Annali di Matematica pura ed applicata
(IV), Vol. CLXV (1993), pp. 351-368

# Boundedness, Periodicity, and Convergence of Solutions in a Retarded Liénard Equation (*).

T. A. BURTON (**) - BO ZHANG (***)

## 1. – Introduction.

In this paper we consider the equation

$$x'' + f(x)\,x' + g(x(t-h)) = e(t)\,, \tag{1.1}$$

where $h$ is a nonnegative constant and $e$ is a bounded function. With appropriate assumptions on $f$ and $g$ we obtain necessary and sufficient conditions for solutions of (1.1) to be uniformly ultimately bounded. Thus, by an asymptotic fixed point theorem, those conditions imply that (1.1) has a $T$-periodic solution whenever $e$ is $T$-periodic. We also give conditions under which all solutions of (1.1) converge.

The book of SANSONE and CONTI [11] gives much history and foundation for (1.1) without a delay, say

$$x'' + f(x)\,x' + g(x) = e(t)\,, \tag{1.2}$$

and for the unforced form

$$x'' + f(x)\,x' + g(x) = 0\,. \tag{1.3}$$

It is shown in BURTON [1] that when

$$f(x) > 0 \quad \text{and} \quad xg(x) > 0 \quad \text{for } x \neq 0 \tag{1.4}$$

(*) Entrata in Redazione il 15 luglio 1991.

Indirizzo degli AA.: Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, IL 62901-4408.

(**) This research was supported in part by a Fulbright grant with number 90-41635.

(***) On leave from Northeast Normal University, Changchun, Jilin, P.R. China.

then the zero solution of (1.5) is globally asymptotically stable if and only if

$$\int_0^{\pm\infty} [f(x) + |g(x)|]\,dx = \pm\infty . \tag{1.5}$$

Recently, that results was improved through relaxation of (1.4) by HARA and YONEYAMA [8], SUGIE ([13], [14]), VILLARI [15], and VILLARI and ZANOLIN [16]; those papers also give extensive bibliographies not listed here.

Condition (1.5) proves to be central. It is shown in BURTON and TOWNSEND [4] THAT WHEN (1.4) IS STRENGTHENED AND WHEN $e(t+T) = e(t)$, then all solutions of (1.2) are bounded and there is a $T$-periodic solution if and only if (1.5) holds. A fairly extensive bibliography for (1.2) and (1.3) is contained in that paper [4] which updates the one from the SANSONE and CONTI book [11]. GRAEF [6] continues the work on (1.2) and reduces (1.4) showing that solutions of (1.2) are bounded and there is a $T$-periodic solution of (1.2) if and only if (1.5) holds: he also updates the bibliography.

KRASOVSKII [9; pp. 173-4] studies a generalization of (1.3) including

$$x'' + f(x)x' + g(x(t-h)) = 0 \tag{1.6}$$

and obtains sufficient conditions for uniform asymptotic stability of the zero solution when at least (1.4) holds. His work generated much interest, as may be seen in the writing of SOMOLINOS [12], YOSHIZAWA [18], MURAKAMI [10], BURTON and HATVANI [3], as well as in the bibliographies given in those references. Recently, ZHANG [19] showed that under suitable conditions on $f$ and $g$, then the zero solution of (1.6) is globally asymptotically stable if and only if (1.5) holds.

Our first result extends the work of Zhang to (1.1) where we show that solutions of (1.1) are uniformly ultimately bounded if and only if (1.5) holds, when (1.4) is suitably modified. When we add the condition that $e(t+T) = e(t)$ then this boundedness will imply that (1.1) has a $T$-periodic solution, as may be seen in HALE [7] or BURTON and ZHANG [5].

In our last result we adapt a technique of WALTMAN and BRIDGLAND [17] (cf. MURAKAMI [10] also) which they used on (1.3) to show that when solutions of (1.2) are uniformly ultimately bounded and $e(t+T) = e(t)$ then all solutions of (1.2) converge to a $T$-periodic solution of (1.2).

When a function is written without its argument, then that argument is $t$.

## 2. - Boundedness and periodicity.

Consider the Liénard equation

$$x''(t) + f(x)x' + g(x(t-h)) = e(t) \tag{2.1}$$

where

(2.2) $$h \geq 0,\ f,\ g,\ e \text{ are continuous and } e(t),\ E(t) = \int_0^t e(s)\,ds \text{ bounded}.$$

Let $F(x) = \int_0^x f(s)\,ds$, $G(x) = \int_0^x g(s)\,ds$,

(2.3) $$|e(t)| \leq m,\ |E(t)| \leq M \text{ for some } m, M > 0.$$

Assume also that

(2.4) there are constants $k > 0$, $N > 1$ such that $xg(x) > 0$, $x(F(x) - Nhg(x) - (\operatorname{sgn} x)NM) > 0$ for $|x| > k$, where $M$ is defined in (2.3).

A system equivalent to (2.1) is

(2.5) $$\begin{cases} x' = y \\ y' = -f(x)y - g(x(t-h)) + e(t). \end{cases}$$

It is known [2] that for $f$, $g$, and $e$ continuous, given a continuous initial function $\phi: [-h, 0] \to R$ and a number $y_0$, then there exists a solution of (2.5) on an interval $[0, \alpha)$ satisfying the initial condition and satisfying (2.5) on $(0, \alpha)$; if the solution remains bounded then $\alpha = \infty$.

DEFINITION 1. – Solutions of (2.5) are uniformly bounded at $t = 0$ (UB) if for each $B_1 > 0$ there is a $B_2 > 0$ such that $\{\phi: [-h, 0] \to R,\ y_0 \in R$ with $\|\phi\| + |y_0| < B_1\}$ imply that $|x(t)| + |y(t)| < B_2$ for all $t \geq 0$ where $(x(t), y(t))$ is any solution of (2.5) satisfying the given initial conditions.

DEFINITION 2. – Solutions of (2.5) are uniformly ultimately bounded for bound $B$ at $t = 0$ (UUB) if for each $B_1 > 0$ there is a $K > 0$ such that $\{\phi: [-h, 0] \to R,\ y_0 \in R$ with $\|\phi\| + |y_0| < B_1\}$ imply that $|x(t)| + |y(t)| < B$ for all $t \geq K$ where $(x(t), y(t))$ is any solution of (2.5) satisfying the given initial conditions.

THEOREM 1. – Suppose that (2.2), (2.3), and (2.4) hold. Then solutions of (2.5) are UB and UUB if and only if

(2.6) $$\limsup_{x \to \pm\infty} [F(x) \pm G(x)] = \pm\infty.$$

PROOF. – We consider a system equivalent to (2.5)

(2.7) $$\begin{cases} x' = z - F(x) + E(t) + \int_{t-h}^{t} g(x(s))\,ds, \\ z' = -g(x) \end{cases}$$

with

$$y = z - F(x) + E(t) + \int_{t-h}^{t} g(x(s))\, ds .$$

Let

$$Q_0 = 2 \sup \{ |G(s)| : |s| \leq k \} .$$

Then there is a constant $p$ with

$$G(-k) + p = G(k) .$$

Let $c > 0$ be chosen so that

$$(2.8) \qquad (N-1)M - 2c > 0 , \qquad k(N+1) - 2hc > 0$$

and define

$$W(x, z) = \begin{cases} z^2 + 2G(x) + Q_0 & \text{for } x > k , \\ (z+c)^2 + 2G(x) + Q_0 + 2p & \text{for } x < -k , \\ (z - (cx/2k) + (c/2))^2 + 2G(k) + Q_0 & \text{for } |x| \leq k . \end{cases}$$

Notice that

$$W(k, z) = z^2 + 2G(k) + Q_0$$

and

$$W(-k, z) = (z+c)^2 + 2G(k) + Q_0 ,$$

$$= (z+c)^2 + 2G(-k) + Q_0 + 2p .$$

This implies that $W(x, z)$ is continuous.

Let $x(t) = x(t, \phi, z_0)$, $z(t) = z(t, \phi, z_0)$ be a solution of (2.7) with $\phi \in C([-h, 0], R)$, $x(s) = \phi(s)$ for $s \in [-h, 0]$ and $z(0) = z_0$. Then $(x(t), z(t))$ exists on $R^+$. We define

$$(2.9) \qquad V(t) = W(x(t), z(t)) + [(N+1)/2] \int_{-h}^{0} \int_{t+s}^{t} g^2(x(v))\, dv\, ds + $$

$$+ [(N-1)/4] \int_{t-h}^{t} g^2(x(s))\, ds .$$

Differentiate (2.9) to obtain

$$V'(t) = W'(x(t), z(t)) + \left(\frac{1}{2}\right) h(N+1)\, g^2(x(t)) -$$

$$-\left(\frac{1}{2}\right)(N+1)\int_{t-h}^{t} g^2(x(s))\, ds + \left(\frac{1}{4}\right)(N-1)\, hg^2(x(t)) - \left(\frac{1}{4}\right)(N-1)\, hg^2(x(t-h)).$$

If $x(t) > k$, then

$$V'(t) = 2zz' + 2g(x)\left[z - F(x) + E(t) + \int_{t-h}^{t} g(x(s))\, ds\right] + \tag{2.10}$$

$$+\left(\frac{1}{2}\right) h(N+1)\, g^2(x(t)) - \left(\frac{1}{2}\right)(N+1)\int_{t-h}^{t} g^2(x(s))\, ds +$$

$$+\left(\frac{1}{4}\right)(N-1)\, hg^2(x(t)) - \left(\frac{1}{4}\right)(N-1)\, hg^2(x(t-h)) \leqslant$$

$$\leqslant -2g(x)F(x) + 2|g(x)|M + hg^2(x) + \int_{t-h}^{t} g^2(x(s))\, ds +$$

$$+\left(\frac{1}{2}\right) h(N+1)\, g^2(x(t)) - \left(\frac{1}{2}\right)(N+1)\int_{t-h}^{t} g^2(x(s))\, ds + \left(\frac{1}{4}\right)(N-1)\, hg^2(x(t)) =$$

$$= -[4/(N+1)]\, g(x)[F(x) - \{(N+1)\, hg(x)/2\} - M(N+1)/2] -$$

$$-\left(\frac{3}{2}\right)[1 - (2/(N+1))][F(x) - (N+1)\, hg(x)/2]\, g(x) -$$

$$-\left(\frac{1}{2}\right)[1 - (2/(1+N))]\, g(x)F(x) - \left(\frac{1}{2}\right)(N-1)\int_{t-h}^{t} g^2(x(s))\, ds \leqslant$$

$$\leqslant -(4/(N+1))\, g(x)[F(x) - Nhg(x) - NM] -$$

$$-\left(\frac{3}{2}\right)[1 - (2/(1+N))](F(x) - Nhg(x))\, g(x) -$$

$$-\left(\frac{1}{2}\right)[1 - (2/(1+N))]\, g(x)F(x) - \left(\frac{1}{2}\right)(N-1)\int_{t-h}^{t} g^2(x(s))\, ds$$

which is negative for $x > k$.

For $x(t) < -k$ we have

$$(2.11)\qquad V'(t) = 2(z + c)z' + 2g(x)\left[z - F(x) + E(t) + \int_{t-h}^{t} g(x(s))\,ds\right] +$$

$$+ \left(\frac{1}{2}\right)h(N + 1)g^2(x(t)) - \left(\frac{1}{2}\right)(N + 1)\int_{t-h}^{t} g^2(x(s))\,ds +$$

$$+ \left(\frac{1}{4}\right)(N - 1)hg^2(x(t)) - \left(\frac{1}{4}\right)(N - 1)hg^2(x(t - h)) \leqslant$$

$$\leqslant -[4/(N + 1)]|g(x)|[|F(x)| - c - ((N + 1)/2)h|g(x)| - (N + 1)M/2] -$$

$$- \left(\frac{3}{2}\right)[1 - (2/(N + 1))](|F(x)| - c - ((N + 1)/2)h|g(x)|)|g(x)| -$$

$$- \left(\frac{1}{2}\right)[1 - (2/(N + 1))](|F(x)| - c)|g(x)| - \left(\frac{1}{2}\right)(N - 1)\int_{t-h}^{t} g^2(x(s))\,ds.$$

Taking into account $(N - 1)M - 2c > 0$, we get

$$(2.12)\qquad V'(t) \leqslant -[4/(N + 1)][|F(x)| - Nh|g(x)| - NM] -$$

$$- \left(\frac{3}{2}\right)[1 - (2/(N + 1))]|g(x)|[|F(x)| - Nh|g(x)| - NM] -$$

$$- \left(\frac{1}{2}\right)[1 - (2/(N + 1))](|F(x)| - c)|g(x)| - \left(\frac{1}{2}\right)(N - 1)\int_{t-h}^{t} g^2(x(s))\,ds$$

for $x(t) < -k$. Thus, $V' < 0$ if $x < -k$.

Next, consider $|x(t)| \leqslant k$. Then

$$(2.13)\qquad V'(t) = 2(z - (cx/2k) + (c/2))\left\{-g(x) - (c/2k)\left[z - F(x) + E(t) + \int_{t-h}^{t} g(x(s))\,ds\right]\right\} +$$

$$+ \left(\frac{1}{2}\right)h(N + 1)g^2(x(t)) - \left(\frac{1}{2}\right)(N + 1)\int_{t-h}^{t} g^2(x(s))\,ds +$$

$$+ \left(\frac{1}{4}\right)(N - 1)hg^2(x(t)) - \left(\frac{1}{4}\right)(N - 1)hg^2(x(t - h)) =$$

$$= 2[z - (cx/2k) + (c/2)][-g(x) + (cF(x)/2k) - (cE(t)/2k)] -$$

$$- (cz^2/k) + (cz/k)[(cx/2k) - (c/2)] -$$

$$- (cz/k) \int_{t-h}^{t} g(x(s))\, ds + (c/k)[(cx/2k) - (c/2)] \int_{t-h}^{t} g(x(s))\, ds +$$

$$+ \left(\frac{h}{2}\right)(N+1)\, g^2(x(t)) - \left(\frac{1}{2}\right)(N+1) \int_{t-h}^{t} g^2(x(s))\, ds +$$

$$+ \left(\frac{1}{4}\right)(N-1)\, hg^2(x(t)) - \left(\frac{1}{4}\right)(N-1)\, hg^2(x(t-h)).$$

Notice that

$$(c/k)|z| \int_{t-h}^{t} |g(x(s))|\, ds \leq (c/2k)\, z^2 + (ch/2k) \int_{t-h}^{t} g^2(x(s))\, ds \leq \tag{2.14}$$

$$\leq (c/2k)\, z^2 + \left(\frac{1}{4}\right)(N+1) \int_{t-h}^{t} g^2(x(s))\, ds$$

by (2.8).

Substituting (2.14) into (2.13) and using the fact that $|x(t)| \leq k$, we obtain

$$V'(t) \leq -\alpha\left(z^2 + \int_{t-h}^{t} g^2(x(s))\, ds\right) + \beta, \tag{2.15}$$

where $\alpha$, $\beta$ are positive constant depending only on $k$ and $c$.

Let $\overline{Q} > 0$ be a constant such that $\overline{Q} - k^2 > 0$ and

$$\alpha(\overline{Q} - k^2) - \beta > 0.$$

We consider

$$|z(t)|^2 + |x(t)|^2 + \int_{t-h}^{t} g^2(x(s))\, ds > \overline{Q}.$$

If $|x(t)| > k$, then $V'(t) < 0$ by (2.10) and (2.12). If $|x(t)| \leq k$ then

$$|z(t)|^2 + \int_{t-h}^{t} g^2(x(s))\, ds > \overline{Q} - k^2.$$

Thus,

$$V'(t) \leq -\alpha(\overline{Q} - k_2) + \beta < 0$$

by (2.15). We then conclude that

$$V'(t) < 0 \tag{2.16}$$

whenever $|z(t)|^2 + |x(t)|^2 + \int_{t-h}^{t} g^2(x(s))\,ds > \overline{Q}$.

Notice also that

$$V(t) \leq L\left[z^2 + |G(x(t))| + \int_{t-h}^{t} g^2(x(s))\,ds\right] + q \tag{2.17}$$

for some positive constants $L$ and $q$.

We now show that solutions of (2.7) are uniformly bounded (the definition is the same as Definition 1 with $y_0$ replaced by $z_0$).

Let $D > 0$ and $\phi \in C([-h, 0], R)$, $z_0 \in R$ such that $\|\phi\| + |z_0| \leq D$. Then there is a $Q = Q(D)$ such that $Q > \overline{Q}$ and

$$|z(0)|^2 + |\phi(0)|^2 + \int_{-h}^{0} |g(\phi(s))|^2\,ds \leq Q.$$

Then either

$$|z(t)|^2 + |x(t)|^2 + \int_{-h}^{0} |g(x(s))|^2\,ds \leq Q \tag{2.18}$$

for all $t \in R^+$ or

$$|z(t_1)|^2 + |x(t_1)|^2 + \int_{t_1-h}^{t_1} |g(x(s))|^2\,ds > Q \tag{2.19}$$

for some $t_1 > 0$.

Let $G^* = \sup\{|G(s)|: |s| \leq Q^{1/2}\}$. If (2.18) holds, then

$$V(t) \leq L(Q + G^*) + q \quad \text{for all } t. \tag{2.20}$$

If (2.19) holds, then there exists a $t_0 \geq 0$ such that $t_0 < t_1$ with

$$|z(t_0)|^2 + |x(t_0)|^2 + \int_{t_0-h}^{t_0} g^2(x(s))\,ds = Q$$

and

$$|z(s)|^2 + |x(s)|^2 + \int_{s-h}^{s} g^2(x(v))\,dv > Q$$

for $s \in (t_0, t_1]$. By (2.16) we have

$$V(t_1) \leqslant V(t_0) \leqslant L(Q + G^*) + q.$$

Thus

$$V(t) \leqslant L(Q + G^*) + q =: Q_1 \quad \text{for all } t \in R^+.$$

This implies that

$$|z(t)| + |G(x(t))| + \int_{t-h}^{t} g^2(x(s))\,ds \leqslant Q_2 \quad \text{for all } t \geqslant 0 \tag{2.21}$$

where $Q_2$ depends only on $Q_1$. If $\lim_{s\to\infty} G(s) = \infty$, then

$$x(t) \leqslant Q_3 \quad \text{for some } Q_3 = Q_3(Q) > 0.$$

If $\limsup_{s\to\infty} F(s) = \infty$, then there exists a $\gamma > 0$ such that

$$F(\gamma) - (Q_2 + h + M) > 0 \quad \text{and} \quad \gamma > Q^{1/2}.$$

From (2.7) we have

$$x'(t) \leqslant -F(x) + |z(t)| + |E(t)| + h + \int_{t-h}^{t} g^2(x(s))\,ds \leqslant -F(x) + (Q_2 + h + M).$$

Since $x(0) \leqslant Q^{1/2} < \gamma$, we claim that $x(t) < \gamma$ for $t \geqslant 0$. In fact, if there exists a $t_1 > 0$ with $x(t_1) = \gamma$ and $x(s) < \gamma$ on $[0, \gamma)$, then

$$0 \leqslant x'(t_1) \leqslant -F(x(t_1)) + Q_2 + M + h = -F(\gamma) + Q_2 + M + h < 0,$$

a contradiction. Thus there exists a $Q_4 = Q_4(Q)$ such that $x(t) \leqslant Q_4$ for all $t \in R^+$. Finally, we have $x(t) \leqslant Q_3 + Q_4$ if

$$\limsup_{s\to\infty} [F(s) + G(s)] = \infty.$$

Using the same argument we can show that there exists a constant $Q_5 > 0$ such that

$$-Q_5 \leqslant x(t) \quad \text{for all } t \in R^+ \text{ if } \limsup_{s\to-\infty} [F(s) - G(s)] = -\infty.$$

We therefore conclude that solutions of (2.7) are uniformly bounded.

Next, we will show that solutions of (2.7) are uniformly ultimately bounded. Let $\|\phi\| + |z_0| \leqslant D$; then there exists a $D_1 > 0$ such that

$$|x(t)| + |z(t)| \leqslant D_1 \quad \text{for all } t \in R^+$$

where $x(t) = x(t, \phi, z_0)$, $z(t) = z(t, \phi, z_0)$ is a solution of (2.7) with initial data $(\phi, z_0)$.

From (2.10), (2.12), and (2.16) we conclude that

$$V'(t) \leqslant -\mu \quad \text{for some } \mu > 0 \tag{2.22}$$

whenever $|z(t)|^2 + |x(t)|^2 + \int_{t-h}^{t} g^2(x(s))\,ds > \overline{Q}$. There exists $D_2 > 0$ such that $V(0) \leqslant \leqslant D_2$. By (2.22), there is a $K = K(D) > 0$ such that

$$|z(t_1)|^2 + |x(t_1)|^2 + \int_{t_1-h}^{t_1} g^2(x(s))\,ds \leqslant \overline{Q}$$

for some $t_1 \in [0, K]$. Now suppose that there exists $t_2 > t_1$ such that

$$V(t_2) = \max\{V(s)\colon t_1 \leqslant s \leqslant t_2\}.$$

Then

$$|x(t_2)|^2 + |z(t_2)|^2 + \int_{t_2-h}^{t_2} g^2(x(s))\,ds \leqslant \overline{Q}$$

by (2.22). This implies that

$$V(t_2) \leqslant B_1 \quad \text{for some } B_1 = B_1(Q) > 0.$$

Consequently, we have

$$V(t) \leqslant B_1 \quad \text{for all } t \geqslant K \geqslant t_1$$

and

$$|z(t)|^2 + G(x(t)) + \int_{t-h}^{t} g^2(x(s))\,ds \leqslant B_2$$

for some $B_2 = B_2(Q) > 0$ and all $t \geqslant K$. Using the same argument following (2.21) and replacing $Q_2$ by $B_2$, we can find a constant $B > 0$ depending only on $Q$ such that

$$|x(t)| + |z(t)| \leqslant B \quad \text{for } t \geqslant K.$$

Hence, solutions of (2.7) are uniformly ultimately bounded. We therefore conclude that solutions of (2.5) are UB and UUB.

Now we show that (2.6) is necessary. Suppose that (2.6) fails. To be definite, we assume that

$$\limsup_{s\to\infty}\,[F(s) + G(s)] < \infty .$$

Then there exists $F^* > 0$ such that

$$|F(s)| \leqslant F^* \quad \text{for all } s \in R^+.$$

Let $\phi \in C([-h, 0], R)$ and $\phi(\xi) \geq k$ for $\xi \in [-h, 0]$, $x_0 = \phi(0)$. (Here, $k$ is defined in (2.4).) Define

$$\tilde{g} = \max\{g(\phi(s)) \colon s \in [-h, 0]\}$$

and

$$y_0 = 2 + h\tilde{g} + 2F^* + \int_{x_0}^{\infty} g(s)\,ds + M.$$

Let $(x(t), y(t))$ be the solution of (2.5) with $x(s) = \phi(s)$ for $s \in [-h, 0]$ and $y(0) = y_0$. We claim that $y(s) > 1$ for all $s \in R^+$.

Now suppose that there exists $t_1 > 0$ such that $y(t_1) = 1$ and $y(s) > 1$ on $[0, t_1)$. Consequently, $x(t)$ is increasing on $[0, t_1)$.

*Case* 1. Suppose that $t_1 \leq h$. Integrate the second equation in (2.5) from 0 to $t_1$ to obtain

$$y(t_1) = y_0 - \int_0^{t_1} g(x(s-h))\,ds - \int_0^{t_1} f(x(s))\,x'(s)\,ds + E(t_1) \geq$$

$$\geq y_0 - h\tilde{g} - \int_{x(0)}^{x(t_1)} f(s)\,ds - M \geq y_0 - h\tilde{g} - 2F^* - M > 1,$$

a contradiction.

*Case* 2. Suppose that $t_1 > h$. Integrating the second equation in (2.5) from 0 to $t_1$, we have

$$y(t_1) = y_0 - \int_0^{h} g(x(s-h))\,ds - \int_h^{t_1} g(x(s-h))\,ds - \int_0^{t_1} f(x(\xi))\,x'(\xi)\,d\xi + E(t_1) \geq$$

$$\geq y_0 - h\tilde{g} - \int_0^{t_1-h} g(x(s))\,ds - \int_0^{t_1} f(x(s))\,x'(s)\,ds + E(t_1) \geq$$

$$\geq y_0 - h\tilde{g} - \int_0^{t_1-h} g(x(s))\,x'(s)\,ds - \int_0^{t_1} f(x(s))\,x'(s)\,ds + E(t_1) \geq$$

$$\geq y_0 - h\tilde{g} - \int_{x_0}^{\infty} g(s)\,ds - \int_{x(0)}^{x(t_1)} f(s)\,ds - M \geq y_0 - h\tilde{g} - \int_{x_0}^{\infty} g(s)\,ds - 2F^* - M > 0,$$

a contradiction. Thus, $y(s) > 1$ on $R^+$ and $x(t) > t + x_0 \to \infty$ as $t \to \infty$. This completes the proof of Theorem 1.

## 3. – Convergence of solutions.

We first consider the half-linear equation

$$x'' + f(x)\,x' + kx(t-h) = e(t) \tag{3.1}$$

where $h \geqslant 0$, $k > 0$ and $f$, $e$ are continuous.

THEOREM 2. – Suppose that

$$f(x) > kh \qquad \text{for all } x \in R. \tag{3.2}$$

Then any pair of bounded solutions $(x_1(t), x_2(t))$ of (3.1) satisfies

$$|x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)| \to 0 \qquad \text{as } t \to \infty. \tag{3.3}$$

PROOF. – A system equivalent to (3.1) is

$$\begin{cases} x' = z - F(x) + \displaystyle\int_{t-h}^{t} kx(s)\,ds + E(t) \\ z' = -kx. \end{cases} \tag{3.4}$$

Let $(x_1(t), z_1(t))$ and $(x_2(t), z_2(t))$ be two bounded solutions of (3.4) and define $X(t) = x_1(t) - x_2(t)$, $Z(t) = z_1(t) - z_2(t)$. Then $(X(t), Z(t))$ satisfies

$$\begin{cases} X' = Z - (F(x_1) - F(x_2)) + \displaystyle\int_{t-h}^{t} kX(s)\,ds \\ Z' = -kX. \end{cases} \tag{3.5}$$

Now define

$$V(t) = kX^2(t) + Z^2(t) + \int_{-h}^{0}\int_{t+s}^{t} k^2X^2(v)\,dv\,ds$$

so that

$$\begin{aligned} V'(t) = 2kX\Bigl[Z - (F(x_1) - F(x_2)) + \int_{t-h}^{t} kX(s)\,ds\Bigr] + \\ + 2Z[-kX] + hk^2X^2(t) - k^2\int_{t-h}^{t} X^2(s)\,ds \leqslant \\ \leqslant -2k\{(F(x_1) - F(x_2))/(x_1 - x_2)\}X^2 + 2hk^2X^2. \end{aligned} \tag{3.6}$$

Since $x_1(t)$ and $x_2(t)$ are bounded, it follows that $V'(t) \leqslant -\mu X^2(t)$ for some $\mu > 0$ by (3.2). This implies that $X(t) \in L^2(0, \infty)$ and, in fact, $X(t) \to 0$ as $t \to \infty$ since $X'(t)$ is bounded. Moreover, by (3.5) and the definition of $V(t)$ it follows that $Z(t) \to C$ as $t \to \infty$ for some constant $C$. Since $x_1(t)$ and $x_2(t)$ are bounded, we conclude that

$$F(x_1) - F(x_2) \to 0 \quad \text{as } t \to \infty .$$

Thus, $C = 0$ by (3.5) and

$$|x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)| = |X(t)| + |X'(t)| \to 0$$

as $t \to \infty$, as required.

We turn now to (2.1) and consider its equivalent system

$$\text{(3.7)} \qquad \begin{cases} x' = z - F(x) + \displaystyle\int_{t-h}^{t} g(x(s))\, ds + E(t), \\ z' = -g(x), \end{cases}$$

where $h \geqslant 0$, $f$, $g'$, $e$ are continuous.

THEOREM 3. – Suppose there are positive constants $p$, $q$, and $\theta$, with $\theta \in (0,1)$, such that

$$\text{(3.8)} \qquad (g'(x) - qf(x) - p)^2 \leqslant 4\theta(pf(x) - qg'(x) - H(x))\, q ,$$

where

$$\text{(3.9)} \qquad \begin{cases} H(x) = \dfrac{1}{\theta}(Nph + h(p + q\delta)|g'(x)|^2), \\ \delta = h/(1-\theta), \qquad N > 1 . \end{cases}$$

Then any pair of bounded solutions $(x_1(t), z_1(t))$, $(x_2(t), z_2(t))$, of (3.7) satisfies

$$|x_1(t) - x_2(t)| + |z_1(t) - z_2(t)| \to 0 \quad \text{as } t \to \infty .$$

PROOF. – Let $X(t) = x_1(t) - x_2(t)$, $Z(t) = z_1(t) - z_2(t)$, so that

$$\text{(3.10)} \qquad \begin{cases} X'(t) = Z(t) - (F(x_1) - F(x_2)) + \displaystyle\int_{t-h}^{t} (g(x_1(s)) - g(x_2(s)))\, ds , \\ Z'(t) = -\big(g(x_1(t)) - g(x_2(t))\big) . \end{cases}$$

Now define

$$V(t) = pX^2(t) - 2qX(t)Z(t) + Z^2(t)$$

so that

$$V'(t) = 2(pX - qZ)\left(Z - F(x_1) + F(x_2) + \int_{t-h}^{t} (g(x_1(s)) - g(x_2(s)))\,ds\right) + \\ + 2(Z - qX)(-g(x_1) + g(x_2)).$$

To simplify the notation, define

$$F_{11}(t) = \begin{cases} (F(x_1(t)) - F(x_2(t)))/(x_1(t) - x_2(t)), & \text{if } x_1(t) \neq x_2(t), \\ f(x_1(t)), & \text{if } x_1(t) = x_2(t), \end{cases}$$

$$g_{11}(t) = \begin{cases} (g(x_1(t)) - g(x_2(t)))/(x_1(t) - x_2(t)), & \text{if } x_1(t) \neq x_2(t), \\ g'(x_1(t)), & \text{if } x_1(t) = x_2(t), \end{cases}$$

Then

$$V'(t) = -2(pF_{11}(t) - qg_{11}(t))X^2 - 2qZ^2 + 2(p + qF_{11}(t) - g_{11}(t))XZ + \\ + 2(pX - qZ)\int_{t-h}^{t} g_{11}(s)X(s)\,ds.$$

Let $A = pF_{11}(t) - qg_{11}(t)$, $B = p + qF_{11}(t) - g_{11}(t)$ so that

$$V'(t) = -2[A(X - (B/2A)Z)^2 + \{(4qA - B^2)Z^2/4A\}] + 2(pX - qZ)\int_{t-h}^{t} g_{11}(s)X(s)\,ds.$$

$$V'(t) = -2[q(Z - (B/2q)X)^2 + \{(4qA - B^2)/4q\}X^2] + 2(pX - qZ)\int_{t-h}^{t} g_{11}(s)X(s)\,ds$$

and

$$V'(t) \leqslant -\{(4qA - B^2)/4q\}X^2 - \{(4qA - B^2)/4A\}Z^2 + 2(pX - qZ)\int_{t-h}^{t} g_{11}(s)X(s)\,ds,$$

$$V'(t) \leqslant -\{(4qA - B^2)/4q\}X^2 - \{(4qA - B^2)/4A\}Z^2 + \\ + phX^2 + p\int_{t-h}^{t} |g_{11}(s)|^2\,|X(s)|^2\,ds + (qh/\delta)Z^2 + q\delta\int_{t-h}^{t} |g_{11}(s)|^2\,|X(s)|^2\,ds.$$

Let

$$V_1(t) = V(t) + (p + q\delta)\int_{-h}^{0}\int_{t+v}^{t} |g_{11}(s)|^2 |X(s)|^2 \, ds\, dv$$

so that

$$V_1'(t) \leqslant -\{(4qA - B^2)/4q\}X^2 - \{(4qA - B^2)/4A\}Z^2 +$$
$$+ h(p + (p + q\delta)|g_{11}(t)|^2)X^2 + q(1-\theta)Z^2 \leqslant$$
$$\leqslant -\{[4q\theta(A - H_{11}) - B^2]/4q\}X^2 - \{(4q\theta A - B^2)/4A\}Z^2 - h(N-1)pX^2(t),$$

where

$$H_{11} = (1/\theta)[Nph + h(p + q\delta)|g_{11}|^2].$$

Notice that

$$F_{11}(t) = \int_0^1 f(sX(t) + x_2(t))\, ds,$$

$$g_{11}(t) = \int_0^1 g'(sX(t) + x_2(t))\, ds.$$

Then

$$B^2 = \left[\int_0^1 (g'(sX(t) + x_2(t)) - qf(sX(t) + x_2(t)) - p)\, ds\right]^2 \leqslant$$

$$\leqslant \int_0^1 (g'(sX(t) + x_2(t)) - qf(sX(t) + x_2(t)) - p)^2\, ds \leqslant$$

$$\leqslant 4\theta q\int_0^1 (pf(sX(t) + x_2(t)) - qg'(sX(t) + x_2(t)) - H(sX(t) + x_2(t))\, ds \leqslant 4\theta(A - H_{11})q.$$

By (3.8) there is a constant $\mu > 0$ such that

$$V_1'(t) \leqslant -\mu(X^2(t) + Z^2(t)).$$

Since $x_1(t)$, $x_2(t)$ are bounded, it follows that

$$|X(t)| + |Z(t)| \to 0 \quad \text{as } t \to \infty.$$

This completes the proof.

We now show that (2.6) is a necessary condition that every pair of solutions $x_1(t)$,

$x_2(t)$ of

$$x'' + f(x)x' + g(x(t-h)) = e(t) \tag{3.11}$$

satisfy

$$|x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)| \to 0 \quad \text{as } t \to \infty . \tag{3.12}$$

THEOREM 4. – Suppose that $f$, $g$, $e$ are continuous with $e(t)$, $E(t)$ bounded and there is a $k \geq 0$ such that $xF(x) \geq 0$, $xg(x) \geq 0$ for $|x| \geq k$. Then every pair of solutions of (3.11) satisfies (3.12) only if (2.6) holds.

PROOF. – Suppose that (2.6) fails. To be definite, we assume that

$$\int_0^\infty [f(s) + |g(s)|]\, ds < \infty .$$

Write (3.11) as the equivalent system

$$\begin{cases} x' = y \\ y' = -f(x)y - g(x(t-h)) + e(t). \end{cases} \tag{3.13}$$

Let $\phi \in C([-h, 0], R)$ and $\phi(\xi) \geq k$ for $\xi \in [-h, 0]$, $x_0 = \phi(0)$. We define

$$\tilde{g} = \max \{g(\phi(\xi)) \colon \xi \in [-h, 0]\}$$

and

$$y_1^0 = 2 + h\tilde{g} + \int_{x_0}^\infty [f(\xi) + g(\xi)]\, d\xi + M$$

where $M$ is defined in (2.3). Let $(x_1(t), y_1(t))$ be the solution of (3.13) with $x_1(\xi) = \phi(\xi)$, $\xi \in [-h, 0]$ and $y_1(0) = y_1^0$. By the proof of Theorem 1 (for the only if part), it follows that $y_1(t) > 1$, $x_1(t) > t + x_0$ for $t \in R^+$.

Next, let

$$y_2^0 = y_1^0 + 2 + \int_{x_0}^\infty [f(s) + g(s)]\, ds + h\tilde{g} .$$

Let $(x_2(t), y_2(t))$ be the solution of (3.13) with $x_2(\xi) = \phi(\xi)$, $\xi \in [-h, 0]$ and $y_2(0) = y_2^0$. Then $y_2(t) > 1$ and $x_2(t) \geq t + x_0$ for $t \in R^+$.

Now consider

$$y_2(t) - y_1(t) = y_2^0 - y_1^0 - \int_0^t f(x_2(s))\, y_2(s)\, ds +$$

$$+ \int_0^t f(x_1(s))\, y_1(s)\, ds - \int_0^t g(x_2(s-h)) + \int_0^t g(x_1(s-h))\, ds,$$

$$y_2(t) - y_1(t) \geqslant y_2^0 - y_1^0 - \int_0^t f(x_2(s))\, y_2(s)\, ds - \int_0^t g(x_2(s-h))\, ds,$$

$$= y_2^0 - y_1^0 - \int_{x_0}^{x_2(t)} f(s)\, ds - \int_{-h}^{t-h} g(x_2(s))\, ds.$$

If $0 \leqslant t \leqslant h$, then

$$y_2(t) - y_1(t) \geqslant y_2^0 - y_1^0 - \int_{x_0}^{\infty} f(s)\, ds - h\tilde{g} \geqslant 1.$$

If $h \leqslant t < \infty$, then

$$y_2(t) - y_1(t) \geqslant y_2^0 - y_1^0 - \int_{x_0}^{\infty} f(s)\, ds - h\tilde{g} - \int_0^{t-h} g(x_2(s))\, ds,$$

$$y_2(t) - y_1(t) \geqslant y_2^0 - y_1^0 - \int_{x_0}^{\infty} f(s)\, ds - h\tilde{g} - \int_{x_0}^{\infty} g(s)\, ds \geqslant 1.$$

Hence, $y_2(t) - y_1(t) \geqslant 1$ and $x_2(t) - x_1(t) \geqslant t$. This completes the proof of Theorem 4.

## REFERENCES


[1] T. A. BURTON, *The generalized Liénard equation*, SIAM J. Control Ser. A, 3 (1965), pp. 223-230.

[2] T. A. BURTON, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, Florida (1985).

[3] T. A. BURTON - L. HATVANI, *Stability theorems for nonautonomous functional differential equations by Liapunov functionals*, Tôhoku Math. J., 41 (1989), pp. 65-104.

[4] T. A. BURTON - C. G. TOWNSEND, *On the generalized Liénard equation with forcing function*, J. Diff. Eq., 4 (1968), pp. 620-633.

[5] T. A. BURTON - BO ZHANG, *Uniform ultimate boundedness and periodicity in functional differential equations*, Tôhoku Math. J., 42 (1990), pp. 93-100.

[6] J. R. GRAEF, *On the generalized Liénard equation with negative damping*, J. Diff. Eq., **12** (1972), pp. 34-62.

[7] J. K. HALE - O. LOPES, *Fixed point theorems and dissipative processes*, J. Diff. Eq., **13** (1973), pp. 391- 402.

[8] T. HARA - T. YONEYAMA, *On the global center of generalized Liénard equation and its application to stability problems*, Funkcialaj Ekvacioj, **28** (1985), pp. 171-192.

[9] N. N. KRASOVSKII, *Stability of Motion*, Stanford University Press, Stanford, California (1963).

[10] S. MURAKAMI, *Asymptotic behavior of solutions of some differential equations*, J. Math. Anal. Appl., **109** (1985), pp. 534-545.

[11] G. SANSONE - R. CONTI, *Non-linear Differential Equations*, MacMillan, New York (1964).

[12] A. SOMOLINOS, *Periodic solutions of the sunflower equation*, Quart. Appl. Math., **35** (1978), pp. 465-478.

[13] J. SUGIE, *On the generalized Liénard equation without the Signum condition*, J. Math. Anal. Appl., **128** (1987), pp. 80-91.

[14] J. SUGIE, *On the boundedness of solutions of the generalized Liénard equation without the Signum condition*, Nonlinear Analysis, **11** (1987), pp. 1391-1397.

[15] G. VILLARI, *On the qualitative behaviour of solutions of Liénard equation*, J. Diff. Eq., **67** (1987), pp. 269-277.

[16] G. VILLARI - F. ZANOLIN, *On a dynamical system in the Liénard plane. Necessary and sufficient conditions for the intersection with the vertical isocline and application*, Funcialaj Ekvacioj, **33** (1990), pp. 19-38.

[17] P. WALTMAN - T. F. BRIDGLAND, *On convergence of solutions of the forced Liénard equation*, J. Math. Phys., **44** (1965), pp. 284-287.

[18] T. YOSHIZAWA, *Asymptotic behavior of solutions of differential equations*, in *Differential Equations qualitative Theory (Szeged, 1984)*, Colloq. Math. Soc. János Bolyai, **47**, North-Holland, Amsterdam (1984), pp. 1141-1172.

[19] BO ZHANG, *On the retarded Liénard equation*, Proc. Amer. Math. Soc., in press.