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Almost Everywhere Convergence of Riesz Means on Certain Noncompact Symmetric Spaces (*).

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Summary. – *Let G/K be a rank one or complex non compact symmetric space of dimension l. We prove that if $f \in L^p$, $1 \leq p \leq 2$, the Riesz means of order z of f with respect to the eigenfunction expansion of the Laplacian converge to f almost everywhere for* $\mathrm{Re}(z) > \delta(l, p)$. *The critical index $\delta(l, p)$ is the same as in the classical result of Stein in the Euclidean case.*

1. – Introduction.

The purpose of this paper is to study the almost everywhere convergence of the Riesz means of the eigenfunction expansion associated to the Laplacian on complex and rank one symmetric spaces. Let Δ_0 be the Laplace-Beltrami operator on the symmetric space G/K, where G is a noncompact semisimple Lie group with finite center and K is a maximal compact subgroup of G. The operator $-\Delta_0$ is elliptic and formally positive on $L^2(G/K)$ that is

$$(-\Delta_0 u, u) \geq \|\rho\|^2 (u, u) \qquad u \in C_c^\infty(G/K),$$

where ρ is the half sum of the positive roots of the pair (G, K) with multiplicities. Let

$$-\Delta = \int_{\|\rho\|^2}^{+\infty} t\, dE_t$$

be the spectral resolution of the unique self-adjoint extension $-\Delta$ of $-\Delta_0$ there. For

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$z \in \boldsymbol{C}$, $\operatorname{Re} z \geqslant 0$, the Riesz means of order z are the operators

$$S_R^z = \int_{\|\rho\|^2}^{+\infty} (1 - t/R)_+^z \, dE_t \qquad R \geqslant \|\rho\|^2 .$$

Since Δ_0 is G-invariant the operator S_R^z is given by convolution on G with a C^∞ kernel s_R^z. Thus $S_R^z f = f * s_R^z$ is well defined also for any distribution f with compact support. If $f \in L^2$ it is clear that $S_R^0 f = E_R f$ tends to f in L^2 as R tends to $+\infty$. Since the classical work of E. M. STEIN (see exposition in [13]) on the Riesz summability for multiple Fourier series, many authors have investigated the L^p norm and almost everywhere convergence for Riesz means for the eigenfunction expansions of elliptic or hypoelliptic differential operators on manifolds [1, 6, 7, 9, 10, 11, 12]. In this paper we shall investigate the almost everywhere convergence of $S_R^z f$ to f as $R \to +\infty$, for $f \in L^p, 1 \leqslant p \leqslant 2$.

On noncompact symmetric spaces the problem of norm summability in $L^p, p \neq 2$, is ill posed since the operators S_R^z are not bounded on $L^p, p \neq 2$, for any z. Indeed the spherical transform of the kernel s_R^z does not extend to a holomorphic function in any tube domain over the dual of the abelian component of the Iwasawa decomposition of the Lie algebra of G. Nevertheless we shall see that in the complex or rank one case $S_R^z, \operatorname{Re} z \geqslant 0$, maps L^p into L^q for all p, q such that $1 \leqslant p \leqslant 2$ and $2p/(2-p) \leqslant q \leqslant \infty$. Moreover if $f \in L^p$, $1 \leqslant p \leqslant 2$, $S_R^z f \to f$ almost everywhere provided that $\operatorname{Re} z \geqslant \delta(l, p) = (l-1)(1/2 - 1/p)$ where $l = \dim(G/K)$. We remark that the critical index $\delta(l, p)$ is the same as in the classical result of Stein for the Euclidean case [13]. Our results are based on estimates for the kernel s_R^z of the form

$$|s_R^z(x)| \leqslant c(z) R^{l/2} (1 + R^{1/2}|x|)^{-(\operatorname{Re} z + (l+1)/2)} (1 + |x|)^\delta e^{-\gamma|x|}, \qquad x \in G,$$

where δ, γ are positive constants which depend on G. These estimates are obtained in Section 3. In the complex case they follow by explicit inversion of the spherical transform of s_R^z. In the rank one case they follow by using the inversion formulae for the Abel transform to express s_R^z as a Weyl fractional integral of kernels of Euclidean Riesz means of appropriate order. In Section 4 we use the estimates of s_R^z to estimate the maximal function $S_*^z f(x) = \sup\{|S_R^z f(x)| : R > \|\rho\|^2\}$. We prove that if $\operatorname{Re} z > (l-1)/2$ then $S_*^z f(x) \leqslant M_1 f(x) + c|f| * k(x)$, where M_1 is the Hardy-Littlewood maximal function over the balls of radius less than 1 and k is a kernel in L^q for every q sufficiently large. Thus, when $\operatorname{Re} z > (l-1)/2$, S_*^z is bounded from L^p to $L^p + L^r$, for $p > 1$ but sufficiently close to 1, and r large. Since S_*^z is bounded on L^2 for every $z \in \boldsymbol{C}$, $\operatorname{Re} z > 0$, we can apply complex interpolation to prove that, for $\operatorname{Re} z > \delta(l, p)$, S_*^z maps L^p into $L^p + L^r$ for every $1 < p \leqslant 2$ and r sufficiently large. For $p = 1$ the same argument shows that S_*^z satisfies a weak type estimate when $\operatorname{Re} z > (l-1)/2$. The almost everywhere convergence results then follow by standard measure theoretic arguments.

Finally we point out that the methods of this paper, in combination with the method of reduction to the complex case introduced by FLENSTED-JENSEN [4] allow us to obtain almost everywhere convergence of the Riesz means of L^p functions, $1 \leq p \leq 2$, also when the Lie algebra of G is a normal real form of its complexification. However, since in this case the critical index that we obtain is not sharp, as it can be seen by considering $SL(2, \boldsymbol{R})$ which is a normal real form of $SL(2, \boldsymbol{C})$, we omit the details.

2. – Preliminaries.

In this section we shall briefly recall some basic facts on the spherical Fourier transform on complex and rank one symmetric spaces. The main references for this section are [5] and [8].

Let G be a semisimple, noncompact Lie group with finite center and Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G and G/K the corresponding symmetric space. Let σ be the Cartan involution of (G, K) and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra of G. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $G = NAK$ and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be corresponding Iwasawa decompositions. For any g let $H(g) \in \mathfrak{a}$ denote the unique element of $\mathfrak{a}$ such that $g \in N \exp H(g) K$. Let Σ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, let Σ^+ denote the subset of positive roots and let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha ,$$

m_α denoting the multiplicity of α. Let $\mathfrak{a}_C^*$ denote the complex dual of $\mathfrak{a}$. The spherical functions on G are the functions

$$\varphi_\lambda(g) = \int_K \exp[(i\lambda + \rho) H(kg)]\, dk \qquad g \in G ,$$

$\lambda \in \mathfrak{a}^*{}_C$. The functions φ_λ are K-biinvariant. Moreover $(\lambda, H) \to \varphi_\lambda(\exp(H))$ is invariant under the action of the Weyl group both as a function of $\lambda \in \mathfrak{a}^*$ and of $H \in \mathfrak{a}$. We shall systematically identify K-invariant functions on G/K with K-biinvariant functions on G. If f is a K-biinvariant function on G its spherical Fourier transform is defined by

$$\tilde{f}(\lambda) = \int_G f(x) \varphi_{-\lambda}(x)\, dx \qquad \lambda \in \mathfrak{a}_C^* .$$

The inversion formula is

$$f(x) = c(G, K) \int_{\mathfrak{a}^*} \tilde{f}(\lambda) \varphi_\lambda(x) |c(\lambda)|^{-2}\, d\lambda , \tag{2.1}$$

where $c(G, K)$ is a constant which depends on the normalization of the measures of G and K and c is the Harish-Chandra c-function. The exponential map $\mathrm{Exp}(X) = \exp(X)K$ is a diffeomorphism of $\mathfrak{p}$ onto G/K. Let J denote its Jacobian. Then for $f \in L^1(G/K)$

$$\int_{G/K} f(x)\, dx = \int_{\mathfrak{p}} f(\mathrm{Exp}(X))J(X)\, dX .$$

Let Log: $G/K \to \mathfrak{p}$ be the inverse of the exponential map. Then for $x \in G/K$ we shall denote by $|x| = \|\mathrm{Log}(x)\|$ the distance from x to the origin $o = eK$ in G/K. Here $\|\cdot\|$ denotes the norm induced by the Killing form on $\mathfrak{p}$.

If G is complex its Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. In this case all Weyl-invariant functions of $H \in \mathfrak{a}$ or $\lambda \in \mathfrak{a}^*$ can be extended uniquely to K-invariant functions of $Z \in \mathfrak{p}$ or $\Lambda \in \mathfrak{p}^*$. In the complex case $\mathfrak{p} = i\mathfrak{k}$, $\mathfrak{p}^* = (i\mathfrak{k})^*$. Moreover one has the following representation for the spherical functions

$$\varphi_\Lambda(Z) = J(Z)^{-1/2} \int_K \exp[i\Lambda(AdkZ)]\, dk ,$$

for $Z \in i\mathfrak{k}$, $\Lambda \in (i\mathfrak{k})^*$, from which one can derive a different expression of the inversion formula

$$f(\exp(Z)) = c(G,K)J(Z)^{-1/2} \int_{(i\mathfrak{k})^*} \tilde{f}(\Lambda)\, e^{i\Lambda(Z)}\, d\Lambda \qquad Z \in i\mathfrak{k} . \tag{2.2}$$

We now turn to a discussion of the inversion formula in the rank one case. Let $\boldsymbol{F} = \boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$, or $\boldsymbol{O}$ be the real numbers, the complex numbers, the quaternions or the Cayley octonions. The rank one symmetric spaces can be realized as the hyperbolic space $H_n(\boldsymbol{F})$. Here the subscript n denotes the dimension over the base field $\boldsymbol{F}$. Notice that $n = 2, 3, 4, \ldots$ for $\boldsymbol{F} = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, but $n = 2$ for $\boldsymbol{F} = \boldsymbol{O}$. Let $d = \dim_{\boldsymbol{R}} \boldsymbol{F}$. Thus the dimension of G/K over $\boldsymbol{R}$ is $l = dn$. The subspace $\mathfrak{a} \subset \mathfrak{p}$ is one-dimensional. We shall denote by α and 2α the positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. They have multiplicities $m_\alpha = d(n-1)$ and $m_{2\alpha} = d - 1$ respectively. Let H be the vector in $\mathfrak{a}$ such that $\alpha(H) = 1$ and write $a_t = \exp(tH)$. The measure on G/K is normalized so that if f is a K-invariant function on G/K then

$$\int_{G/K} f(x)\, dx = \frac{2\pi^{l/2}}{\Gamma(l/2)} \int_0^{+\infty} f(a_t)(\sinh t)^{m_\alpha}(\sinh 2t)^{m_{2\alpha}}\, dt . \tag{2.3}$$

Let $f^{\#}\colon [1, +\infty) \to \boldsymbol{C}$ be the function such that $f(a_t) = f^{\#}(\cosh t)$. The Abel transform of f is

$$\mathcal{Q}f(a_t) = \int_{\boldsymbol{R}^{m_\alpha}} dy \int_{\boldsymbol{R}^{m_{2\alpha}}} dx f^{\#}[(\cosh t + |x|^2/2)^2 + |y|^2)^{1/2}], \tag{2.4}$$

Then the spherical Fourier transform is the composition of the ordinary Fourier transform $\mathcal{F}$ on $\boldsymbol{R}$ and of the Abel transform, i.e. $\tilde{f}(\lambda) = \mathcal{F}\mathcal{Q}f(\lambda)$, where $\lambda \in \mathfrak{a}^* \cong \boldsymbol{R}$. Thus to invert the spherical transform on G/K we only need to invert the Abel transform. The inversion formulae for the Abel transform are the following (see for instance [7]). Let $D_y = -(2\pi \sinh y)^{-1} d/dy$. Then

i) if $\boldsymbol{F} = \boldsymbol{R}$ and l is odd

$$F(a_y) = D_y^{(l-1)/2} \mathcal{Q}F(a_y) \tag{2.5}$$

ii) if $\boldsymbol{F} = \boldsymbol{R}$ and l is even

$$F(a_y) = (2\pi)^{1/2} \int_y^{+\infty} [\cosh(x) - \cosh(y)]^{-1/2} D_x^{l/2} \mathcal{Q}F(a_x) \sinh(x)\, dx, \tag{2.6}$$

iii) if $\boldsymbol{F} = \boldsymbol{C}$, $\boldsymbol{H}$ or $\boldsymbol{O}$ there exist constants $c_1, c_2 \ldots, c_{d/2}$ such that

$$F(a) = \sum_{j=1}^{d/2} c_j \int_y^{+\infty} [\cosh^2(x) - \cosh^2(y)]^{-1/2} (\cosh x)^{j+1-d} D_x^{j+(m_\alpha/2)} \mathcal{Q}F(a_x) \sinh x\, dx.$$

Let Δ be the Laplace-Beltrami operator on the symmetric space G/K. Then for every smooth function f with compact support on G/K

$$(\Delta f)^{\sim}(\lambda) = -(\|\rho\|^2 + \|\lambda\|^2)\tilde{f}(\lambda) \tag{2.7}$$

$\lambda \in \mathfrak{a}^*$. For every $R > \|\rho\|^2$ and $z \in \boldsymbol{C}$ such that $\operatorname{Re} z \geq 0$ let s_R^z be the K-biinvariant function on G whose spherical transform is

$$(s_R^z)^{\sim}(\lambda) = (1 - (\|\rho\|^2 + \|\lambda\|^2)/R)_+^z. \tag{2.8}$$

Then for every function $\varphi \in C_c(G/K)$ the Bochner-Riesz means of order z are given by $S_R^z \varphi = \varphi * s_R^z$. Notice that, since Δ is elliptic, $S_R^z = (I + \Delta)_+^z$ maps $L^2(G/K)$ into $C^\infty(G/K)$ continuously, by the spectral theorem. Thus $s_R^z \in C^\infty(G)$ and $S_R^z \varphi$ is well defined whenever φ is a distribution with compact support on G/K.

3. – Estimates for the kernel of the Riesz means.

In this section we shall estimate the decay at infinity of the kernels s_R^z. Our main results are Propositions 3.1 and 3.3 below that show that, as a function of the hyperbolic distance from the origin, s_R^z behaves essentially as the product of a function of exponential decay and of the Bochner-Riesz kernel of the same order for the Euclidean space $\boldsymbol{R}^l$ where $l = \dim(G/K)$. Let $\mathfrak{J}_\nu(t) = t^{-\nu} J_\nu(t)$, $t > 0$, where J_ν is the Bessel function of order ν. We shall begin with the complex case for which the result is trivial.

PROPOSITION 3.1. – Let G be complex, $l = \dim(G/K)$. Then for $z \in \boldsymbol{C}$, $\operatorname{Re} z \geqslant 0$, $R \geqslant \|\rho\|^2$, $Z \in i\mathfrak{k}$:

$$s_R^z(\exp Z) = c(G,K) 2^z (2\pi)^{l/2} \Gamma(z+1) R^{-z} (R - \|\rho\|^2)^{l/2+z} \mathfrak{J}_{l/2+z}[(R - \|\rho\|^2)^{1/2} \|Z\|] J^{-1/2}(Z).$$

PROOF. – The proof is a straightforward consequence of (2.2) and (2.8).

Next we assume that G/K has rank one. Since

$$(s_R^z)^{\sim}(\lambda) = \mathcal{F}(\mathcal{Q} s_R^z)(\lambda) = (1 - (\|\rho\|^2 + \|\lambda\|^2)/R)_+^z ,$$

we have

$$(\mathcal{Q} s_R^z)(a_t) = c(l,z) R^{-z} (R - \|\rho\|^2)^{z+1/2} \mathfrak{J}_{z+1/2}((R - \|\rho\|^2)^{1/2} t) \tag{3.0}$$

where $c(l,z) = c(G,K,l) 2^z \Gamma(z+1)$.

LEMMA 3.2. – Let $k > 0$, $f_z(t) = \mathfrak{J}_{z+1/2}(k^{1/2} t)$. For every positive integer μ there exist constants c_j and functions ψ_j in $C^\infty(\boldsymbol{R}^+)$, $j = 1, \ldots, \mu$, such that

$$D_t^\mu f_z(t) = \sum_{j=1}^{\mu} k^j f_{z+j}(t) \psi_j(t) \qquad t \in \boldsymbol{R}_+ . \tag{3.1}$$

Moreover the functions ψ_j satisfy the estimates

$$|\psi_j(t)| \leqslant c_j (1+t)^j e^{-\mu t} \qquad t \in \boldsymbol{R}_+ . \tag{3.2}$$

PROOF. – Let $\Phi_1(t) = t/\sinh t$, $\Phi_r(t) = D_t \Phi_{r-1}(t)$, $r \in \boldsymbol{N}$. Using the identity $\mathfrak{J}'_\nu(t) = -t \mathfrak{J}_{\nu+1}(t)$ it is now easy to prove by induction on μ that (3.1) holds with functions ψ_j which are sums of products $\Phi_{r_1} \cdot \Phi_{r_2} \cdot \ldots \cdot \Phi_{r_j}$, where $\sum_{i=1}^{j} r_i = \mu$ and with coefficients which do not depend on z.

To prove the estimate (3.2) observe that $\Phi_r(t) = \Phi_0^{(2r+1)}(a_t)$ where $\Phi_0^{(2r+1)}$ is the spherical function on $H_{2r+1}(\boldsymbol{R})$ corresponding to the eigenvalue $-\|\rho\|^2$. Thus by Harish-Chandra's estimate [5] we have

$$|\Phi_r(t)| \leqslant c(r)(1+t) e^{-rt}, \qquad t > 0, \quad r \in \boldsymbol{N},$$

from which (3.2) follows at once.

REMARK. – It follows from the proof of the previous lemma that $\psi_\mu(t) = (t/\sinh t)^\mu$.

From now on $c(z)$, $c_k(z)$ will denote constants that grow at most exponentially in $\operatorname{Im} z$ when $\operatorname{Re} z$ is in a bounded subset of $[0, +\infty)$ and that may vary from line to line.

To prove the desired estimate of the asymptotic decay of the kernels s_R^z we shall make use of the following (elementary) facts.

A) By the integral representation of Poisson type of Bessel functions [3] we have

$$|\mathfrak{J}_z(t)| \leq c(z) \qquad t > 0. \tag{3.3}$$

Moreover by the asymptotic expansions for Hankel's functions [3] we have

$$|\mathfrak{J}_z(t)| \leq c(z) t^{-\operatorname{Re} z - 1/2}, \tag{3.4}$$

$$\mathfrak{J}_z(t) = t^{-z-1/2}\left(c_1(z) e^{i(t-\varphi)} + c_2(z) e^{-i(t-\varphi)} + R_z(t)\right), \tag{3.5}$$

where $|R_z(t)| \leq c(z) t^{-1}, t > 0$ and $\varphi = \left[z - \dfrac{1}{2}\right]\dfrac{\pi}{2}$.

B) For every $k > 0$ the function $t \to (1+t)(1+k^{-1/2}t)^{-1}$ is a monotone function on $[0, +\infty)$. It is increasing or decreasing according as $k < 1$ or $k > 1$. Thus

$$\min(1, k^{-1/2}) \leq (1+t)(1+k^{1/2}t)^{-1} \leq \max(1, k^{-1/2}), \tag{3.6}$$

C) The functions $u \to \cosh(\sqrt{u}))$ and $u \to \cosh^2(\sqrt{u})$ are convex on $[0, +\infty)$. Thus for every $x \geq y \geq 0$

$$\cosh x - \cosh y \geq (\sinh y/2y)(x^2 - y^2), \tag{3.7}$$

$$\cosh^2 x - \cosh^2 y \geq (\sinh y \ \cosh y/y)(x^2 - y^2). \tag{3.8}$$

PROPOSITION 3.3. – Let $R \geq \|\rho\|^2$, $l = \dim(G/K)$, $\operatorname{Re} z \geq 0$. Then for every $t \in \boldsymbol{R}$

$$|S_R^z(a_t)| \leq c(z) R^{l/2}(1 + R^{1/2}|t|)^{-\operatorname{Re} z - (l+1)/2}(t/\sinh t)^{(l-1)/2}(\cosh t)^{m_{2\alpha}/2}, \tag{3.9}$$

where $c(z)$ is a constant that grows at most exponentially in $\operatorname{Im} z$ when $\operatorname{Re} z$ is in a bounded subset of $[0, +\infty)$.

PROOF. – Let $m = l/2$, $\beta = \operatorname{Re} z$, $k = R - \|\rho\|^2$. Since $t \to S_R^z(a_t)$ is even, we can assume that $t \geq 0$. Suppose $k \geq 1$. If $d = 1$ and l is odd the proof of estimate (3.9) is very simple. From (2.5), (3.0), Lemma 3.2 and the estimates of Bessel functions we get

$$|S_R^z(a_t)| \leq c(z) R^{-\beta} k^{\beta+1/2} \sum_{j=1}^{m-1/2} k^j (1 + k^{1/2}|t|)^{-\beta-j-1}(1+|t|)^j e^{-(m-1/2)t}.$$

From (3.6), $k(1+|t|)(1+k^{1/2}|t|)^{-1}\geq 1$, so

$$\sum_{j=1}^{m-1/2}\left\{\frac{k(1+|t|)}{1+k^{1/2}|t|}\right\}^j \leq \left(m-\frac{1}{2}\right)\left\{\frac{k(1+|t|)}{1+k^{1/2}|t|}\right\}^{m-1/2}.$$

Next we consider the case $d=1$, l even. By the inversion formula (2.6) for the Abel transform and Lemma 3.2 we have

$$S_R^z(a_t)=c(z)R^{-z}\sum_{j=1}^{m} c_j k^{z+j+1/2}\times \tag{3.10}$$

$$\times\int_t^{+\infty}(\cosh x-\cosh t)^{-1/2}\mathfrak{J}_{z+j+1/2}(k^{1/2}x)\,\psi_j(x)\sinh x\,dx.$$

We decompose the integral over $[t,+\infty)$ into the sum $A_{R,j}^z(t)+B_{R,j}^z(t)$ where $A_{R,j}^z(t)$ is the integral over $[t,t+k^{-1/2}]$ and we estimate the two terms separately. We break the proof up into several lemmata.

LEMMA 3.4. – *Let* $k=R-\|\rho\|^2\geq 1$, $\beta=\operatorname{Re} z$. *Then*

$$|k^{z+j+1/2}A_{R,j}^z(t)|\leq c(z)R^{m+\beta}(1+R^{1/2}t)^{-\beta-m-1/2}(t/\sinh t)^{m-1/2},$$

for $1\leq j\leq m$.

PROOF. – Since $k\geq 1$, by (3.10), (3.7), the estimates for Bessel functions and the fact that the dominant term in the sum is that with $j=m$, we have

$$|k^jA_{R,j}^z(t)|\leq c(z)(t/\sinh t)^{1/2}\cdot\int_t^{t+k^{-1/2}} k^m(1+k^{1/2}x)^{-\beta-m-1}\left(\frac{x}{\sinh x}\right)^{m-1}(x^2-t^2)^{-1/2}x\,dx\leq$$

$$\leq c(z)\left(\frac{t}{\sinh t}\right)^{m-1/2}k^{m-1/2}(1+k^{1/2}t)^{-\beta-m-1/2}.$$

Since $(1+\|\rho\|^2)^{-1/2}R\leq k\leq R$ the conclusion follows.

LEMMA 3.5. – *Let* $k=R-\|\rho\|^2\geq 1$, $\beta=\operatorname{Re} z$. *Then*

$$|k^{z+j+1/2}B_{R,j}^z(t)|\leq c(z)R^{m+\beta}(1+R^{1/2}t)^{-\beta-m-1/2}(t/\sinh t)^{m-1/2}$$

for $1\leq j\leq m-1$.

PROOF. – By (3.2) and (3.4) we get

$$|k^{z+j+1/2}B_{R,j}^z(t)|\leq c(z)k^{(\beta+j)/2}[\cosh(t+k^{-1/2})-\cosh t]^{-1/2}\int_{t+k^{-1/2}}^{+\infty}x^{-\beta-m}\left(\frac{x}{\sinh x}\right)^{m-1}dx.$$

Hence by (3.7)

$$|k^{z+j+1/2}B^z_{R,j}(t)| \leqslant c(z)k^{\beta+(m+j+1)/2}(1+k^{1/2}t)^{-\beta-m-1/2}\left(\frac{t}{\sinh t}\right)^{1/2} \times$$

$$\times \int\limits_{t+k^{1/2}}^{+\infty} \left(\frac{x}{\sinh x}\right)^{m-1} dx \leqslant c(z)k^{\beta+m}(1+k^{1/2}t)^{-\beta-m-1/2}\left(\frac{t}{\sinh t}\right)^{m-1/2}.$$

Thus we only need to estimate $B^z_{R,m}$. This we shall do by integrating by parts, exploiting the oscillatory properties of Bessel functions.

LEMMA 3.6. – Let $k = R - \|\rho\|^2 \geqslant 1$, $\beta = \operatorname{Re} z$. Then

$$|k^{z+m+1/2}B^z_{R,m}(t)| \leqslant c(z)R^{m+\beta}(1+R^{1/2}t)^{-\beta-m-1/2}(t/\sinh t)^{m-1/2}.$$

PROOF. – By using the asymptotic expansion (3.5), we write $B^z_{R,m}(t) = p(t) + r(t)$, where

$$(3.11) \qquad p(t) = k^{-(z+m+1)/2}\int\limits_{t+k^{-1/2}}^{+\infty} (\cosh x - \cosh t)^{-1/2} x^{-z-1} \times$$

$$\times [c_1(z)e^{i(\sqrt{k}x-\varphi)} + c_2(z)e^{-i(\sqrt{k}x-\varphi)}](\sinh x)^{1-m}\, dx = p_1(t) + p_2(t)$$

and

$$(3.12) \qquad |r(t)| \leqslant c(z)k^{-(\beta+m+2)/2}\int\limits_{t+k^{-1/2}}^{+\infty} (\cosh x - \cosh t)^{-1/2} x^{-\beta-2}(\sinh x)^{1-m}\, dx.$$

The estimate of r is straightforward. By (3.7) we have

$$|r(t)| \leqslant c(z)k^{-(\beta+m+2)/2}\left(\frac{t}{\sinh t}\right)^{m-1/2}\int\limits_{t+k^{-1/2}}^{+\infty} x^{-\beta-m-1}(x^2-t^2)^{1/2}\, dx \leqslant$$

$$\leqslant c(z)k^{-1/2}(1+k^{1/2}t)^{-\beta-m-1/2}\left(\frac{1}{\sinh t}\right)^{m-1/2}.$$

Finally to estimate p it is enough to consider p_1. Let $H(x,t) = (\cosh x -$

$-\cosh t)^{-1/2} x^{-\beta-1} (\sinh x)^{1-m}$. By an integration by parts we have

$$p_1(t) = c_1(z) k^{-(z+m+1)/2} e^{-i\varphi}(-ik^{-1/2}) \Bigg[-e^{i(1+\sqrt{k}t)} (k^{-1/2}+t)^{-i\,\mathrm{Im}\,z} \times$$

$$\times H(t+k^{-1/2}, t) - \int_{t+k^{-1/2}}^{+\infty} e^{i\sqrt{k}x} \frac{\partial}{\partial x}(x^{-i\,\mathrm{Im}\,z} H(x,t))\, dx \Bigg].$$

Now

$$\left| \frac{\partial}{\partial x}(x^{-i\,\mathrm{Im}\,z} H(x,t)) \right| \leqslant |\mathrm{Im} z| x^{-1} H(x,t) - \frac{\partial}{\partial x}(H(x,t))$$

then

$$|p_1(t)| \leqslant c_1(z) k^{-(\beta+m+2)/2} \Bigg[2H(t+k^{-1/2}, t) + |\mathrm{Im},\, z| \int_{t+k^{-1/2}}^{+\infty} x^{-1} H(x,t)\, dx \Bigg].$$

The desired estimate follows at once.

The proof of Proposition 3.3 in the case $R \geqslant \|\rho\|^2 + 1$ is an immediate consequence of Lemma 3.4 and Lemma 3.6.

If $k \leqslant 1$, i.e. $\|\rho\|^2 \leqslant R \leqslant \|\rho\|^2 + 1$, then we must prove that

$$|S_R^z(a_t)| \leqslant c(z)(1+|t|)^{-\beta-m} \left(\frac{t}{\sinh t} \right)^{m-1/2}. \tag{3.13}$$

This estimate is obvious if $0 < t < 1$. If $t > 1$, it follows from (3.6) and the hypothesis $k \leqslant 1$ that

$$[k(1+y)/(1+k^{1/2}y)]^{\beta+j+1} \leqslant k^{(\beta+j+1)/2} \leqslant 1.$$

Then

$$|S_R^z(a_t)| \leqslant c(z)(\sinh t)^{-1/2}(1+t)^{-\beta-1} \int_t^{+\infty} (x-t)^{-1/2} e^{-(m-1)x}\, dx \leqslant c(z) e^{-(m-1/2)t}(1+t)^{-\beta-1}.$$

For $d = 2, 4, 8$ the proof is analogous. We simply use (3.8) instead of (3.7).

COROLLARY 3.7. – If G is complex or G/K has rank 1 there exist constants $\gamma, \delta > 0$ such that

$$|S_R^z(x)| \leqslant c(z) R^{l/2} (1 + R^{1/2}|x|)^{-\mathrm{Re}\,z - (l+1)/2} (1+|x|)^{\delta} e^{-\gamma|x|},$$

for all x in G/K. If G has rank 1, $\gamma = \|\rho\|$ and $\delta = (l-1)/2$.

In both cases $S_R^z \in L^q(G/K)$ for every q, $2 \leqslant q \leqslant +\infty$.

4. – Estimates for the maximal operator.

In this section we shall study the boundedness of the maximal operator

$$S_*^z f(x) = \sup_{R>0} |S_R^z f(x)| \qquad \text{for } f \text{ in } L^p(G/K), \qquad 1 \leq p \leq 2.$$

We begin with the L^2 estimates, using an idea introduced in [2].

LEMMA 4.1. – If $f \in L^2(G/K)$ we have, for fixed z with $\operatorname{Re} z > 0$,

$$\|S_*^z f\|_2 \leq c(z) \|f\|_2 .$$

PROOF. – Let $w_R, R > 0$, be the heat kernel on G/K, i.e. $e^{R\Delta} f = f * w_R$, $f \in L^p(G/K)$. Since the heat maximal operator $f \to W_* f = \sup\{|f * w_R| : R > 0\}$ is bounded on $L^2(G/K)$, it suffices to prove the boundedness of $(S^z - W)_*$. By using the spectral theorem for Δ and the Mellin transform we can write

$$(S_R^z - W_R) f = \int_{\boldsymbol{R}} R^{-i\gamma} c(z, \gamma)(-\Delta)^{i\gamma} f \, d\gamma . \tag{4.1}$$

for f in L^2 where

$$c(z, \gamma) = [\Gamma(z+1)\Gamma(-i\gamma)\Gamma^{-1}(z+1-i\gamma) - \Gamma(-i\gamma)]/2\pi .$$

Hence $|c(z, \gamma)| \leq c(z)(1 + |\gamma|)^{-(\operatorname{Re} z + 1)}$ and the integral in (4.1) converges. Since $L^2(G/H)$ is a complete Banach lattice we can write

$$(S^z - W)_* f = \sup_{R>0} |(S_R^z - W_R) f| \leq c(z) \int_{\boldsymbol{R}} (1 + |\gamma|)^{-(\operatorname{Re} z + 1)} |(-\Delta)^{i\gamma} f| \, d\gamma .$$

Thus

$$\|(S^z - W)_* f\|_2 \leq c(z) \|f\|_2$$

since the operators $(-\Delta)^{i\gamma}$, $\gamma \in \boldsymbol{R}$, are isometries on $L^2(G/K)$.

Next we turn to the L^p estimates for p close to 1. For every p, q, $1 \leq p, q \leq \infty$, we denote by $(L^p + L^q)(G/K)$ the Banach space of all functions f on G/K which admit a decomposition $f = g + h$ with $g \in L^p$ and $h \in L^q$. The norm of f is given by

$$\|f\|_{p+q} = \inf\{\|g\|_p + \|h\|_q : f = g + h\} .$$

The space (weak-$L^1 + L^q$)(G/K) is the space of all functions $f = g + h$ with $g \in$ weak-L^1 and $h \in L^q$.

LEMMA 4.2. – Let $\operatorname{Re} z > (l-1)/2$. There exists $q_0 \geq 2$ such that if $1 < p \leq q_0'$ then S_*^z maps $L^p(G/K)$ continuously into $(L^p + L^r)(G/K)$ for every $r \in [q_0 p'/(p' - q_0), +\infty]$. Moreover S_*^z maps $L^1(G/K)$ continuously into (weak-$L^1 + L^r$)(G/K) for every

$r \in [q_0, +\infty]$. The norm of S_*^z grows at most polynomially in $\operatorname{Im} z$ if $\operatorname{Re} z$ is in a bounded subset of $((l-1)/2, +\infty)$.

The constant q_0 depends only on G/K.

PROOF. – Let $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be the function $\varphi(t) = (1+t)^\delta (t/2)^{-l} e^{-\gamma t/2}$ where γ, δ are as in Corollary 3.7. Consider the maximal operator on functions on G/K

$$\mathcal{M}_\varphi f(x) = \sup_{t>0} \varphi(t) \int_{B(x,t)} |f(y)|\, dy.$$

We shall prove that if $\operatorname{Re} z > (l-1)/2$ then

$$|S_*^z f(x)| \leq c(z)\, \mathcal{M}_\varphi f(x), \tag{4.3}$$

for every $f \in C_c(G/K)$ and x in G/K. Since both S_*^z and $\mathcal{M}_\varphi$ commute with the action of G it is enough to prove (4.3) at the origin. Now by Corollary 3.7

$$|S_R^z f(0)| \leq \sum_{\nu=-\infty}^{+\infty} \int_{2^\nu \leq |x| < 2^{\nu+1}} |f(y)||s_R^z(y)|\, dy \leq$$

$$\leq c(z) \sum_{\nu=-\infty}^{+\infty} (R^{1/2} 2^\nu)^l (1 + R^{1/2} 2^\nu)^{-\operatorname{Re} z - (l+1)/2} \varphi(2^{\nu+1}) \int_{|x| \leq 2^{\nu+1}} |f(y)|\, dy \leq c(z) \mathcal{M}_\varphi f(0),$$

since for $\operatorname{Re} z > (l-1)/2$ the series converges and is a bounded function of R.

Now we can write $\mathcal{M}_\varphi f(x) \leq \mathcal{N}_\varphi f(x) + |f| * k(x)$, where $\mathcal{N}_\varphi f$ has the same definition as $\mathcal{M}_\varphi f$, but the sup is taken only for $t \in [0, 1]$, k is the function given by $k(x) = \min(c, \varphi(|x|))$ and c is some positive constant. Since $\varphi(t) \sim \text{const}|B(0,t)|$ for $t \to 0^+$, a classical covering lemma argument shows that the maximal operator $\mathcal{N}_\varphi$ is weak type 1-1 and is bounded on $L^p(G/K)$ for every $1 < p \leq \infty$. On the other hand the function k is in $L^q(G/K)$ for every $q \geq q_0$, for some $q_0 \geq 2$, which depends only on G/K. Thus the operator $f \to |f| * k$ maps $L^p(G/K)$ for $1 \leq p \leq q_0'$ continuously into $L^r(G/K)$ for every $r \in [q_0 p'/(p' - q_0), +\infty]$. This proves the lemma.

THEOREM 4.3. – Let $1 \leq p \leq 2$. If $\operatorname{Re} z > (2/p - 1)(l-1)/2$ then for every $r \geq p q_0/(2 - p + p q_0 - q_0)$

$$\|S_*^z f\|_{p+r} \leq c(z) \|f\|_p,$$

for every $f \in L^p(G/K)$.

PROOF. – We use Stein's complex interpolation theorem [13], interpolating between the L^p result for p close to 1 (Lemma 4.2) and the L^2 result (Lemma 4.1). Notice that Stein's theorem extends to the setting of the spaces $L^p + L^q$ with almost the same proof.

COROLLARY 4.4. – Let $1 \leq p \leq 2$. If $\operatorname{Re} z > (2/p - 1)(l - 1)/2$ then

$$\lim_{R \to +\infty} S_R^z f(x) = f(x) \qquad \text{a.e.} \qquad \text{for } f \in L^p .$$

PROOF. – The proof is a straightforward consequence of Lemma 4.2 for $p = 1$ and of Theorem 4.3 for $1 < p \leq 2$.

REFERENCES

[1] J. L. CLERC, *Sommes de Riesz et multiplicateurs sur un groupe de Lie compact*, Ann. Inst. Fourier, **24** (1974), pp. 149-172.

[2] M. G. COWLING, *Harmonic analysis on semigroups*, Ann. of Math., **117** (1983), pp. 267-283.

[3] A. ERDÉLYI - M. MAGNUS - F. OBERHETTINGER - F. G. TRICOMI, *Higher trascendental functions*, Vol. II, McGraw-Hill, New York, 1953.

[4] M. FLENSTED-JENSEN, *Spherical functions on a real semisimple Lie group. A method of reduction to the complex case*, J. Funct. Anal., **30** (1978), pp. 106-146.

[5] S. HELGASON, *Lie groups and geometric analysis*, Academic Press (1984).

[6] L. HÖRMANDER, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, in *Some Recent Advances in the Basic Sciences*, Yeshiva University, New York, 1966.

[7] A. HULANICKI - J. JENKINS, *Almost everywhere summability on nilmanifolds*, Trans. Amer. Math. Soc., **278** (1983), pp. 703-715.

[8] N. LOHOUÉ - TH. RYCHENER, *Die Risolvent von Δ auf Symmetrischen Raeumen von Nichtkompakten Typ*, Comment. Math. Helv., **57** (1982), pp. 445-468.

[9] G. MAUCERI, *Riesz means for the eigenfunction expansions for a class of hypoelliptic differential operators*, Ann. Inst. Fourier, **31** (1981), pp. 115-140.

[10] G. MAUCERI, *Maximal operators and Riesz means on stratified groups*, Sympos. Math., **29** (1984), pp. 47-62.

[11] C. D. SOGGE, *On the convergence of Riesz means on compact manifolds*, Ann. of Math., **126** (1987), pp. 439-447.

[12] R. STANTON, *Mean convergence of Fourier series on compact Lie groups*, Trans. Amer. Math. Soc., **217** (1976), pp. 61-87.

[13] E. M. STEIN - G. WEISS, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton University Press, Princeton N.J., 1971.