ is unique for any $\hat{P}$. First, one can see that we can replace the $\ell + 1^{\text{st}}$ column with any set of columns $K = \{C_k\}_{k=\ell+1}^{m}$, where $\max_{k>\ell} c_{ik} = b_i$ for every i. This is because μ^* would still be the "diagonal", and the core equations would change only inasmuch as new redundant constraints would be added. Second, we can clearly rearrange the columns of $C(\hat{P})$ without changing the u-space core. [Rearranging the columns changes μ^* and the v-space core, but not the u-space core.] Finally, we surmise that equations (3.3)–(3.5) may produce many c_{ij}'s for which $u_i + v_j > c_{ij}$ for every (u,v) in the core. In this case, the c_{ij} could be perturbed without changing the core. In fact, we conjecture that there is some link between the set of c_{ij}'s which behave in this way, and the set of d_{ik}'s which can be set equal to $-M$ when determining a "$P(C)$" [section 2].

Thus, there may exist a result such as: "Given $\hat{P} \in P^{\ell}$, there is a unique $C(\hat{P})$ which satisfies:

1) Core $(C(\hat{P})) = \hat{P}$.
2) $C(\hat{P})$ contains $\ell + 1$ buyers (or ℓ buyers if $\hat{P} \in P^{\ell z}$).
3) A maximal matching for $C(\hat{P})$ is $\mu^* : \mu^* (i) = i, i \in 1, \ldots, \ell$.
4) The $c_{ij}'s$, $j \neq i$, are "as high as possible".

In such an issue may lie the basis for future work.

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