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Structure of Equilibria in N-person Non-cooperative Games

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Abstract: Here we study the structure of Nash equilibrium points for N-person games. For two-person games we observe that exchangeability and convexity of the set of equilibrium points are synonymous. This is shown to be false even for three-person games. For completely mixed games we get the necessary inequality constraints on the number of pure strategies for the players. Whereas the equilibrium point is unique for completely mixed two-person games, we show that it is not true for three-person completely mixed game without some side conditions such as convexity on the equilibrium set. It is a curious fact that for the special three-person completely mixed game with two pure strategies for each player, the equilibrium point is unique; the proof of this involves some combinatorial arguments.

1. Introduction

NASH [1951] developed the theory of non-cooperative N-person games by introducing the concept of equilibrium points. He showed that every non-cooperative finite N-person game has at least one equilibrium point in mixed strategies. When $N = 2$ and the game zero-sum, this reduces to the well-known minimax theorem or VON NEUMANN. For the zero-sum two-person games KAPLANSKY [1945] introduced the notion of completely mixed strategies and showed that in games where both players have only completely mixed optimal strategies, the payoff matrix is a square and each player has a unique optimal strategy. RAGHAVAN [1970] extended this result to the non-zero-sum bimatrix games.

In this paper we try to see, how far the results in two-person games extend to general N-person games. In two-person zero-sum games, the optimal strategies are exchangeable. We observe that the convexity of the set of equilibrium points implies the exchangeability of equilibrium points in bimatrix games.

Convexity is no more adequate for the exchangeability in 3-person games. It is well-known that in zero-sum as well as in non-zero-sum two-person games, if one player has more pure strategies than the other, then the game is not completely mixed. In a 3-person game, if a player has more strategies compared to the other two players, the game may be completely mixed. In fact we give an example to substantiate this statement. If a 3-person game is completely mixed then the sum of the number of pure strategies for the three players is greater than

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twice the number of pure strategies for any player. In a completely mixed N-person game, the equilibrium set may contain more than one point. However, if the set is convex, there is only one equlibrium point. It is a curious fact to note that for 3-person completely mixed games with two pure strategies for each player, the equilibrium point is unique. We are unable to say anything about the cardinality of the equilibrium set for completely mixed N-person games in general.

2. Exchangeability

Let $I_1, I_2, \ldots, I_N$ be N finite sets and $X_1, X_2, \ldots, X_N$ be the sets of probability vectors on $I_1, I_2, \ldots, I_N$ respectively. An element of I_j is called a pure strategy for player $j = 1,2,\ldots,N$ and an element of X_j is called a mixed strategy for player j. Let $K_1, K_2, \ldots, K_N$ be N real valued functions on $I_1 \times I_2 \times \cdots \times I_N$. If $i_1, \ldots, i_N$ are the pure strategy choices of the N-players then player j receives an income equal to $K_j(i_1, i_2, \ldots, i_N)$. We again denote by $K_j(x_1, x_2, \ldots, x_N)$ the expected income to the j^{th} player when $x_1, x_2, \ldots, x_n$ are the mixed strategy choices of the N-players.

A point $(x_1^0, x_2^0, \ldots, x_N^0) \in X_1 \times X_2 \times \cdots \times X_N$ is called a NASH equilibrium point if

$$K_j(x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0, x_{j+1}^0, \ldots, x_N^0)$$
$$\geq K_j(x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j, x_{j+1}^0, \ldots, x_N^0)$$

for all $x_j \in X_j$ and $j = 1,2,\ldots,N$.

For a proof of the existence of a NASH equilibrium point see NASH [1951] or PARTHASARATHY and RAGHAVAN [1971].

Let $\varepsilon \subseteq X_1 \times X_2 \times \cdots \times X_N$ be the set of all equilibrium points. We call ε exchangeable if for $(x_1^0, x_2^0, \ldots x_N^0) \in \varepsilon$, $(x_1^1, x_2^1, \ldots, x_N^1) \in \varepsilon$, we have $(x_1^{i_1}, \ldots, x_N^{i_N})$ $(x_1^{i_2}, \ldots, x_N^{i_N}) \in \varepsilon$ where $i_j = 0$ or 1 for all j.

When ε is exchangeable ε is always convex. For $N = 2$ we have the following.

Theorem 1:

In a two person game ε is exchangeable if ε is convex.

Proof:

We will prove ε is exchangeable if ε is convex. We will take X and Y as the mixed strategy sets for the players I and II. Let (x^0, y^0) and (x', y') be in ε. By assumption for $0 \leq \lambda \leq 1$, $(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y') \in \varepsilon$. That is

$$K_1(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y') \geq K_1(x, \lambda y^0 + (1-\lambda)y') \quad \text{for all} \quad x \in X.$$
$$K_2(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y') \geq K_2(\lambda x^0 + (1-\lambda)x', y) \quad \text{for all} \quad y \in Y.$$

By putting $x = x^0$ and x' in the first inequality we get

$$K_1(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y') \geq K_1(x^0, \lambda y^0 + (1-\lambda)y')$$
$$K_1(\lambda x^0 + (1-\lambda)x', \quad y^0 + (1-\lambda)y') \geq K_1(x', \lambda y^0 + (1-\lambda)y')$$

and hence

$$K_1(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y') = K_1(x^0, \lambda y^0 + (1-\lambda)y')$$
$$= K_1(x', \lambda y^0 + (1-\lambda)y')$$

for all $0 \leq \lambda \leq 1$. Putting $\lambda = 0$ and $\lambda = 1$ we get

$$K_1(x^0, y') = K_1(x', y') \quad \text{and} \quad K_1(x^0, y^0) = K_1(x', y^0).$$

Similarly we get $K_2(x^0, y^0) = K_2(x^0, y')$ and $K_2(x', y^0) = K_2(x', y')$. This implies (x^0, y') and (x', y^0) are in ε. This completes the proof of theorem 1.

Remark:

Convexity alone is not sufficient for the exchangeability of equilibrium points in $N(\geq 3)$ person games. We construct below a 3-person game with ε convex but not exchangeable.

Example:

Consider a three-person game in which each player has two pure strategies. We will define the payoffs K_1, K_2, K_3 in such a way that ε is precisely the line segment joining $\{(1,0),(1,0),(1,0)\}$ and $\{(0,1),(0,1),(0,1)\}$. Here (1,0) for any player stands for choosing the first pure strategy with probability one and (0,1) means choosing the second pure strategy with probability one. In other words we will have ε to be equal to

$$\varepsilon = \{(\lambda, 1-\lambda), (\lambda, 1-\lambda), (\lambda, 1-\lambda) : 0 \leq \lambda \leq 1\}.$$

Let $K_r(i,j,k)$ stand for the payoff to the r^{th} player when i,j,k are the pure strategies chosen by players 1,2, and 3 respectively.

Now we define

$$a_1 = K_1(1,1,1) - K_1(2,1,1),\ b_1 = K_2(1,1,1) - K_2(1,2,1),\ c_1 = K_3(1,1,1) - K_3(1,1,2)$$
$$a_2 = K_1(1,1,2) - K_1(2,1,2),\ b_2 = K_2(1,1,2) - K_2(1,2,2),\ c_2 = K_3(1,2,1) - K_3(1,2,2)$$
$$a_3 = K_1(1,2,1) - K_1(2,2,1),\ b_3 = K_2(2,1,1) - K_2(2,2,1),\ c_3 = K_3(2,1,1) - K_3(2,1,2)$$
$$a_4 = K_1(1,2,2) - K_1(2,2,2),\ b_4 = K_2(2,1,2) - K_2(2,2,2),\ c_4 = K_3(2,2,1) - K_3(2,2,2)$$

Now we define the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

We will show that the equilibrium points are of the required type. For simplicity we will write (p^0, q^0, r^0) instead of $(p^0, 1-p^0), (q^0, 1-q^0), (r^0, 1-r^0)$. If $(p^0, q^0, r^0) \in \varepsilon$ we have

$$K_1(p^0, q^0, r^0) \geq K_1(p, q^0, r^0)$$

That is $p^0 K_1(1,q^0,r^0) + (1-p^0)K_1(2,q^0,r^0) \geq pK_1(1,q^0,r^0) + (1-p)K_1(2,q^0,r^0)$. Here $K_1(2,q^0,r^0)$ for example refers to the expected income to player 1 if he uses the second strategy, player 2 uses the mixed strategy $(q^0, 1-q^0)$ and player 3 uses the mixed strategy $(r^0, 1-r^0)$.

We get from the above inequality

$$(p^0 - p)((K_1(1,q^0,r^0) - K_1(2,q^0,r^0)) \geq 0.$$

Namely $(p^0 - p)\,(q^0 r^0 a_1 + q^0(1-r^0)a_2 + (1-q^0)r^0 a_3 + (1-q^0)(1-r^0)a_4) \geq 0.$

In our example this reduces to

$$(p^0 - p)(q^0 - r^0) \geq 0 \quad \text{for all} \quad 0 \leq p \leq 1 \quad \longrightarrow \ 1$$

Similarly by considering the K_2 and K_3 functions we get

$$(q^0 - q)(r^0 - p^0) \geq 0 \quad \text{for all} \quad 0 \leq q \leq 1 \quad \longrightarrow \ 2$$

$$(r^0 - r)(p^0 - q^0) \geq 0 \quad \text{for all} \quad 0 \leq r \leq 1 \quad \longrightarrow \ 3.$$

From inequality (1) we have either (a) $q^0 = r^0$ or (b) $q^0 - r^0 > 0$ and $p^0 = 1$ or (c) $q^0 - r^0 < 0$. Case (b) and (c) are not possible if they have to satisfy (2) and (3). Thus $q^0 = r^0$. Similarly from (2) we infer $p^0 = r^0$. Thus $p^0 = q^0 = r^0$. Thus any equilibrium point is of the required type. Also it is clear that any $p^* = q^* = r^*$ where $0 \leq p^* \leq 1$ satisfy the three inequalities and that $(p^*,q^*,r^*) \in \varepsilon$. Clearly $(p=1, q=0, r=0)$ is not in ε though $(p=0, q=0, r=0)$ and $(p=1, q=1, r=1)$ are in ε. In other words ε is convex but not exchangeable.

3. Completely Mixed Games

Let $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, l$ be the set of pure strategies for players 1, 2, and 3. Let $K_r(i,j,k)$, $r = 1,2,3$ be the payoff to the r^{th} player when i,j,k are their strategy choices. Let x,y,z denote mixed strategies for 1,2, and 3. A mixed strategy x is called completely mixed if each co-ordinate of x is positive. As before we use $K_r(x,y,z)$ to denote the expected income to the r^{th} player when x,y,z are used by the respective players. For an equilibrium point (x^0,y^0,z^0) let A,B be $n \times m$ and $l \times m$ matrices defined by

$$a_{ji} = K_2(i,j,z^0) \quad j = 1,2,\ldots,n, \quad i = 1,2,\ldots,m.$$

$$b_{ki} = K_3(i,y^0,k) \quad k = 1,2,\ldots,l, \quad i = 1,2,\ldots,m.$$

We define $s(y^0,z^0) = \{x' : (x',y^0,z^0) \in \varepsilon\}$. We say $s(y^0,z^0) > 0$ if every x' in it is completely mixed.

Theorem 2:

Let $(x^0,y^0,z^0) \in \varepsilon$ and $v_1 = v_2 = v_3 = 0$ where $v_r = K_r(x^0,y^0,z^0)$, $r = 1,2,3$. Let $s(y^0,z^0)$ be completely mixed. Then the rank of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is m or $m-1$. In case it is $m-1$, x^0 is the only element in $s(y^0,z^0)$.

Proof:

Clearly the rank p of $\begin{bmatrix} A \\ B \end{bmatrix} \leq m$. If possible $p \leq m - 2$. Then there exists a Π independent of x^0 such that $A\Pi = 0$, $B\Pi = 0$. Since $(Ax^0)_j = K_2(x^0, j, z^0) \leq v_2 = 0$ for all j w have $(Ax^0) \leq 0$. (Here we mean each coordinate of (Ax^0) is less than or equal to 0). Similarly $Bx^0 \leq 0$.

As in KAPLANSKY's proof [1945] consider $x^* = x^0 - \lambda\Pi$ in case $\Sigma\Pi_i = 0$ or $x^* = (1+\lambda)x^0 - \lambda\Pi$ in case $\Sigma\Pi_i = 1$ (we can assume this when $\Sigma\Pi_i \neq 0$). For suitable $\lambda > 0$, x^* is a mixed strategy but not completely mixed. Thus $K_2(x^*, j, z^0) = K_2(x^0 - \lambda\Pi, j, z^0)$ or $K_2((1+\lambda)x^0 - \lambda\Pi, j, z^0)$ and that $K_2(x^*, j, z^0) \leq 0$. Similarly $K_3(x^*, y^0, k) \leq 0$. Also we have $K_2(x^*, y^0, z^0) = K_3(x^*, y^0, z^0) = 0$. But $K_1(i, y^0, z^0) = 0$ for all i as x^0 is completely mixed. Thus $(x^*, y^0, z^0) \in \varepsilon$ and $s(y^0, z^0)$ is not completely mixed. This contradicts our assumption regarding $s(y^0, z^0)$.

Hence $p = m - 1$ or m. Lastly in case it is $m - 1$, $A\Pi = 0$, $B\Pi = 0$ has a solution Π. We claim that Π is a multiple of x^0; for otherwise if Π and x^0 are independent we can repeat the above proof verbatim and contradict our assumption on $s(y^0, z^0)$. Hence the last assertion in the theorem.

Theorem 3:

Let $m \geq n + l$. Then given any $(x^0, y^0, z^0) \in \varepsilon$ there is always an (x^*, y^0, z^0) in ε such that x^* is not completely mixed.

Proof:

In case x^0 is not completely mixed we have nothing to prove. Let x^0 be completely mixed. Without loss of generality $K_r(x^0, y^0, z^0) = v_r = 0$, $r = 1,2,3$. Consider the matrix

$$C = \begin{bmatrix} A & -1 & 0 \\ B & 0 & -1 \end{bmatrix}$$

where 1 denotes an appropriate column vector with all entries unity. The rank of C is at most $n + l \leq m$. We have at least 2 linearly independent solutions to

$$\begin{bmatrix} A & -1 & 0 \\ & & \\ B & 0 & -1 \end{bmatrix} \begin{bmatrix} \Pi \\ \alpha \\ \beta \end{bmatrix} = 0$$

and clearly one of them is independent of the vector $(x^0, 0, 0)$. Let (Π, α, β) be that solution. Clearly $\Pi \neq 0$ for otherwise $\alpha = \beta = 0$ and that (Π, α, β) is the trivial vector. We get $A\Pi = \alpha \cdot 1$, $B\Pi = \beta \cdot 1$.

Since $Ax^0 \leq 0$ with equality attained for some co-ordinate and $Bx^0 \leq 0$ with equality attained for some co-ordinate x^0 and Π are independent if α or $\beta \neq 0$. If $\alpha = \beta = 0$, by our very choice x^0 and Π are independent. As in theorem 2 we can construct an x^* such that x^* is not completely mixed and $(x^*, y^0, z^0) \in \varepsilon$. This completes the proof of theorem 3.

Corollary:

If a three person game is completely mixed we should have $m + n + l > \max(2m, 2n, 2l)$.

Proof:

This follows easily from the previous theorem.

Remark:

The inequality condition in theorem 3 cannot be weakened as the following example shows.
Let $m = 3, n = 2, l = 2$. Let $K_1(i,j,k) \equiv$ constant.
We define

$$K_2(i,j,1) = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad K_2(i,j,2) = \begin{bmatrix} 0 & 0 & -3 \\ -2 & -1 & 0 \end{bmatrix}.$$

Here i refers to column and k refers to row.

$$K_3(i,1,k) = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}, \quad K_3(i,2,k) = \begin{bmatrix} 0 & -2 & -1 \\ 1 & -2 & -2 \end{bmatrix}.$$

Here i refers to column and k refers to row.

We have $x^0 = (1/3, 1/3, 1/3)$, $y^0 = (1/2, 1/2)$, $z^0 = (1/2, 1/2)$ as an equilibrium point and x^0 is the only element in $s(y^0, z^0)$. Note that $m = n + l - 1$ and $s(y^0, z^0)$ is completely mixed. Of course the game is not completely mixed, for the strategy $(1,1,2)$ is a pure equilibrium point. We will later construct an example with $m = 3, n = 2, l = 2$ where all the equilibrium points are completely mixed. We see that the assertion in theorem 3 is sharp. However with some added condition we can prove the following.

Theorem 4:

Let in a three person game $m = n + l - 1$ and let $(x^0, y^0, z^0) \in \varepsilon$. If y^0 or z^0 is not completely mixed, then $s(y^0, z^0)$ is not completely mixed.

Proof:

As before we assume $K_r(x^0, y^0, z^0) = v_r = 0$, $r = 1,2,3$. Suppose, say z_l^0, the last co-ordinate of z^0 is zero. Consider

$$C = \begin{bmatrix} A & -1 & 0 \\ B_1 & 0 & -1 \end{bmatrix}$$

where B_1 is the submatrix of B with the last row omitted. Since C has $n + l - 1$ rows and $n + l + 1$ columns, there is a non-trivial solution to

$$\begin{bmatrix} A & -1 & 0 \\ B_1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \Pi \\ \alpha \\ \beta \end{bmatrix} = 0$$

which is linearly independent of $(x^0,0,0)$. Now $\Pi \neq 0$ for otherwise (Π,α,β) is a trivial vector. If $\alpha \neq 0$ and since $K_2(x^0,j,z^0) = (Ax^0)_j = 0$ for some j, x^0 and Π are independent. Suppose $\alpha = 0$ and $\beta = 0$ then evidently by our choice of (Π,α,β), Π is independent of x^0. Lastly let $\alpha = 0$, but $\beta \neq 0$. Since $z_k^0 > 0$ for some $k < l$ we have $v_3 = k_3(x^0,y^0,k) = 0$ and that $(B_1x^0)_k = 0$ for that k. Therefore x^0 and Π are independent.

Let b_l be the last row of B. We have $(b_l\Pi) < \beta$ or $(b_l\Pi) \geq \beta$. Let $x^* = x^0 - \lambda\Pi$, $\hat{x} = x^0 + \lambda\Pi$, $x' = (1+\lambda)x - \lambda\Pi, \tilde{x} = (1-\lambda)x^0 + \lambda\Pi$. We will choose one of them depending on $\Sigma\Pi_i = 0$ or 1 and $(B_l\Pi) < \beta$ or $(b_l\Pi) \geq \beta$. If $\Sigma\Pi_i = 0$, for suitable positive λ's $x^*,\hat{x}$ are both mixed strategies but not completely mixed. If $\Sigma\Pi_i = 1$ (without loss of generality when $\Sigma\Pi_i \neq 0$), similar statement can be made on x' and $\hat{x}$.

If $\Sigma\Pi_i = 0, b_l\Pi < \beta$, choose $\hat{x}$; if $\Sigma\Pi_i = 0, b_l\Pi \geq \beta$ choose x^*.

If $\Sigma\Pi_i = 1, b_l\Pi < \beta$, choose $\tilde{x}$; if $\Sigma\Pi_i = 1, b_l\Pi \geq \beta$ choose x'.

Depending on these cases we have

$$Ax^* \leq -\lambda\alpha,\ A\hat{x} \leq \lambda\alpha,\ Ax' \leq -\lambda\alpha,\ A\tilde{x} \leq \lambda\alpha$$

$$B_1x^* \leq -\lambda\beta,\ B_1\hat{x} \leq \lambda\beta,\ B_1x' \leq -\lambda\beta,\ B_1\tilde{x} \leq \lambda\beta$$

$$b_lx^* \leq -\lambda\beta,\ b_l\hat{x} \leq \lambda\beta,\ b_lx' \leq -\lambda\beta,\ b_l\tilde{x} \leq \lambda\beta\ .$$

This shows, say for the case $K_2(x^*,j,z^0) \leq -\lambda\alpha$ for all j, there is a j^0 with $K_2(x^*,j^0,x^0) = -\lambda\alpha$. In fact when $y_j^0 > 0$ the equality is attained. Similarly $K_3(x^*,y^0,k) \leq -\lambda\beta$ for all k and for some $k^0 < l$, $K_3(x^*,y^0,k^0) = -\lambda\beta$. This shows that $(x^*,y^0,z^0) \in \varepsilon$. Similarly the other cases can be disposed of. Hence $s(y^0,z^0)$ is not completely mixed.

Remark:

When $m = n + l - 2$ and when player 2 or 3 misses a pure strategy in an equilibrium (x^0,y^0,z^0), we cannot conclude $s(y^0,z^0)$ is not completely mixed. We have simple examples even when $m = n = l = 2$.

4. Non-uniqueness of Equilibrium Points in Completely Mixed Games

We call a game completely mixed if every equilibrium is completely mixed. In otherwords no player can be in equilibrium with the other player if he misses a pure strategy. In two-person zero-sum as well as non-zero-sum completely mixed games, the equilibrium set has just one point in it. For games with $N \geq 3$ this is no more true. We construct below a 3-person completely mixed game where ε consists of precisely two points. This demonstrates that without further conditions, we cannot assert the uniqueness in general.

We introduce the following notations. Player 1 has 3 pure strategies. Player 2 and 3 have two pure strategies each.

Let $K_r(i,j,k)$ denote the payoff to the r^{th} player when i,j,k are the pure strategy choices of 1, 2 and 3 respectively. We define

$$K_1(1,1,1) - K_1(2,1,1) = a_1,\ K_1(2,1,1) - K_1(3,1,1) = a'_1$$
$$K_1(1,1,2) - K_1(2,1,2) = a_2,\ K_1(2,1,2) - K_1(3,1,2) = a'_2$$
$$K_1(1,2,1) - K_1(2,2,1) = a_3,\ K_1(2,2,1) - K_1(3,2,1) = a'_3$$
$$K_1(1,2,2) - K_1(2,2,2) = a_4,\ K_1(2,2,2) - K_1(3,2,2) = a'_4$$

We define $a''_i = -(a_i + a'_i)$ for $i = 1,2,3,4$.
Let

$$K_2(1,1,1) - K_2(1,2,1) = b_1,\ K_3(1,1,1) - K_3(1,1,2) = c_1$$
$$K_2(1,1,2) - K_2(1,2,2) = b_2,\ K_3(1,2,1) - K_3(1,2,2) = c_2$$
$$K_2(2,1,1) - K_2(2,2,1) = b_3,\ K_3(2,1,1) - K_3(2,1,2) = c_3$$
$$K_2(2,1,2) - K_2(2,2,2) = b_4,\ K_3(2,2,1) - K_3(2,2,2) = c_4$$
$$K_2(3,1,1) - K_2(3,2,1) = b_5,\ K_3(3,1,1) - K_3(3,1,2) = c_5$$
$$K_2(3,1,2) - K_2(3,2,2) = b_6,\ K_3(3,2,1) - K_3(3,2,2) = c_6$$

In fact we will choose these constants such that ε contains precisely the following two elements

$$\{(1/3,\ 1/3,\ 1/3), (1/2,\ 1/2), (1/2,\ 1/2)\}$$

and

$$\{(1/3,\ 1/3,\ 1/3), (1/3,\ 2/3), (1/3,\ 2/3)\}\,.$$

Consider the following matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & & \\ a'_1 & a'_2 & a'_3 & a'_4 & & \\ a''_1 & a''_2 & a''_3 & a''_4 & & \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \quad \begin{bmatrix} 2 & -3.5 & .5 & 1 & & \\ 2 & -4 & 1 & 1 & & \\ -4 & 7.5 & -1.5 & -2 & & \\ -10 & 1 & 9 & -2 & 1 & 1 \\ -1 & -1 & -3 & -3 & 4 & 4 \end{bmatrix}$$

One can check that this game is completely mixed.

We will now establish that ε contains precisely

$\{(1/3,\ 1/3,\ 1/3), (1/2,\ 1/2), (1/2,\ 1/2)\}$ and $\{(1/3,\ 1/3,\ 1/3), (1/3,\ 2/3), (1/3,\ 2/3)\}$.

In order that $\{(p_1, p_2, 1 - p_1 - p_2), (q, 1 - q), (r, 1 - r)\}$ to be an equilibrium point in our example we should have

$$K_1(1,(q,1-q),(r,1-r)) = K_1(2,(q,1-q),(r,1-r)) = K_1(3,(q,1-q),(r,1-r))$$

i.e.,

$$6qr - 4.5q - .5r + 1 = 0$$
$$6qr - 5q + 1 = 0\,.$$

By eliminating q we have $6r^2 - 5r + 1 = 0$, i.e., $r = 1/2$ or $1/3$. Similarly $q = 1/2$ or $1/3$. When $q = 1/2$, we have $r = 1/2$ and when $q = 1/3$ we have $r = 1/3$. We will show in either case $p_1 = p_2 = 1/3$.

Consider

$$K_2((p_1,p_2,1-p_1-p_2),1,(1/2,1/2)) = K_2((p_1,p_2,1-p_1-p_2),2,(1/2,1/2))$$

$$K_3((p_1,p_2,1-p_1-p_2),(1/2,1/2),1) = K_3((p_1,p_2,1-p_1-p_2),(1/2,1/2),2)$$

i.e.,

$$-11p_1 + 5p_2 + 2 = 0$$
$$-10p_1 - 14p_2 + 8 = 0.$$

We get $p_1 = p_2 = (1 - p_1 - p_2) = 1/3$. Similarly, we settle the other case when $q = r = 1/3$. This completes the construction of the required example.

5. Uniqueness of the Equilibrium Point

Here we prove the following

Theorem 5:

Let in a three-person game the set of equilibrium points be completely mixed and convex. Then ε has just one element.

Proof:

As before let $K_r(x,y,z)$ be the expected payoff to player r when x,y,z are the mixed strategy choices of players 1, 2, and 3. Let, if possible, ε have two equilibriums (x^0,y^0,z^0) and (x',y',z'). By assumption $(\lambda x^0 + (1-\lambda)x', \lambda y^0 + (1-\lambda)y', \lambda z^0 + (1-\lambda)z') \in \varepsilon$ for $0 \le \lambda \le 1$. Since ε is completely mixed

$$K_1(i,\lambda y^0 + (1-\lambda)y', \lambda z^0 + (1-\lambda)z') \equiv c \quad \text{for all pure strategies } i$$

i.e.

$$\lambda^2 K_1(i,y^0,z^0) + (1-\lambda)^2 K_1(i,y',z') + \lambda(1-\lambda)\{K_1(i,y^0,z') + K_1(i,y',z^0)\} \equiv c.$$

Since

$$K_1(i,y^0,z^0) \equiv v_1^0, \quad \text{and} \quad K_1(i,y',z') \equiv v_1'$$

we have

$$K_1(i,y^0,z') + K_1(i,y',z^0) \equiv \alpha_1 \quad \text{say.}$$

Similarly

$$K_2(x^0,j,z') + K_2(x',j,z^0) \equiv \alpha_2$$

and

$$K_3(x^0,y',k) + K_3(x',y^0,k) \equiv \alpha_3.$$

Consider

$$x^* = (1+t)x^0 - tx', y^* = (1+t)y^0 - ty', z^* = (1+t)z^0 - tz'.$$

For $t > 0$ and small, x^*,y^*,z^* are all mixed strategies. Since $(x^0,y^0,z^0) \neq (x',y',z')$ there is a $t > 0$ for which x^*,y^*,z^* are mixed strategies but at least one of them is not completely mixed. Further we will show $(x^*, y^*, z^*) \in \varepsilon$. Consider

$$\begin{aligned} K_1(i,y^*,z^*) &= (1 + t)^2 K_1(i,y^0,z^0) + t^2 K_1(i,y',z') \\ &\quad - t(1 + t)(K_1(i,y^0,z') + K_1(i,y',z^0)) \\ &\equiv (1 + t^2)v_1^0 + t^2 v_1' - t(1 - t)\alpha_1 . \end{aligned}$$

Similarly we establish the identities for K_2 and K_3. Hence $(x^*,y^*,z^*) \in \varepsilon$. This contradicts the assumption that ε is completely mixed. Hence the proof.

Remark 1:

The proof only uses the existence of a line segment. The theorem says that ε when completely mixed cannot contain line segments.

Remark 2:

We have to go to $m = 3$, $n = 2$, $k = 2$ to construct a counter example for non-uniqueness of equilibrium points in completely mixed games; for when $m = n = k = 2$ we have the following curious

Theorem 6:

Let a three-person game have two pure strategies for each player. If the game is completely mixed, then the equilibrium point is unique.

We will start with some preliminaries for the proof of this theorem.

As before let

$$\begin{aligned} a_1 &= K_1(1,1,1) - K_1(2,1,1), & b_1 &= K_2(1,1,1) - K_2(1,2,1), & c_1 &= K_3(1,1,1) - K_3(1,1,2) \\ a_2 &= K_1(1,1,2) - K_1(2,1,2), & b_2 &= K_2(1,1.2) - K_2(1,2,2), & c_2 &= K_3(1,2,1) - K_3(1,2,2) \\ a_3 &= K_1(1,2,1) - K_1(2,2,1), & b_3 &= K_2(2,1,1) - K_2(2,2,1), & c_3 &= K_3(2,1,1) - K_3(2,1,2) \\ a_4 &= K_1(1,2,2) - K_1(2,2,2), & b_4 &= K_2(2,1,2) - K_2(2,2,2), & c_4 &= K_3(2,2,1) - K_3(2,2,2) \end{aligned}$$

Let $(p,(1 - p)),(q,(1 - q)),(r,(1 - r))$ be mixed strategies for the three players. For simplicity we denote the above choices of mixed strategies by (p,q,r). Let $(p^0,q^0,r^0) \in \varepsilon$ and $0 < p^0,q^0,r^0 < 1$. Then we have

$$\begin{aligned} q^0 r^0(a_1 - a_2 - a_3 + a_4) + q^0(a_2 - a_4) + r^0(a_3 - a_4) + a_4 &= 0 \\ p^0 r^0(b_1 - b_2 - b_3 - b_4) + p^0(b_2 - b_4) + r^0(b_3 - b_4) + b_4 &= 0 \\ p^0 q^0(c_1 - c_2 - c_3 - c_4) + p^0(c_2 - c_4) + q^0(c_3 - c_4) + c_4 &= 0 \end{aligned}$$

We will write

$$\begin{aligned} &\lambda_1 = a_1 - a_2 - a_3 - a_4, \; \lambda_2 = a_2 - a_4, \; \lambda_3 = a_3 - a_4, \; \lambda_4 = a_4 \\ &\mu_1 = b_1 - b_2 - b_3 - b_4, \; \mu_2 = b_2 - b_4, \; \mu_3 = b_3 - b_4, \; \mu_4 = b_4 \\ &v_1 = c_1 - c_2 - c_3 - c_4, \; v_2 = c_2 - c_4, \; v_3 = c_3 - c_4, \; v_4 = c_4 \end{aligned}$$

Lemma 1:

When $m = n = k = 2$ and the game completely mixed with $(p^0,q^0,r^0) \in \varepsilon$, p^0 and q^0 are unique for r^0; namely if $(p^*,q^*,r^0) \in \varepsilon$ then $p^* = p^0$, $q^* = q^0$. Similar assertions hold for p^0 and q^0.

Proof:

Consider the equations

$$qr\lambda_1 + q\lambda_2 + r\lambda_3 + \lambda_4 = 0$$
$$pr\mu_1 + p\mu_2 + r\mu_3 + \mu_4 = 0$$
$$pq\nu_1 + p\nu_2 + q\nu_3 + \nu_4 = 0$$

(p^0,q^0,r^0) is a solution to them and any (p,q,r) with $0 \leq p,q,r \leq 1$ satisfying the above lies in ε. Clearly $r_0\lambda_1 + \lambda_2, r_0\mu_1 + \mu_2, q_0\nu_1 + \nu_2$ cannot all be zero; for then $(1,q^0,r^0)$ satisfies the above equations contradicting the fact that the game is completely mixed. Thus at least one of them is not 0. Say $r_0\lambda_1 + \lambda_2 \neq 0$. Then

$$q^0(r_0\lambda_1 + \lambda_2) + (r_0\lambda_3 + \lambda_4) = 0$$
$$q^*(r_0\lambda_1 + \lambda_2) + (r_0\lambda_3 + \lambda_4) = 0$$

i.e., $q^* = q^0$. We have $(p^*,q^0,r^0),(p^0,q^0,r^0) \in \varepsilon$ and $p^* = p^0$ for otherwise ε will contain the line segment joining (p^*,q^0,r^0) and (p^0,q^0,r^0). This contradicts that ε is completely mixed.

Similarly if $r^0\mu_1 + \mu_2 \neq 0$ we have $p^* = p^0, q^* = q^0$. We will show that $r^0\lambda_1 + \lambda_2 = 0$ and $r^0\mu_1 + \mu_2 = 0$ is not possible. Suppose so. We saw that in this case $q^0\nu_1 + \nu_2 \neq 0$. We have

$$p^0 = -\frac{(q^0\nu_3 + \nu_4)}{(q^0\nu_1 + \nu_2)} .$$

Consider

$$p = -\frac{(q\nu_3 + \nu_4)}{(q\nu_1 + \nu_2)} .$$

p is a continuously differentiable function in a neighborhood Δ of q^0 with $0 < p < 1$, when $q \in \Delta$.

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{(\nu_1\nu_4 - \nu_2\nu_3)}{(q\nu_1 + \nu_2)^2} \text{ for } q \in \Delta .$$

If $\nu_1\nu_4 - \nu_2\nu_3 = 0$, then $p \equiv p^0$ for $q \in \Delta$. That is, the line segment $\{(p^0,q,r^0) : q \in \Delta\} \subseteq \varepsilon$, which is clearly not possible.

If $\nu_1\nu_4 - \nu_2\nu_3 > 0$, in this case p is an increasing function of q in Δ. Consider $p_1 = \sup\{p : (p,q,r^0) \in \varepsilon\}$. Clearly there is a q_1 with $(p_1,q_1,r^0) \in \varepsilon$. Also $p_1 < 1, q_1 < 1$. Since $r^0\lambda_1 + \lambda_2 = r^0\mu_1 + \mu_2 = 0$, we know $q_1\nu_1 + \nu_2 \neq 0$. This shows that $\left.\frac{\mathrm{d}p}{\mathrm{d}q}\right|_{q_1} > 0$ and hence we have a $(p_2, q_2, r^0) \in \varepsilon$ with $p_2 > p_1$. This contradicts the maximality of p_1. Similar contradiction can be arrived at when $\nu_1\nu_4 - \nu_2\nu_3 < 0$. This completes the proof of the lemma.

Lemma 2:

Let ε be completely mixed and let

$$f(r) = (r\lambda_3 + \lambda_4)(r\mu_3 + \mu_4)v_1 - (r\lambda_3 + \lambda_4)(r\mu_1 + \mu_2)v_2$$
$$- (r\lambda_1 + \lambda_2)(r\mu_3 + \mu_4)v_3 + (r\lambda_1 + \lambda_2)(r\mu_1 + \mu_2)v_4 \equiv 0$$

for all r. Then ε has a unique point.

Proof:

Let if possible (p^0,q^0,r^0), (p^*,q^*,r^*) be two distinct elements in ε. By lemma 1, $p^0 \neq p^*, q^0 \neq q^*$ and $r^0 \neq r^*$. We will analyze different cases to establish the uniqueness.

Case (i):

$r^0\lambda_1 + \lambda_2 = 0$, $r^0\mu_1 + \mu_2 = 0$. Clearly $(p^*,q^*,r^0) \in \varepsilon$ and by lemma 1 $p^* = p^0$ and $q^* = q^0$ a contradiction to our assumption.

Case (ii):

$r^1\lambda_1 + \lambda_2 \neq 0, r^1\mu_1 + \mu_2 \neq 0$ for any $(p^1,q^1,r^1) \in \varepsilon$. Let $\varepsilon_3 = \{r^1 : (p^1,q^1,r^1) \in \varepsilon\}$. ε_3 is closed in $[0,1]$. We will prove ε_3 is open. Let $r^1 \in \varepsilon_3$. There is an open interval Δ_{r^1} such that $p(r) = -\dfrac{(r\mu_3 + \mu_4)}{r\mu_1 + \mu_2}$ and $q(r) = -\dfrac{(r\lambda_3 + \lambda_4)}{r\lambda_1 + \lambda_2}$ with $0 < p(r)$, $q(r) < 1$ for all $r \in \Delta_{r^1}$.

Also $(p(r),q(r),r)$ satisfies the equations (1) and (2) of lemma 1. Further by assumption $f(r) \equiv 0$ and that

$$(p(r)q(r)v_1 + p(r)v_2 + q(r)v_3 + v_4)(r\lambda_1 + \lambda_2)(r\ \mu_1 + \mu_2) \equiv 0$$

for all $r \in \Delta_{r^1}$. Also $r\lambda_1 + \lambda_2 \neq 0$, $r\mu_1 + \mu_2 \neq 0$. Thus $p(r), q(r)$ satisfies the third equation. Hence $(p(r),q(r),r) \in \varepsilon$ for all $r \in \Delta_{r^1}$, i.e., ε_3 is open. Hence $\varepsilon_3 = [0,1]$, as $\varepsilon_3 \neq \emptyset$. This contradicts ε is completely mixed.

Case (iii):

$$r^0\lambda_1 + \lambda_2 \neq 0 \quad r^0\mu_1 + \mu_2 = 0$$
$$r^*\lambda_1 + \lambda_2 \neq 0 \quad r^*\mu_1 + \mu_2 = 0$$

since $r^0 \neq r^*$, $\mu_1 = \mu_2 = 0$. Also $\mu_3 = \mu_4 = 0$. Such a game cannot be completely mixed.

Case (iv):

$$r^0\lambda_1 + \lambda_2 = 0 \quad r^0\mu_1 + \mu_2 \neq 0$$
$$r^*\lambda_1 + \lambda_2 = 0 \quad r^*\mu_1 + \mu_2 \neq 0.$$

This is similar to case (iii).

Case (v):

$$r^0\lambda_1 + \lambda_2 \neq 0 \quad r^0\mu_1 + \mu_2 = 0$$
$$r^*\lambda_1 + \lambda_2 = 0 \quad r^*\mu_1 + \mu_2 \neq 0.$$

Here $\mu_1 \neq 0$ for otherwise $\mu_2 = 0$ and this contradicts our assumption $r^*\mu_1 + \mu_2 \neq 0$. Similarly $\mu_2, \mu_3, \mu_4 \neq 0$ (observe $r^0\mu_3 + \mu_4 = 0$, $r^*\mu_3 + \mu_4 \neq 0$).

Further $(\mu_1 \mu_2), (\mu_3, \mu_4)$ are linearly dependent as $r^0 \mu_1 + \mu_2 = 0$ and $r^0 \mu_3 + \mu_4 = 0$. Therefore $(\mu_3, \mu_4) = \alpha(\mu_1, \mu_2)$ for some $\alpha \neq 0$, Further

$$p^* = -\frac{r^* \mu_3 + \mu_4}{r^* \mu_1 + \mu_2} = -\alpha .$$

Consider

$$p(r) = -\frac{r\mu_3 + \mu_4}{r\mu_1 + \mu_2}$$

$$q(r) = -\frac{r\lambda_3 + \lambda_4}{r\lambda_1 + \lambda_2}$$

in a neighbourhood Δ_r of the point r^0 except at r^0.

Since $\mu_1, \lambda_1 \neq 0$ and since $r^0 \mu_1 + \mu_2 = 0$, in this neighbourhood both $p(r)$ and $q(r)$ are defined except at $r = r^0$. Further $(p(r), q(r), r)$ satisfies equations 1,2 and also 3, by our assumption on $f(r)$ for all $r \in \Delta_{r_0}$ except for $r = r^0$. Also $p(r) = p^*$ at all those points. This contradicts lemma 5.1.

Case (vi):

$$r^0 \lambda_1 + \lambda_2 \neq 0 \quad r^0 \mu_1 + \mu_2 = 0$$
$$r^* \lambda_1 + \lambda_2 \neq 0 \quad r^* \mu_1 + \mu_2 \neq 0 .$$

As before $\mu_1 \neq 0$, $\mu_2 \neq 0$, $\mu_3 \neq 0$, $\mu_4 \neq 0$. Also

$$p^* = -\frac{(r\mu_3 + \mu_4)}{(r\mu_1 + \mu_2)} \quad \text{for all} \quad r \neq r^0.$$

This is similar to case (v). Hence the proof of the lemma.

Remark:

If $(p^0, q^0, r^0) \in \varepsilon$ then one can easily check that $f(r^0) = 0$.

Proof of Theorem 6:

Consider the matrix

$$A = \begin{bmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \end{bmatrix}$$

associated with the completely mixed game. Since it has no pure equilibrium, for each one of the vectors

$$v_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} a_2 \\ b_2 \\ -c_1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} a_3 \\ -b_1 \\ c_2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} a_4 \\ -b_2 \\ -c_2 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} -a_1 \\ b_3 \\ c_3 \end{bmatrix}, \quad v_6 = \begin{bmatrix} -a_2 \\ b_4 \\ -c_3 \end{bmatrix}, \quad v_7 = \begin{bmatrix} -a_3 \\ -b_3 \\ c_4 \end{bmatrix}, \quad v_8 = \begin{bmatrix} -a_4 \\ -b_4 \\ -c_4 \end{bmatrix}$$

at least one coordinate is negative.

We will assume for the present no entry in A is zero. This means there will not be any equilibrium where some two players use pure strategies.

For any given matrix A we can consider which of the following 6 conditions are satisfied.

$$S_1 : b_1 b_2 < 0 \text{ and } c_1 c_2 < 0, \quad S_2 : b_3 b_4 < 0 \text{ and } c_3 c_4 < 0,$$
$$S_3 : a_1 a_2 < 0 \text{ and } c_1 c_3 < 0, \quad S_4 : a_3 a_4 < 0 \text{ and } c_2 c_4 < 0,$$
$$S_5 : a_1 a_3 < 0 \text{ and } b_1 b_3 < 0, \quad S_6 : a_2 a_4 < 0 \text{ and } b_2 b_4 < 0.$$

It is quite possible that none are satisfied.

Let us consider the equations

$$qr\lambda_1 + q\lambda_2 + r\lambda_3 + \lambda_4 = 0.$$
$$pr\mu_1 + p\mu_2 + r\mu_3 + \mu_4 = 0$$
$$pq\nu_1 + p\nu_2 + q_3 + \nu_4 = 0.$$

Let $(p_0, q_0, r_0) \in \varepsilon$. We have

$$q_0(r_0\lambda_1 + \lambda_2) + (r_0\lambda_3 + \lambda_4) = 0 \longrightarrow 1$$
$$p_0(r_0\mu_1 + \mu_2) + (r_0\mu_3 + \mu_4) = 0 \longrightarrow 2$$
$$(r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)(p_0 q_0 \nu_1 + p_0\nu_2 + q_0\nu_3 + \nu_4) = 0.$$

That is

$$p_0(r_0\mu_1 + \mu_2)q_0(r_0\lambda_1 + \lambda_2)\nu_1 + p_0(r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)\nu_2$$
$$+ q_0(r_0\mu_1 + \mu_2)(r_0\lambda_1 + \lambda_2)\nu_3 + (r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)\nu_4 = 0.$$

Using 1 and 2 we get

$$(r_0\mu_3 + \mu_4)(r_0\lambda_3 + \lambda_4)\nu_1 - (r_0\mu_3 + \mu_4)(r_0\lambda_1 + \lambda_2)\nu_2$$
$$- (r_0\lambda_3 + \lambda_4)(r_0\mu_1 + \mu_2)\nu_3 + (r_0\lambda_1 + \lambda_2)(r_0\mu_1 + \mu_2)\nu_4 = 0.$$

Thus the function

$$f(r) = (r\mu_3 + \mu_4)(r\lambda_3 + \lambda_4)\nu_1 - (r\mu_3 + \mu_4)(r\lambda_1 + \lambda_2)\nu_2$$
$$- (r\lambda_3 + \lambda_4)(r\mu_1 + \mu_2)\nu_3 + (r\lambda_1 + \lambda_2)(r\mu_1 + \mu_2)\nu_2$$

has a root at $r = r_0$. By lemma 2 we can assume that $f(r) \not\equiv 0$ and that $f(r)$ is either linear or quadratic. If it is linear, then for any $(p_0, q_0, r_0) \in \varepsilon$, r_0 is unique and from lemma 1 we have a unique equilibrium point. The non-trivial case is when $f(r)$ is a quadratic. We will without loss of generality assume that the three quadratics exist. Similarly one gets the quadratics in p and q. Let f_p, f_q, f_r be the three quadratics.

One easily computes

$$f_p(0) = b_4 c_4 a_1 - b_3 c_4 a_2 - b_4 c_3 a_3 + b_3 c_3 a_4$$
$$f_p(1) = b_2 c_2 a_1 - b_1 c_2 a_2 - b_2 c_1 a_3 + b_1 c_1 a_4$$
$$f_q(0) = a_4 c_4 b_1 - a_3 c_4 b_2 - a_4 c_2 b_3 + a_3 c_2 b_4$$

$$f_q(1) = a_2 c_3 b_1 - a_1 c_3 b_2 - a_2 c_1 b_3 + a_1 c_1 b_4$$
$$f_r(0) = a_4 b_4 c_1 - a_2 b_4 c_2 - a_4 b_2 c_3 + a_2 b_2 c_4$$
$$f_r(1) = a_3 b_3 c_1 - a_1 b_3 c_2 - a_3 b_1 c_3 + a_1 b_1 c_4$$

One can also compute the following expressions which we need in the sequel.

$$f_p\left(\frac{c_3}{c_3 - c_1}\right) = (c_1 c_4 - c_2 c_3) f_q(1) \quad \text{when} \quad c_1 c_3 < 0$$
$$f_p\left(\frac{b_3}{b_3 - b_1}\right) = (b_1 b_4 - b_2 b_3) f_r(1) \quad \text{when} \quad b_1 b_3 < 0$$
$$f_p\left(\frac{c_4}{c_4 - c_2}\right) = (c_1 c_4 - c_2 c_3) f_q(0) \quad \text{when} \quad c_2 c_4 < 0$$
$$f_p\left(\frac{b_4}{b_4 - b_2}\right) = (b_1 b_4 - b_2 b_3) f_r(0) \quad \text{when} \quad b_2 b_4 < 0$$
$$f_q\left(\frac{c_2}{c_2 - c_1}\right) = (c_1 c_4 - c_2 c_3) f_p(1) \quad \text{when} \quad c_1 c_2 < 0$$
$$f_q\left(\frac{c_4}{c_4 - c_3}\right) = (c_1 c_4 - c_2 c_3) f_p(0) \quad \text{when} \quad c_3 c_4 < 0$$
$$f_q\left(\frac{a_3}{a_3 - a_1}\right) = (a_1 a_4 - a_2 a_3) f_r(1) \quad \text{when} \quad a_1 a_3 < 0$$
$$f_q\left(\frac{a_4}{a_4 - a_2}\right) = (a_1 a_4 - a_2 a_3) f_r(0) \quad \text{when} \quad a_2 a_4 < 0$$
$$f_r\left(\frac{a_4}{a_4 - a_3}\right) = (a_1 a_4 - a_2 a_3) f_q(0) \quad \text{when} \quad a_3 a_4 < 0$$
$$f_r\left(\frac{b_4}{b_4 - b_3}\right) = (b_1 b_4 - b_2 b_3) f_p(0) \quad \text{when} \quad b_3 b_4 < 0$$
$$f_r\left(\frac{a_2}{a_2 - a_1}\right) = (a_1 a_4 - a_2 a_3) f_q(1) \quad \text{when} \quad a_1 a_2 < 0$$
$$f_r\left(\frac{b_2}{b_2 - b_1}\right) = (b_1 b_4 - b_2 b_3) f_p(1) \quad \text{when} \quad b_1 b_2 < 0$$

We call two matrices equivalent if their entries are non-zero and they have the same sign structure. We have 2^{12} distinct matrices modulo equivalence. Of these, certain matrices may possess pure equilibrium and hence can be omitted; for example if the first column is $\begin{bmatrix} + \\ + \\ + \end{bmatrix}$ it can be omitted. For certain matrices one can directly check that $f_p(0) f_p(1) \leq 0$ or $f_q(0) f_q(1) \leq 0$ or $f_r(0) f_r(1) \leq 0$ in which case there is only one positive root for one of the quadratics and by lemma 1, the equilibrium is unique. For example, if the sign structure is

$$\begin{bmatrix} -, & +, & -, & + \\ +, & +, & -, & - \\ +, & -, & +, & - \end{bmatrix}, \quad f_r(0) f_r(1) < 0 .$$

We computerized the matrices and in total we were left with 762 cases which do not fall in the above category. We have checked all the cases and the proof of the theorem depends on the location of the root in an appropriate quadratic. We will discuss a few of the typical cases which repeat in all other cases.

Case (i):

None of the conditions S_1 or $S_2 \ldots$ or S_6 satisfied.

Consider for example the case

$$A = \begin{bmatrix} -, & +, & -, & + \\ +, & +, & -, & - \\ +, & -, & +, & - \end{bmatrix}$$

One directly checks that $f_r(0) < 0$ and $f_r(1) > 0$, and that there is a unique root for the r-quadratic in (0,1). Hence by lemma 1 the equilibrium point is unique.

Remark:

In fact when none of the conditions S_1 to $\ldots, S_6$ are satisfied and when no entry in A is zero, then one can check that each row has exactly two positive and two negative signs, when ε is completely mixed. We can also assert that $a_1 a_4 < 0$, $b_1 b_4 < 0, c_1 c_4 < 0, a_2 a_3 < 0, b_2 b_3 < 0, c_2 c_3 < 0$.

Case (ii):

Let one of the conditions S_1 or$\ldots$ S_6 be satisfied. Say $a_4 a_2 < 0$ and $b_4 b_2 < 0$. For example consider

$$A = \begin{bmatrix} -, & -, & -, & + \\ -, & +, & -, & - \\ -, & -, & +, & - \end{bmatrix}$$

clearly in this case

$$0 < \frac{a_4}{a_4 - a_2}, \frac{b_4}{b_4 - b_2} < 1 .$$

Let

$$p = \frac{b_4}{b_4 - b_2} \; q = \frac{a_4}{a_4 - a_2} r = 0 .$$

We have

$$K_1(1, q, 2) = K_1(2, q, 2)$$
$$K_2(p, 1, 2) = K_2(p, 2, 2)$$

and hence

$$K_3(p,q,2) < K_3(p,q,1)$$

since ε is completely mixed,

i.e.,

$$K_3(p,q,1) - K_3(p,q,2) = \frac{(a_4 b_4 c_1 - a_2 b_4 c_2 - a_4 b_2 c_3 + a_2 b_2 c_4)}{(a_4 - a_2)(b_4 - b_2)} > 0\,.$$

But

$$(a_4 - a_2)(b_4 - b_2) < 0\,.$$

Hence

$$a_4 b_4 c_1 - a_2 b_4 c_2 - a_4 b_2 c_3 + a_2 b_2 c_4 = f_r(0) < 0\,.$$

We will also assume $f_r(1) < 0$; for otherwise there is only one root in (0,1) to the r-quadratic and the proof was already discussed for that case. Since $a_1 a_4 - a_2 a_3 < 0$ and $a_2 a_4 < 0$ we have $f_q\left(\frac{a_4}{a_4 - a_2}\right) = (a_1 a_4 - a_2 a_3) f_r(0) > 0$.

With the first row having the sign structure $(-,\ -,\ -,\ +)$

$$r_0(q_0 a_1 + (1 - q_0)a_3) + (1 - r_0)(q_0 a_2 + (1 - q_0)a_4) = 0$$

(Here $(p_0,q_0,r_0) \in \varepsilon$) implies $q_0 a_2 + (1 - q_0)a_4 > 0$,

i.e.,

$$a_4 > q_0(a_4 - a_2) \quad \text{and since} \quad a_4 - a_2 > 0\,,\ q_0 < \frac{a_4}{a_4 - a_2}$$

i.e., for any

$$(p,q,r) \in \varepsilon\,.\,q < \frac{a_4}{a_4 - a_2}\,.$$

Thus if $f_q(0) < 0$, from the assertion $f_q\left(\frac{a_4}{a_4 - a_2}\right) > 0$ we know there is only one admissible root for the q-quadratic in the interval $\left(0, \frac{a_4}{a_4 - a_2}\right)$. In this case ε has a unique element. Suppose $f_q(0) > 0$; we can assume $f_q(1) > 0$.

One checks

$$f_p\left(\frac{c_3}{c_3 - c_1}\right) = (c_1 c_4 - c_2 c_3) f_q(1) > 0$$

and

$$f_p\left(\frac{b_4}{b_4 - b_2}\right) = (b_1 b_4 - b_2 b_3) f_r(0) < 0$$

and also for any $(p_0,q_0,r_0) \in \varepsilon$, $\frac{b_4}{b_4 - b_2} < p_0 < \frac{c_3}{c_3 - c_1}$ and that p_0 is the unique positive root for the quadratic in this interval. Hence ε has a unique element.

Case (iii):

Two of the conditions S_1 or S_2 or ... or S_6 are satisfied. Say

$$A = \begin{bmatrix} +, & +, & -, & - \\ +, & -, & -, & + \\ -, & +, & +, & - \end{bmatrix}$$

Here S_1 and S_2 are satisfied. As before we demand

$$\left(1, \frac{c_2}{c_2 - c_1}, \frac{b_2}{b_2 - b_1}\right) \notin \varepsilon \quad \text{and} \quad \left(0, \frac{c_4}{c_4 - c_3}, \frac{b_4}{b_4 - b_3}\right) \notin \varepsilon,$$

i.e.,

$$\frac{(b_2 c_2 a_1 - b_1 c_2 a_2 - b_2 c_1 a_3 + b_1 c_1 a_4)}{(b_2 - b_1)(c_2 - c_1)} = \frac{f_p(1)}{(b_2 - b_1)(c_2 - c_1)} < 0$$

and

$$\frac{(b_4 c_4 a_1 - b_3 c_4 a_2 - b_4 c_3 a_3 + b_3 c_3 a_4)}{(b_4 - b)(c_4 - c_3)} = \frac{f_p(0)}{(b_4 - b_3)(c_4 - c_3)} > 0.$$

We get $f_p(1) f_p(0) < 0$, and hence as before ε has a unique element.

One deals with all the other matrices in a similar fashion. This completes the proof of the theorem when A has only non-zero entries.

Remark 1:

If some four of the conditions $S_1, S_2, \ldots, S_6$ are satisfied and if A has all the entries non-zero, then the game has a pure equilibrium. This we observed from the computer output.

Remark 2:

If we allow zeros in some of the entries, the same type of proof goes through. It would be interesting if one could give a simpler proof avoiding these combinatorial arguments.

Remark 3:

It would be nice to know whether the uniqueness holds for general n person games with two pure strategies for each player.

6. Further Remarks on N-person Games

Some of the theorems proved in the previous sections are extendable to general N-person games as follows.

Theorem 7:

Let $m_1, m_2, \ldots, m_N$, $(N \geq 3)$ be the number of pure strategies for N-players in an N-person non-cooperative game. Further let $m_i \geq \sum_{j \neq i} m_j - (N - 3)$ for some i.

Then for any equilibrium $(x_1^0, x_2^0, \ldots, x_i^0, \ldots, x_N^0) \in \varepsilon$, we can find another equilibrium $(x_1^0, x_2^0, \ldots, x_{i-1}^0, x_i^*, x_{i+1}^0, \ldots, x_N^0)$ where x_i^* is not completely mixed.

Theorem 8:

In the above theorem suppose $m_i = \sum_{j \neq i} m_j - (N-2)$ for some i. If in the equilibrium $(x_1^0, x_2^0, \ldots, x_i^0, \ldots, x_N^0)$ some one of the x_j^0 is not completely mixed for $j \neq i$, then we can find another equilibrium $(x_1^0, x_2^0, \ldots, x_{i-1}^0, x_i^*, x_{i+1}^0, \ldots, x_N^0)$ where x_i^* is not completely mixed.

Theorem 9:

Let an N-person game be completely mixed. If the equilibrium set is convex then the set contains just one element.

Remark 1:

It would be interesting to extend the theorem on the exchangeability of the equilibrium points for two-person games to N-person games, under suitable conditions.

Remark 2:

For a general N-person completely mixed game nothing is known about the cardinality of the equilibrium set. We believe that it is finite or uncountably infinite.

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