Annali di Matematica pura ed applicata
(IV), Vol. CLVII (1990), pp. 285-367

Sharp Regularity Theory for Second Order Hyperbolic Equations of Neumann Type (*).

PART I. – L_2 Nonhomogeneous Data.

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Summary. – *We consider the mixed problem for a general, time independent, second order hyperbolic equation in the unknown u, with datum $g \in L_2(\Sigma)$ in the Neumann B.C., with datum $f \in L_2(Q)$ in the right hand side of the equation and, say, initial conditions $u_0 = u_1 = 0$. We obtain sharp regularity results for u in Q and $\dot{u}|_\Sigma$ in Σ, by a pseudo-differential approach on the half-space.*

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(*) Entrata in Redazione il 31 ottobre 1988.

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The content of the present paper was partially presented by the authors in various places including: (i) a series of lectures at the Scuola Normale Superiore di Pisa, Italy, during the summer 1985; (ii) the International Conference on « Differential Equations in Banach spaces » held at University of Bologna, Italy, July 1985, see [L-1]; (iii) the IFIP Conference on Optimal Control for systems governed by partial differential equations, held at the University of Santiago de Campostela, Spain, July 1987, see [T.2]. Moreover, the main results are reported in [L-T.3], [L-T.7].

Research partially supported by the National Science Foundation under Grants DMS-83-016668 and DMS-87-96320.

1. – Regularity problem, preliminaries, and statement of main results.

Let $x > 0$ be a scalar positive variable, t be a real variable, and $y = [y_1, \ldots, y_{n-1}]$ be an $(n-1)$-dimensional vector with real components. In symbols: $x \in R^1_{x+}$; $t \in R^1_t$; $y \in R^{n-1}_y$. Let

$$\Omega \equiv R^1_{x+} \times R^{n-1}_y\,, \qquad \Gamma \equiv R^{n-1}_y = \Omega|_{x=0} \qquad \dim \Omega = n \geqslant 2\,. \tag{1.1}$$

be, respectively, an n-dimensional half-space Ω with boundary Γ. On Ω we consider the second order differential operator

$$P(x, y; D_t, D_x, D_y) \equiv -aD_t^2 + \sum_{i,j=1}^{n-1} a_{ij} D_{y_i} D_{y_j} + 2\sum_{j=1}^{n-1} a_{nj} D_{y_j} D_x + D_x^2 \tag{1.2}$$

with space-dependent, but time-independent coefficients

$$a \equiv a(x, y)\,, \qquad a_{ij} \equiv a_{ij}(x, y)\,; \qquad [x, y] \in \Omega\,;\ i, j = 1, \ldots, n-1 \tag{1.3}$$

satisfying the symmetricity condition $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n-1$. Here and throughout we use the notation

$$D_t \equiv \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}; \quad D_x \equiv \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x}, \quad D_{y_j} \equiv \frac{1}{\sqrt{-1}} \frac{\partial}{\partial y_j} \quad \text{etc.}.$$

On Γ, the boundary of the half-space Ω, we consider the first order operator

$$B(y; D_x, D_y) \equiv D_x + \sum_{j=1}^{n-1} b_j D_{y_j} \quad \text{on } x = 0 \tag{1.4}$$

with space-dependent, but time-independent coefficients

$$b_j \equiv b_j(y), \quad y \in \Gamma. \tag{1.5}$$

The present paper investigates *regularity properties* of the solution $u(t, x, y)$ of the following second order hyperbolic mixed problem with Neumann boundary conditions

$$P(x, y; D_t, D_x, D_y)u = f(t, x, y) \quad \text{on } \Omega, t > 0, \tag{1.6a}$$

$$B(y; D_x, D_y)u = g(t, y) \quad \text{on } \Gamma, t > 0, \tag{1.6b}$$

$$u|_{t=0} = u_0; \quad D_t u|_{t=0} = u_1 \quad \text{on } \Omega, t = 0, \tag{1.6c}$$

at least for a few specific fundamental function spaces for f and g. Other classes of functions spaces are examined in a subsequent paper [L-T.5]. Generally, we are interested in the continuity of the map from the data (u_0, u_1, f, g) in preassigned function spaces (possibly, subject to compatibility conditions) into the solution u, u_t, ... and possibly its trace $u|_\Gamma$, ... in suitable (optimal) function spaces. Throughout the paper, problem (1.6) will be subject to the following *assumptions*:

(i) the coefficients a, a_{ij}, a_{nj} of P and b_j of B are assumed real, time independent, sufficiently smooth in the space variables, and constant outside a compact set $\mathcal{K}_{xy}$ of $R^1_{x^+} \times R^{n-1}_y = \Omega$;

(ii) the boundary Γ $(x = 0)$ is non-characteristic for P and P is «regularly hyperbolic with respect to t», i.e. the characteristic polynomial of P.

$$p(x, y; \tau, \xi, \eta) \equiv -a\tau^2 + \sum_{i,j=1}^{n-1} a_{ij}\eta_i\eta_j + 2\xi \sum_{j=1}^{n-1} a_{nj}\eta_j + \xi^2, \tag{1.7a}$$

$$p(x, y; \tau, \xi, \eta) \equiv -a\tau^2 + \left[\xi + \sum_{j=1}^{n-1} a_{nj}\eta_j\right]^2 + \sum_{i,j=1}^{n-1} a_{ij}\eta_i\eta_j - \left(\sum_{j=1}^{n-1} a_{nj}\eta_j\right)^2 \tag{1.7b}$$

has *two real* and *distinct roots* in τ, for $(x, y) \in \Omega$ and (ξ, η) on the unit sphere

$\xi^2 + |\eta|^2 = 1$, where $|\eta|^2 = \sum_{j=1}^{n-1} \eta_j^2$. If we consider $\eta = 0$ and $\xi = 1$, this requirement yields the condition

$$\min a(x, y) > 0 \quad \text{in } \Omega ; \tag{1.8}$$

moreover, if we consider the points of the unit sphere in (ξ, η) which lie also on the hyperplane $\xi + \sum_{j=1}^{n-1} a_{nj}\eta_j = 0$, this requirement yields the necessary condition, which is plainly also sufficient, that the quadratic form in η

$$d(x, y; \eta) \equiv a^2(x, y) \left\{ \sum_{i,j=1}^{n-1} a_{ij}(x, y)\eta_i\eta_j - \left(\sum_{j=1}^{n-1} a_{nj}(x, y)\eta_j \right)^2 \right\} \tag{1.9}$$

(independent of ξ) be positive definite

$$d(x, y; \eta) > c|\eta|^2 \quad \text{uniformly in } (x, y) \in \Omega\,,\ c > 0\,; \tag{1.10}$$

(iii) the first order operator $\tilde{D}_x$ defined by

$$\tilde{D}_x \equiv D_x + \sum_{j=0}^{n-1} a_{nj}(x, y) D_{y_j} \tag{1.11}$$

restricted on the boundary Γ, coincides with B; i.e.

$$B \equiv \tilde{D}_x|_{x=0}\,; \quad \text{i.e. } b_j(y) \equiv a_{nj}(0, y)\,,\ j = 1, \dots, n-1\,. \tag{1.12}$$

The following results are known and provide the a-priori regularity needed in the subsequent development.

LEMMA 1.1. – Let $u_0 = u_1 = 0$ in (1.6c) and let $0 < T < \infty$.

a) Let $g \equiv 0$ and $f \in L_1(0, T; L_2(\Omega))$ in (1.6). Then

$$u \in C([0, T]; H^1(\Omega))\,, \qquad u_t \in C([0, T]; L_2(\Omega))$$

(a fortiori $u \in H^1([0, T] \times \Omega)$) continuously.

b) Let $f \equiv 0$ and $g \in L_2(0, T; L_2(\Gamma))$ in (1.6). Then

$$u \in C([0, T]; H^{1/2}(\Omega))\,, \qquad u_t \in C([0, T]; H^{-1/2}(\Omega))$$

(a fortiori $u \in H^{1/2}([0, T] \times \Omega)$).

(LIONS-MAGENES, vol. II, p. 120 provide only $L_2(0, T; \cdot)$; but this can be improved to $C([0, T]; \cdot)$ with the same space regularity, as e.g. in [L-T.2], [L-T.4], [L-T.5].) □

Trace theory applied to Lemma 1.1*a*) then gives

$$\left.\begin{aligned} &f\in L_1(0, T; L_2(\Omega))\\ &g=0\\ &u_0=u_1=0\end{aligned}\right\} \to u|_\Sigma \in C([0, T]; H^{1/2}(\Gamma)) \text{ continuously}. \tag{1.13}$$

A main goal of the present paper si to show the following results when $\dim\Omega\geqslant 2$.

MAIN THEOREM 1.2. – Let $g=0$ and $u_0=u_1=0$ and let $f\in L_2(Q_+)$, $Q_+=R^1_{t+}\times\Omega$. Then,

a) if $\Sigma_+=R^1_{t+}\times\Gamma$, the trace $u|_\Sigma$ of the solution to (1.6) satisfies $u|_\Sigma\in H^{3/5}(\Sigma_+)$ continuously: there is a constant $C>0$ independent of f such that

$$\|u|_\Sigma\|_{H^{3/5}(\Sigma_+)}\leqslant C\|f\|_{L_2(Q_+)}. \tag{1.14}$$

b) In the special cases where the coefficients a_{ij}, $i, j=1, ..., n-1$; a_{nj}, $j=1, ..., n-1$ either do *not* depend on x, or else do *not* depend on y, then $u|\Sigma\in H^{2/3}(\Sigma_+)$ continuously: there is a constant $C>0$ independent of f such that

$$\|u|_\Sigma\|_{H^{2/3}(\Sigma_+)}\leqslant C\|f\|_{L_2(Q_+)}. \tag{1.15}$$

REMARKS 1.1.

(i) The general case (1.14) represents an improvement by «1/10» ($1/2+1/10=3/5$) in the space regularity of the trace over (1.13).

(ii) Let Ω be a smooth open bounded domain in R^n, $\dim\Omega\geqslant 2$. Then, Theorem 1.2 provides regularity results for a general Ω, Eq. (1.14), as well as for the case where the coefficients of the spatial partial differential operator a_{ij} depend near the boundary either only on the tangential direction, or else only on the direction normal to the boundary, Eq. (1.15). In addition to these the following results for specialized geometries hold true, when the operator P in (1.6*a*) is $Pu=\Box u=u_{tt}-\Delta u$ (i.e. the spatial differential operator on Ω is the Laplacian) and $g=0$: the map

$$\{f, u_0, u_1\}\to u|_\Sigma\colon L_2(Q_+)\times H^1(\Omega)\times L_2(\Omega)\to Y$$

is continuous, where

a) $Y=H^{3/4-\varepsilon}(\Sigma_+)$, $\forall\varepsilon>0$, when $\Omega=$ parallelopiped;

b) $Y=H^{2/3}(\Sigma_+)$, when $\Omega=$ sphere;

(while $Y=H^{3/5}(\Sigma_+)$ and $Y=H^{2/3}(\Sigma_+)$ in the cases of Theorem 1.2 *a*), *b*), respectively).

The proof of parts *a*), *b*) can be given by use of the same techniques (eigenfunction expansion for the solution followed by Fourier transform in time) which were employed in [L-T.1] to obtain corresponding results for the *interior* regularity under non homogeneous boundary conditions $g \in L_2(\Sigma_+)$, i.e. the corresponding *dual* problem. (These interior results will be stated explicitly in Remarks 1.2 (ii), (iii), below.)

(iii) Addition of a *first* order differential operator to P does not affect the results. □

A second main result of this paper is the following

MAIN THEOREM 1.3. – Let $f = 0$, $u_0 = u_1 = 0$, and $g \in L_2(\Sigma_+)$. Then, continuously for any $\varepsilon > 0$:

a)

(1.16) $\quad u \in H^{3/5-\varepsilon}(Q_+) \quad$ (improvement by $1/10 - \varepsilon$ over Lemma 1.1 *b*))

and

(1.1) $$u|_\Sigma \in H^{1/5-\varepsilon}(\Sigma_+)\,.$$

b) In the special cases where the coefficients a_{ij}, a_{nj}, $i, j = 1, \ldots, n-1$, *either do not depend on* x, or *else do not depend on* y, then

(1.18) $$u \in H^{2/3}(Q_+)$$

and

(1.19) $$u|_\Sigma \in H^{1/3}(\Sigma_+)\,. \qquad \square$$

REMARKS 1.2.

(i) For dim $\Omega \geqslant 2$ and the Laplacian case, one can show that $u \notin H^{3/4+\varepsilon}(Q)$, $\forall \varepsilon >$ [L-T.3].

(ii) Result (1.17) is a *regularity result.* Trace theory applied to interior regularity (1.16) gives only $H^{3/5-\varepsilon-1/2=1/10-\varepsilon}(\Sigma_+)$, a result worse than (1.17) by « 1/10 ». Similarly, trace theory applied to (1.18) gives $H^{2/3-1/2=1/6}(\Sigma_+)$, a result worse than (1.19) by « 1/6 ».

(iii) The regularity in (1.18)-(1.19) coincides with that proved *directly*, by eigenfunction expansions, for the Laplacian Δ on a *sphere* $= \Omega$ [L-T.1].

(iv) Direct computations, by eigenfunction expansions, with the Laplacian on a parallelepiped Ω produced $u \in H^{3/4-\varepsilon}(Q_+)$, $u|_\Sigma \in H^{2/3-\varepsilon}(\Sigma_+)$, $\varepsilon > 0$ [L-T.1].

(v) When Ω is a smooth bounded domain in R^n, one obtains a fortiori from (1.17) that with $f = u_0 = u_1 = 0$, the map $g \to u|_\Sigma$ is *compact* from $L_2(\Sigma_+)$ to $H^{1/5-\delta}(\Sigma_+)$, for any fixed $\delta > 0$. The special case $\delta = 1/5$, i.e. compactness of said map from $L_2(\Sigma_+)$ into itself, plays an important role in the study of the quadratic cost optimal control problem with control function $L_2(\Sigma_+)$ and with boundary « observation » $u|_\Sigma$ and related differential, operator—Riccati equation, see [L-T.6]. □

The proofs of Theorem 1.2 and 1.3 are very lengthy and technical and are given in the subsequent sections.

Acknowledgement. – We wish to thank J. L. LIONS for some correspondence exchanged during May 1984 which included a proof by J. L. Lions and a different proof by the authors of the trace result

$$f = u_0 = u_1 = 0\,, \qquad g \in L_2(\Sigma) \to u|_\Sigma \in L_2(\Sigma)\,,$$

see Remark 7.1.

2. – Comparison with the case of compactly supported data.

The present article is a companion paper to our work [L-T.3], which is chiefly devoted to the important special case of problem (1.6) where, say $g \equiv 0$, and where in addition the data f, u_0, u_1 are compactly supported away from the boundary Γ. This case was previously studied also in [S.3] by different methods. A study with general data f, g—both smoother than, or less smooth than, L_2 in time and space—is carried out in [L-T.5].

Let $g \equiv 0$ in (1.6*b*). Then, Theorem 1.2, complemented by Remarks 1.1 (ii), points out the property that the regularity of the trace $u|_\Sigma$ depends in general on the geometry. (This is in contrast with the corresponding *Dirichlet problem*, see Remark 2.1 below). It is instructive to compare these results with those that one obtains when the assumption is added that the data are compactly supported away from the boundary. In this case it suffices to take $Pu = \Box u = u_{tt} - \Delta u$. By the principle of superposition, we may consider the following two cases.

THEOREM 2.1 [S.3], [L-T.3]. – Consider problem (1.6) with $Pu = \Box u = u_{tt} - \Delta u$, Ω as in (1.1) and $u_0 = u_1 = 0$. Assume that $f \in L_2(Q)$, $Q = \Omega \times (0, T]$ and, moreover, that

$$f \text{ has compact support contained in } \Omega\,. \tag{2.1}$$

Then, for any $0 < T < \infty$, the trace $u|_\Sigma$ of the solution u satisfies:

$$u|_\Sigma = u|_{x=0} \in H^1(\Sigma)\,; \qquad \Sigma = \Gamma \times (0, T]\,. \quad \Box \tag{2.2}$$

THEOREM 2.2 [S.3], [L-T.3]. – Consider problem (1.6) with $Pu = \Box u = u_{tt} - \Delta u$, Ω as in (1.1) and $f = 0$. Assume that $u_0 \in H^1(\Omega)$ and $u_1 \in L_2(\Omega)$ and that, moreover,

$$u_0 \text{ and } u_1 \text{ have compact support contained in } \Omega . \tag{2.3}$$

Then, for any $0 < T < \infty$, the trace $u|_\Sigma$ of the solution u satisfies

$$u|_\Sigma = u|_{x=0} \in H^1(\Sigma) . \qquad \Box \tag{2.4}$$

REMARK 2.1 (Sharp trace regularity of the corresponding hyperbolic problem of *Dirichlet* type). – In an attempt to find, in addition to (1.13), a second limitation for the trace $u|_\Sigma$ of the Neumann problem (1.6), this time from below, we next consider the corresponding second order hyperbolic problem of *Dirichlet* type, which consists Eqts. (1.6*a*)-(1.6*c*) and of the homogeneous boundary condition

$$u(x, t) \equiv 0 \qquad \text{on } \Sigma = \Gamma \times (0, T] , \tag{2.5}$$

replacing (1.6*b*) on a smooth bounded $\Omega \subset R^n$, $\dim \Omega \geqslant 1$. The Dirichlet problem (1.6*a*), (1.6*c*), (2.5) admits the following trace regularity result, which was established recently (in fact, even in the case of sufficiently smooth time dependent coefficients of the spatial differential operator, see [L-2], [L-T.1], [L-T.2] and [L-L-T.1]): the map

$$\{f, u_0, u_1\} \to \frac{\partial u}{\partial \nu_{\mathcal{A}}} \colon L_2(Q) \times H_0^1(\Omega) \times L_2(\Omega) \to L_2(\Sigma); \qquad Q = \Omega \times (0, T] \tag{2.6}$$

is continuous. (Actually, the space $L_1(0, T; L_2(\Omega))$ may replace the space $L_2(Q)$ in (2.6)). In (2.6) we are considering the conormal derivative with respect to the spatial differential operator $\mathcal{A}$, which becomes the regular normal derivative $\partial/\partial \nu$, ν being an outward unit vector to Γ, when $\mathcal{A} = -\Delta$.

Since the *interior* regularity of the solution to the Dirichlet problem (1.6*a*), (1.6*c*), (2.5) is the same as for the Neumann problem (1.6*a*)-(1.6*c*), i.e., is described by

$$\{f, u_0, u_1\} \in L_2(Q) \times H_0^1(\Omega) \times L_2(\Omega) \to u \in C([0, T]; H_0^1(\Omega)) , \tag{2.7}$$

with H^1 of Lemma 1.1 *a*) replaced by H_0^1 now, we see that (2.6) is an independent regularity result, *not* obtainable by applying (formally) trace theory to the interior optimal regularity (2.7). In fact, (2.6) shows that the Neumann trace of the solution of the hyperbolic problem of Dirichlet type (1.6*a*), (1.6*c*), (2.5) behaves in the space variable « 1/2 better » (in Sobolev space order) than what one would obtain by applying formally trace theory to the interior regularity (2.7). $\Box$

REMARK 2.2 (A conjecture on the Neumann problem). – On the basis of Remark 2.1, and by analogy with the more established elliptic and parabolic theory, the following conjecture has been advanced that in the case of the Neumann problem (1.6*a*)-(1.6*c*), we may perhaps have

$$\{f, u_0, u_1\} \to u|_\Sigma : L_2(Q) \times H^1(\Omega) \times L_2(\Omega) \to H^1(\Sigma)\,. \tag{2.8}$$

As a reinforcement, one may notice that statement (2.8) is precisely the one that one would obtain, if the Dirichlet trace $u|_\Sigma$ of the solution u to the Neumann problem (1.6*a*)-(1.6*c*) as in Lemma 1.1 *a*) would likewise behave «1/2 better» (as it is true for the Dirichlet case (1.6*a*), (1.6*c*), (2.5) described in Remark 2.1) than the regularity that we would get by application of trace theory as in (1.13). □

Our studies reveal that conjecture (2.8) is false in general, except for the one dimensional case, where for $Pu = u_{tt} - \Delta u$ and $\Omega = (0, +\infty)$, the half-space, where the regularity (2.8) can be verified by direct computations as in section 3 of [L-T.3]. In the general case $\dim \Omega > 1$, the situation is much more complex and is described by Theorem 1.2 and Remark 1.1 (ii).

In contrast, Theorems 2.1 and 2.2 establish that under the additional assumption that the data have compact support in Ω, conjecture (2.8) holds true. This latter assumption is *crucial* for the validity of conjecture (2.8), as shown by

THEOREM 2.3 [L-T.3]. – Consider problem (1.6) with $Pu = \Box u = u_{tt} - \Delta u$ and Ω as in (1.1). Then for any $0 < T < \infty$, the map

$$\{f, u_0, u_1\} \to u|_\Sigma : L_2(Q) \times H^1(\Omega) \times L_2(\Omega) \to H^{3/4}(\Sigma) \tag{2.9}$$

is continuous. Moreover, for $\dim \Omega \geqslant 2$

$$u|_\Sigma \notin H^{3/4+\varepsilon}(\Sigma)\,, \qquad \forall \varepsilon > 0\,. \quad \Box \tag{2.10}$$

We now let $f \equiv 0$ and $g \neq 0$ in problem (1.6). By duality or transposition, Theorem 2.3 shows that, say with $u_0 = u_1 = 0$, and for any $0 < T < \infty$, we have in general

$$g \in H^{-3/4-\varepsilon}(\Sigma) \to u \notin L_2(Q)\,, \qquad \varepsilon > 0\,. \tag{2.11}$$

A more satisfactory statement for our purposes that the gain from boundary to interior regularity, from g to u, cannot exceed 3/4 is obtained when g is in $L_2(\Sigma)$ as in the following counterexample given by the authors in 1984, in response to the proposed conjecture of Remark 2.2. This counterexample preceded [L-T.3] and in fact it is easier than the one from f to $u|_\Sigma$ given by Eq. (2.10) of Theorem 2.3; see [L.1].

COUNTEREXAMPLE. – Consider the following two dimensional problem

$$u_{tt} = u_{xx} + u_{yy} \quad \text{in } Q_+ = \Omega \times (0, \infty)\,, \tag{2.12a}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega\,, \tag{2.12b}$$

$$u_x|_{x=0} = g \quad \text{in } \Sigma_+ = \Gamma \times (0, \infty)\,, \tag{2.12c}$$

where $\Omega = \{(x, y) \in R^2 \colon x > 0\}$ and $\Gamma = \{(x, y) \in R^2 \colon x = 0\}$. With reference to problem (2.12) we shall prove that: given any $\varepsilon > 0$, there exists g_ε

$$g_\varepsilon \in L_2(\Sigma_+) \quad \text{such that } u \notin H^{3/4+\varepsilon}(Q_+)\,; \tag{2.13}$$

To the end we use the Fourier-Laplace transform, Laplace in time $t \to \tau = \gamma + i\sigma$, $\gamma > 0$, $\sigma \in R^1$, Fourier in $y \to i\eta$, $\eta \in R^1$, leaving x as a parameter

$$\hat{u}(\tau, x, \eta) = (2\pi)^{-2} \int_{R^2_{ty}} \exp[-(\gamma + i\sigma)t] \exp[-iy\eta]\, u(t, x, y)\, dt\, dy \tag{2.14}$$

where we extend the initial and boundary data to vanish identically for $t < 0$. We obtain

$$\begin{cases} \tau^2 \hat{u} = \hat{u}_{xx} - \eta^2 \hat{u} \\ \hat{u}_x(\tau, 0, \eta) = \hat{g}(\tau, \eta) \end{cases} \quad \text{or} \quad \hat{u}(\tau, x, \eta) = -\frac{\exp[-\sqrt{\tau^2 + \eta^2}\, x]}{\sqrt{\tau^2 + \eta^2}}\, \hat{g}(\tau, \eta)\,. \tag{2.15}$$

Hence

$$\int_0^\infty |\hat{u}(\tau, x, \eta)|^2\, dx = \frac{|\hat{g}(\tau, \eta)|^2}{|\tau^2 + \eta^2| 2 \operatorname{Re} \sqrt{\tau^2 + \eta^2}}\,. \tag{2.16}$$

Since

$$\tau^2 + \eta^2 = (\gamma^2 + \eta^2 - \sigma^2) + 2i\gamma\sigma \tag{2.17}$$

then for fixed $\gamma > 0$, we define the region $\mathfrak{R}^+_\gamma$, say in the first quadrant of the (σ, η) plane, by

$$\mathfrak{R}^+_\gamma = \{(\sigma, \eta) \in R^2 \colon 2\gamma\sigma \geqslant 1,\ \eta > 0\,;\ |\gamma^2 + \eta^2 - \sigma^2| \leqslant 1\} \tag{2.18}$$

comprised between the equilateral hyperbolas $\gamma^2 + \eta^2 - \sigma^2 = \pm 1$, around the equilateral hyperbola $\operatorname{Re}(\tau^2 + \eta^2) = \gamma^2 + \eta^2 - \sigma^2 = 0$ for $\sigma \geqslant 1/2\gamma$. Note that in $\mathfrak{R}^+_\gamma$, where $\sigma \sim \eta$, we have

$$2\gamma\sigma \leqslant |\tau^2 + \eta^2| = \{(\gamma^2 + \eta^2 - \sigma^2)^2 + 4\gamma^2\sigma^2\}^{1/2} \leqslant 2\sqrt{2}\,\gamma\sigma \tag{2.19a}$$

(which we shall re-write in short as $|\tau^2 + \eta^2| \sim \sigma$, as usual)

$$\arg(\tau^2 + \eta^2) \to \frac{\pi}{2} \quad \text{as } \sigma \to +\infty \tag{2.19b}$$

(more precisely:

$$\arg(\tau^2 + \eta^2) \uparrow \pi/2 \quad \text{as } \sigma \to +\infty \text{ for } \gamma^2 + \eta^2 - \sigma^2 > 0\,,$$

while

$$\arg(\tau^2 + \eta^2) \downarrow \pi/2 \quad \text{as } \sigma \to +\infty \text{ for } \gamma^2 + \eta^2 - \sigma^2 < 0)\,,$$

so that in $\mathcal{R}_\gamma^+$ we have:

$$|\tau^2 + \eta^2| \sim \sigma \sim \eta\,; \qquad \operatorname{Re}\sqrt{\tau^2 + \eta^2} \sim \sigma^{1/2} \sim \eta^{1/2} \tag{2.20a}$$

$$c_{2\gamma}\eta^{3/2} \leqslant c_{1\gamma}\sigma^{3/2} \leqslant |\tau^2 + \eta^2| \operatorname{Re}\sqrt{\tau^2 + \eta^2} \leqslant C_{1\gamma}\sigma^{3/2} \leqslant C_{2\gamma}\eta^{3/2} \quad \text{as } \sigma, \eta \to +\infty \tag{2.20b}$$

for $0 < c_{2\gamma} < C_{2\gamma} < \infty$. Thus, by (2.20), we re-write (2.16) on $\mathcal{R}_\gamma^+$ as

$$\int_0^\infty |\hat{u}(\tau, x, \eta)|^2\, dx \sim \eta^{-3/2}\, |\hat{g}(\tau, \eta)|^2 \sim \sigma^{-3/2}\, |\hat{g}(\tau, \eta)|^2 \quad \text{on } \mathcal{R}_\gamma^+\,. \tag{2.21}$$

Next, to prove Eq. (2.13), we notice that if it were true that « $g \in L_2(\Sigma_+)$ implies $u \in H^{3/4+\varepsilon}(Q_+)$ », we would equivalently have

$$\left(|\tau|^{3/4+\varepsilon} + |\eta|^{3/4+\varepsilon}\right)|\hat{u}| \in L^2(Q_+)\,, \quad \text{or } \int_{Q^+} \left(|\tau|^{3/4+\varepsilon} + |\eta|^{3/4+\varepsilon}\right)^2 |\hat{u}|^2\, dQ < \infty\,, \tag{2.22}$$

But the validity of (2.22) is contradicted by

$$\int_{\mathcal{R}_\gamma^+} \int_0^\infty |\eta|^{3/2+2\varepsilon}\, |\hat{u}(\tau, x, \eta)|^2\, dx\, d\sigma\, d\eta \sim \int_{\mathcal{R}_\gamma^+} \frac{\eta^{3/2+2\varepsilon}}{\eta^{3/2}}\, |\hat{g}(\tau, \eta)|^2\, d\sigma\, d\eta = \infty \tag{2.23}$$

which follows from (2.21) when given $\varepsilon > 0$, we choose $\hat{g} = \hat{g}_\varepsilon$ defined by

$$L_2(R_{\sigma\eta}) \ni \hat{g}_\varepsilon(\tau, \eta) = \begin{cases} 1/\eta^{\varepsilon/2} & \text{in } \mathcal{R}_\gamma^+\,, \\ 0 & \text{outside } \mathcal{R}_\gamma^+\,. \end{cases} \tag{2.24}$$

Similarly, the integral for $|\sigma|^{3/2+2\varepsilon}\, |\hat{u}|^2$ is infinite over $\mathcal{R}_\gamma^+ \times (0, \infty)$. This proves (2.13). □

We observe from (2.16) that in the « good » regions of, say, the quarter space $\sigma, \eta > 0$ where either $\sigma \leqslant c_1 \eta$, $c_1 < 1$; or else $\sigma \geqslant c_2 \eta$, $c_2 > 1$ which avoid the « bad » region $\mathcal{R}_\gamma^+$, we do have H^1 regularity for u.

We observe from (2.21) that for any $\hat{g} \in L_2(R_{\sigma\eta})$ we have

$$\int_{\mathcal{R}_\gamma^+} \int_0 (\sigma^{3/4} + \eta^{3/4})^2 |\hat{u}(\tau, x, \eta)|^2 \, dx \, d\sigma \, d\eta \sim \int_{\mathcal{R}_\gamma^+} |\hat{g}(\tau, \eta)|^2 \, d\sigma \, d\eta < \infty . \tag{2.25}$$

It can be seen that in this special case of problem (2.12) involving the Laplacian, we actually have that $\mathcal{R}_\gamma^+$ is the « bad » region in the first quadrant. (Similar considerations apply to other quadrants of the (σ, η)-plane with the regions around the hyperbola $\gamma^2 + \eta^2 - \sigma^2 = 0$ being the « bad » regions outside which the solution behaves « better ».) It can be proved in fact, that in this case of the special problem (2.12), we have

$$g \in L^2(\Sigma_+) \to u \in H^{3/4}(Q_+) \tag{2.26}$$

see [L-T.3] by use of the same techniques of splitting the (σ, η)-plane in « good » and « bad » regions that will be employed in the general case in the subsequent sections. (Albeit in a much simplified form by use of Fourier transform analysis plus Plancherel-theorem rather than pseudo-differential operators analysis.) Indeed, problem (2.12), in any dimension, worked initially as a testing ground of the techniques developed for the general case in the subsequent sections. For future purposes, note that the role of $\gamma^2 + \eta^2 - \sigma^2 = 0$ in identifying the « bad » regions, will be played in the general case by $d_1(x, y; \sigma, \eta) = 0$ with d_1 defined in (3.13*a*), (3.11*a*) below. Indeed in the case of the Laplacian we have $d_1 = \sigma^2 - \eta^2 - \gamma^2$. The above example is enlightening in that it shows what are the regions of the dual variables σ, η which are crucial for the loss of regularity of u, where a finer analysis is needed. This will be carried out in the subsequent sections in the general non constant coefficient case (in the space variable).

3. – Localized problem.

The auxiliary boundary value problem associated with (1.6) is

$$P(x, y; D_t, D_x, D_y) u = f(t, x, y) \quad \text{on } \Omega, \ -\infty < t < \infty , \tag{3.1a}$$

$$B(y; D_x, D_y) u = g(t, y) \quad \text{on } \Gamma, \ -\infty < t < \infty , \tag{3.1b}$$

where the original functions f and g are extended by zero for negative times.

Multiplying problem (3.1) by $\exp[-\gamma t]$, $\gamma > 0$, and using identity

$$\exp[-\gamma t] D_t^2 u = (D_t - i\gamma)^2 (\exp[-\gamma t] u) , \quad i = \sqrt{-1} , \tag{3.2}$$

we re-write problem (3.1) as

(3.3a) $$P(x, y; D_t - i\gamma, D_x, D_y)u_\gamma = f_\gamma(t, x, y) \quad \text{on } \Omega, \ -\infty < t < \infty,$$

(3.3b) $$B(y; D_x, D_y)u_\gamma = g_\gamma(t, y) \quad \text{on } \Gamma, \ -\infty < t < \infty,$$

(3.3c) $$u_\gamma \equiv \exp[-\gamma t]u\,; \quad f_\gamma \equiv \exp[-\gamma t]f\,; \quad g_\gamma \equiv \exp[-\gamma t]g\,, \quad \gamma > 0\,.$$

For $\gamma > 0$ (fixed), $\sigma \in R^1_\sigma$ and with $y\cdot\eta = \sum_{j=1}^{n-1} y_j \eta_j$, we set

(3.4a) $$\hat{h}(\gamma + i\sigma, x, \eta) \equiv [\mathcal{F}_{t,y} h(t, x, y)](\gamma + i\sigma, x, \eta) = \hat{h}_\gamma(\sigma, x, \eta)\,,$$

(3.4b) $$\hat{h}(\gamma + i\sigma, x, \eta) \equiv [\mathcal{F}_{t,y} h_\gamma(t, x, y)](\sigma, x, \eta) \equiv$$
$$\equiv (2\pi)^{-n} \int_{R^n_{ty}} \exp[-(\gamma + i\sigma)t] \exp[-iy\cdot\eta] h(t, x, y)\, dt\, dy\,,$$

where $\mathcal{F}_{t,y} = \hat{}$ is the Laplace-Fourier transform on h (or Fourier transform on $h_\gamma \equiv \exp[-\gamma t]h$). Thus, $D_t \to \tau = \sigma - i\gamma$ (or $\partial/\partial t \to \gamma + i\sigma$), $D_t h = \tau\hat{h}$, $\widehat{D^\alpha_y} \to \eta^\alpha$, $\widehat{D^\alpha_y h} = \eta^\alpha \hat{h}$ and $\hat{h}_\gamma(\sigma, x, \eta) = \hat{h}(\gamma + i\sigma, x, \eta)$. Using the inversion formula

(3.5) $$D^\beta_t D^\alpha_y h_\gamma(t, x, y) = (2\pi)^{-n} \int_{R^n_{\sigma\eta}} \exp[i(\sigma t + y\cdot\eta)]\, \tau^\beta \eta^\alpha \hat{h}_\gamma(\sigma, x, \eta)\, d\sigma\, d\eta$$

with $\beta = 2$ we obtain

(3.6) $$[P(x, y; D_t, D_x, D_y)u_\gamma](t, x, y) =$$
$$= (2\pi)^{-n} \int_{R^n_{\sigma\eta}} \exp[i(\sigma t + y\cdot\eta)] p(x, y, \tau, D_x, \eta)\, \hat{u}_\gamma(\sigma, x, \eta)\, d\sigma\, d\eta\,.$$

We shall next recall Hörmander symbol class $S^m_{\varrho,\delta}$ [H.1], [T.1].

DEFINITION 3.1. – Let z and ζ be two k-dimensional variables and let $v(z, \zeta)$ be a C^∞-function in z running in the open set $\mathcal{O}$ of R^k_z and in ζ running in all of R^k_ζ. Let $m, \varrho, \delta \in R$, with $0 \leqslant \varrho, \delta \leqslant 1$. Then, $s(z, \zeta)$ is said to belong to the symbol class $S^m_{\varrho,\delta}(\mathcal{O})$, $s(z, \zeta) \in S^m_{\varrho,\delta}(\mathcal{O})$, in case: for any compact set $K \in \mathcal{O}$, any multi-indeces α and β, there exists a constant $C_{K,\alpha,\beta}$ such that

(3.7) $$|D^\beta_z D^\alpha_\zeta s(z, \zeta)| \leqslant C_{K,\alpha,\beta}(1 + |\zeta|)^{m - \varrho|\alpha| + \delta|\beta|}$$

for all $z \in K$ and all $\zeta \in R^k$.

Then, the pseudo-differential operator

$$V(z)h(z) = (2\pi)^{-k}\int_{R^k_\zeta} v(z,\zeta)\exp[-iz\cdot\zeta](\mathcal{F}_z h)(\zeta)\,d\zeta \tag{3.8}$$

is likewise said to be of class $OPS^m_{\varrho,\delta}(\mathcal{O})$. □

Thus, from (1.7*a*) with $z = [t, x, y]$ and $\zeta = [\sigma, \xi, \eta]$ and $\mathcal{O}$ an open set of $R^1_t \times R^{n-1}_y \times R^1_{x^+}$, we have plainly

$$p(x, y; \sigma, \xi, \eta) \in S^2_{1,0}(\mathcal{O})\,; \qquad P(x, y; D_t - i\gamma, D_x, D_y) \in OPS^2_{1,0}(\mathcal{O})\,. \tag{3.9}$$

In addition, the following symbols and corresponding operators, defined via (3.8), will be frequently used in the sequel

$$\tilde{\xi}(x, y; \xi, \eta) = \xi + \sum_{j=1}^{n-1} a_{nj}(x, y)\eta_j \in S^1_{1,0}(R^{n-1}_y \times R_{x^+}) \tag{3.10a}$$

with corresponding operator

$$\tilde{D}_x = D_x + \sum_{j=1}^{n-1} a_{nj}(x, y) D_{y_j} \in OPS^1_{1,0}\,, \tag{3.10b}$$

$$d(x, y; \eta) = a^2(x, y)\left\{\sum_{i,j=1}^{n-1} a_{ij}(x, y)\eta_i\eta_j - \left(\sum_{j=1}^{n-1} a_{nj}(x, y)\eta_j\right)^2\right\} \in S^2_{1,0}(R^{n-1}_y \times R^1_{x^+})\,, \tag{3.11a}$$

with corresponding operator

$$D = a^2(x, y)\left\{\sum_{i,j=1}^{n-1} a_{ij}(x, y) D_{y_i} D_{y_j} - \left(\sum_{j=1}^{n-1} a_{nj}(x, y) D_{y_j}\right)^2\right\}, \tag{3.11b}$$

$$d_2(x, y; \sigma) = a(x, y)\sigma \in S^1_{1,0}(R^n_{ty} \times R^1_{x^+})\,, \tag{3.12a}$$

with corresponding operator $D_2 \sim a(x, y)(D_t + i\gamma)$

$$D_2 h = (2\pi)^{-1}\int_{R^1_\sigma} d_2(x, y; \sigma)\exp[-i\sigma t]\hat{h}(\sigma)\,d\sigma\,, \qquad \hat{} = \mathcal{F}_t\,, \tag{3.12b}$$

$$d_1(x, y; \sigma, \eta) = a(x, y)(\sigma^2 - \gamma^2) - \frac{1}{a^2(x, y)}\, d(x, y; \eta) \in S^2_{1,0}(R^n_{ty} \times R^1_{x^+}) \tag{3.13a}$$

with corresponding operator

$$D_1 h = (2\pi)^{-n}\int_{R^n_{\sigma\eta}} d_1(x, y; \sigma, \eta)\exp[-i[\sigma t + y\cdot\eta]]\hat{h}(\sigma, \eta)\,d\sigma\,d\eta, \qquad = \mathcal{F}_{ty}\,. \tag{3.13b}$$

Thus, from (1.7*b*), (3.10*a*), (3.12*a*), (3.13*a*) we obtain

$$p(x, y; \tau = \sigma - i\gamma, \xi, \eta) = \tilde{\xi}^2(x, y; \xi, \eta) - \\ - [d_1(x, y; \sigma, \eta) - 2i\gamma d_2(x, y; \sigma)] \in S^2_{1,0}(\Omega \times R^1_t)\,, \tag{3.14a}$$

with corresponding operator

$$P(x, y; D_t, D_x, D_y) = \tilde{D}^2_x - (D_1 - 2i\gamma D_2)\,. \tag{3.14b}$$

Finally, the symbol corresponding to the boundary operator B in (1.4) is

$$b(y; \xi; \eta) = \xi + \sum_{j=1}^{n-1} b_j(y)\eta_j\,. \tag{3.15}$$

In the sequel (sections 4, 5 and 6), we shall encounter the following modification of the situation described in Definition 3.1—where the symbol v depends on a parameter—which we formalize in another definition.

DEFINITION 3.2. - Let $v(x, y, \sigma, \eta)$ be a C^∞-function in all of its variables, x being a parameter. Let $m, \varrho, \delta \in R$ with $0 \leqslant \varrho, \delta \leqslant 1$. We shall say that

$$v(x, y, \sigma, \eta) \in S^m_{\varrho,\delta}(R^n_{ty x})\,, \quad \text{uniformly in } x \in R^1_{x^+}\,, \tag{3.16a}$$

in case: for any compact set K in R^{n-1}_y and any multi-indeces α, β, there exists a constant $C_{K,\alpha,\beta}$ such that

$$|D^\beta_{\substack{x\\y}} D^\alpha_{\substack{\sigma\\\eta}} v(x, y, \sigma, \eta)| \leqslant C_{K,\alpha,\beta}(|\sigma| + |\eta|)^{m-\varrho|\alpha|+\delta|\beta|}, \\ \text{as } |\sigma|, |\eta| \to \infty \text{ for all } x \in R^1_{x^+}\,, \; y \in K\,, \tag{3.16b}$$

where the constant $C_{K,\alpha,\beta}$ does not depend on $x \in R^1_{x^+}$. For the corresponding pseudodifferential operator V, defined by v through the following corresponding version of (3.8),

$$V(x, y)h(t, y) = (2\pi)^{-n}\int_{R^n_{\sigma\eta}} v(x, y, \sigma, \eta) \exp[-i(\sigma t + \eta\cdot y)](\mathcal{F}_{ty}h)(\sigma, \eta)\,d\sigma\,d\eta \tag{3.16c}$$

we shall then write that

$$V \in OPS^m_{\varrho,\delta}(R^n_{tyx})\,, \quad \text{uniformly in } x \in R^1_{x^+}\,. \qquad \square \tag{3.16d}$$

REMARK 3.1. - Actually, in our analysis below (sections 5, 6 and 7), we shall encounter the even more specialized situation where the symbol $v(x, y, \sigma, \eta)$ is *constant in the space variables x and y outside a compact set $\mathcal{K}_{xy}$ of $R^1_{x^+} \times R^{n-1}_y \equiv \Omega$,* a consequence of the assumption (i) above (1.7*a*) in section 1. As a result, the constant $C_{K,\alpha,\beta}$ will then be independent of $y \in R_y^{n-1}$ as well, in which case we shall simply write $C_{\alpha,\beta}$ (uniformly in $x, y \in R^1_{x+} \times R_y^{n-1} \equiv \Omega$). □

Thus, returning to (3.11)-(3.14) we have that the symbols $d(x, y; \eta)$; $d_2(x, y; \sigma, \eta)$; $d_1(x, y; \sigma, \eta)$; $p(x, y; \tau, \xi, \eta)$ and $b(y, \xi, \eta)$ are in their respective classes uniformly in $(x, y) \in R^1_{x+} \times R_y^{n-1} = \Omega$.

The following consequences of Definition 3.2-Remark 3.1 will be often invoked in the sequel, and thus are stated only in the cases of interest.

LEMMA 3.1. – Let the symbol $v(x, y, \sigma, \eta) \in S^m_{\varrho,\delta}(R^n_{ty})$ uniformly in $x \in R^1_{x+}$ (see (3.16)) with $1 \geqslant \varrho > \delta \geqslant 0$ and let v be, in addition, constant in x and y outside a compact set of Ω. Then, if $Q \equiv \Omega \times R^1_t$ as in section 1, and V is the corresponding operator (defined via (3.16*c*)), we have for $0 \leqslant m \leqslant 1$ and $0 \leqslant s \leqslant 1$:

a) If $m = 0$ and $0 \leqslant s \leqslant 1$

$$V\colon \text{continuous } H^s(Q) \to H^s(Q)\,. \tag{3.17a}$$

b) If $m = 1$

$$V\colon \text{continuous } H^1(Q) \to L_2(Q)\,. \tag{3.17b}$$

c) If $0 < m < 1$

$$V\colon \text{continuous } H^1(Q) \to L_2(R^1_{x+}; H^{1-m}(R^n_{ty}))\,. \tag{3.17c}$$

PROOF. – Let first $m = s = 0$. The assumption on v then implies

$$V\colon \text{continuous } L_2(R^n_{ty}) \to L_2(^n_{ty})\,, \qquad \text{uniformly in } x \in R^1_{x+}\,; \tag{3.18a}$$

i.e. if $h(t, x, y) \in L_2(Q)$, then

$$\|(Vh)(t, x, y)\|^2_{L_2(R^n_{ty})} \leqslant C\|h(t, x, y)\|^2_{L_2(R^n_{ty})} \tag{3.18b}$$

with C independent of $x \in R^1_{x+}$. This is seen by applying the argument in the proof of [T-1, Proposition 6.1, p. 49] under the additional assumption that v is constant in x and y outside a compact set of Ω, whereby $|D^\alpha_{\substack{x\\y}} v| \leqslant \text{const}_\alpha$ as required in that proof. Integrating both sides of (3.18*b*) over R^1_{x+} yields the desired conclusion for $m = s = 0$. The case $m = 1$ of part *b*) is similar.

Next, take $m = 0$, $s = 1$; then

$$V\colon \text{continuous } H^1(R^n_{ty}) \to H^1(R^n_{ty})\,, \qquad \text{uniformly in } x \in R^1_{x+} \tag{3.19a}$$

i.e. for $h(t, x, y) \in H^1(Q)$, $Q = \Omega \times R^1_t$, we have

$$\|D_t Vh\|^2_{L_2(R^n_{ty})} + \sum_j \|D_{y_j} Vh\|^2_{L_2(R^n_{ty})} \leqslant C\|h\|^2_{H^1(R^n_{ty})} \tag{3.19}$$

with C independent of $x \in R^1_{x^+}$. This again follows as in the proof of [T.1, Chapt. II, § 6] with the remark that the specification « loc » in [T.1] can be dispensed with now, because of the assumption that v be constant in x and y outside a compact set of Ω. Integrating both sides of (3.19*b*) over $R^1_{x^+}$ yields $D_t Vh$ and $D_{y_j} Vh \in L_2(Q)$, as desired. To show that $D_x Vh \in L_2(Q)$ as well, we note that the symbol of the operator $D_x V$ is precisely $D_x v(x, y, \sigma, \eta)$ (via (3.16*c*)). Then, the assumption (3.16*b*- implies

$$|D^\beta_{\substack{x\\y}} D^\alpha_{\substack{\sigma\\\eta}} D_x v(x, y, \sigma, \eta)| \leqslant C(|\sigma| + |\eta|)^{-|\alpha|\varrho + (|\beta|+1)\delta} \leqslant C(|\sigma| + |\eta|)^{1-|\alpha|\varrho + |\beta|\delta} \tag{3.20a}$$

as $|\sigma|, |\eta| \to \infty$ uniformly in $R^1_{x^+}$,

i.e.

$$D_x V \in OPS^1_{\varrho,\delta}(R^n_{ty}), \quad \text{uniformly in } x \in R^1_{x^+}. \tag{3.20b}$$

The case $m = 1$ of part *b* treated before applies and gives

$$D_x V: \text{ continuous } H^1(Q) \to L_2(Q) \tag{3.20c}$$

as desired. The case $m = 0$, $s = 1$ is complete. The other cases of part *a*) follow by interpolation. The proof of part *c*) is similar. □

REMARK 3.2 (On neglecting « loc »). – In the sequel we shall often consider an operator $A \in OPS^m_{\varrho,\delta}(\Omega)$ $0 \leqslant \delta < \varrho \leqslant 1$ with corresponding symbol which is constant in x and y outside a compact set of Q. As a consequence, we shall then conclude from [T.1, Theorem 6.5, p. 51] that in such cases

$$A: H^s(\Omega) \to H^{s-m}(\Omega)$$

where the qualification « loc » in [T.1] can now accordingly be discarded. If in addition the symbol of A is also time-independent—as it will be the case in the sequel—then the above result holds true with Ω replaced by Q. This remark will be used freely below. □

In the sequel the class of symbols—and corresponding pseudo-differential operators—singled out in Definition 3.2 and Remark 3.1 will cover, in particular, the crucial class of so-called *symbols of localization* and corresponding *operators of localization* (localizers). These will be quantitatively defined in sections 4-5. For now, qualitatively, a symbol $\chi(x, y, \sigma, \eta)$ of localization will be a C^∞-function in all of its variables, constant in x and y outside a compact set $\mathcal{K}_{xy}$ of $\Omega \equiv R^1_{x+} \times R^{n-1}_y$ (the same as the compact set for the coefficients of P and B, mentioned in assumption (i) in section 1, above (1.7*a*)), such that, for given x and y, the symbol χ is identically one in a certain (x, y)-dependent region of $R^n_{\sigma\eta}$ and decreases smoothly to vanish identically in another (x, y)-dependent region of $R^n_{\eta\sigma}$. Then $\boldsymbol{\chi}$ will be the corresponding localizer defined by

$$\boldsymbol{\chi}(x, y)h = (2\pi)^{-n}\int_{R^n_{\sigma\eta}} \chi(x, y, \sigma, \eta) \exp[i[\sigma t + \eta\cdot y]](\mathcal{F}_{ty}h)(\sigma, \eta)\, d\sigma\, d\eta \tag{3.21}$$

in agreement with (3.16*c*).

Specific symbols of localization will be defined precisely in sections 4-5. Let, for now, $\chi(x, y, \sigma, \eta)$ be any such symbol. Applying the operator $\exp[\gamma t]\chi$, with χ the corresponding localizer, on the auxiliary problem (3.3) and using (3.2) yields the following *localized problem*

$$P(x, y; D_t, D_x, D_y)(\boldsymbol{\chi}_\gamma u) = f_{\chi_\gamma}, \quad \text{in } \Omega, \ -\infty < t < \infty, \tag{3.22}$$

$$B(y; D_x, D_y)(\boldsymbol{\chi}_\gamma u) = g_{\chi_\gamma}, \quad \text{in } \Gamma, \ -\infty < t < \infty, \tag{3.22b}$$

$$f_{\chi_\gamma} \equiv \boldsymbol{\chi}_\gamma f + [P, \boldsymbol{\chi}_\gamma]u, \tag{3.22c}$$

$$g_{\chi_\gamma} \equiv \boldsymbol{\chi}_\gamma g|_{x=0} + [B, \boldsymbol{\chi}_\gamma]u|_{x=0}, \tag{3.22d}$$

where $[\cdot, \cdot]$ denotes the commutator operator and P and B in (3.22*c*)-(3.22*d*) are the operators at the left of (3.22*a*)-(3.22*b*), respectively, and where

$$\exp[\gamma t]\boldsymbol{\chi}(x, y)\exp[-\gamma t] \equiv \boldsymbol{\chi}_\gamma(x, y). \tag{3.22e}$$

REMARK 3.3. – The operators $\boldsymbol{\chi}(x, y)$ and $\boldsymbol{\chi}_\gamma(x, y) = \exp[\gamma t]\boldsymbol{\chi}(x, y)\exp[-\gamma t]$ belong to the same class. Thus, for simplicity, we shall accordingly drop the subscript and work with $\boldsymbol{\chi}$ henceforth. □

We close this section by noting a result essentially from [T.1, (4.7), p. 46], although not explicitly stated there, which will be invoked repeatedly. We shall use the notation of [T.1].

LEMMA 3.2. – Let $p(x, D) \in OPS^m_{\varrho',\delta'}$ and $q(x, D) \in OPS^\mu_{\varrho'',\delta''}$ be properly supported. Suppose $0 \leqslant \delta'' < \varrho \leqslant 1$, where

$$\varrho = \min(\varrho', \varrho'') \quad \text{and} \quad \delta = \max(\delta', \delta'').$$

Then for the commutator $[p(x, D), q(x, D)]$ we have

$$\text{symbol of } [p(x, D), q(x, D)] \in S^{m+\mu-m_1}_{\varrho,\delta}, \quad m_1 = \min[\varrho' - \delta'', \varrho'' - \delta']. \qquad \square$$

4. – Trace theorem and « energy » equalities for the localized problem (3.22).

The aim of the present section is to state and prove a trace theorem along with some fundamental energy equalities (inequalities) for the localized problem (3.22). Their version for the auxiliary problem (3.1) is need first. These results will be crucially used in the sequel.

4.1. *Statement of trace theorem and energy equalities for auxiliary problem* (3.1).

We recall the notation to be used throughout.

$$Q \equiv R_t^1 \times R_x^1 \times R_y^{n-1} \equiv R_t^1 \times \Omega\,; \qquad \Sigma \equiv R_t^1 \times R_y^{n-1} \equiv R_t^1 \times \Gamma\,,$$

$(\,,\,)_Q$, $(\,,\,)_\Omega$: $L_2(Q)$, $L_2(\Omega)$-inner products with norms $\| \;\|_Q$, $\|\cdot\|_\Omega$,

$\langle \,,\, \rangle_\Sigma$, $\langle \,,\, \rangle_\Gamma$: $L_2(\Sigma)$, $L_2(\Gamma)$-inner products with norms $| \;|_\Sigma$, $| \;|_\Gamma$.

Any other norm will be specified by a self-explanatory subindex; thus $\| \;\|_{H^\theta(Q)} =$ = norm of space $H^\theta(Q) = H^{\theta,\theta}(Q) = L_2(-\infty, \infty; H^\theta(\Omega)) \cap H^\theta(-\infty, \infty; L_2(\Omega))$ and similarly for $| \;|_{H^\theta(\Sigma)}$.

Finally, to state the trace theorem, we shall need the operator $\Lambda^\theta \equiv \Lambda^\theta_{t,y}$ defined by

$$\Lambda^\theta h \equiv (2\pi)^{-n}\int_{R^n_{\sigma\eta}} (\gamma^2 + \sigma^2 + |\eta|^2)^{\theta/2} \exp[-i(t\sigma + y\cdot\eta)]\,\hat{h}(\sigma,\eta)\,d\sigma\,d\eta \in \in OPS^\theta_{1,0}(R_t^1 \times R_y^{n-1})\,, \tag{4.1a}$$

We have [T.1, p. 51] for any $s \in R$ and any $\theta \in R$

$$\Lambda^\theta\text{: isomorphism } H^s(R^n_{ty}) \to H^{s-\theta}(R^n_{ty})\,. \tag{4.1b}$$

THEOREM 4.1 (Trace theorem).

a) For $u \in H^1(\Omega)$, we have

$$|u|^2_\Gamma = 2\,\mathrm{Im}\,(\tilde{D}_x u, u)_\Omega + i(u, wu)_\Omega\,, \tag{4.2a}$$

$$w(x,y) \equiv \sum_{j=1}^{n-1} D_{y_j} a_{n_j}(x,y)\,, \tag{4.2b}$$

$$|(u, wu)_\Omega| \le \mathrm{const}_w \|u\|^2_\Omega\,, \qquad \mathrm{const}_w = \sup_\Omega |w(x,y)| < \infty\,. \tag{4.2c}$$

b) For $v \in H^{1+\theta}(Q)$, $\theta > 0$

$$|v|^2_{H^\theta(\Sigma)} = 2\,\mathrm{Im}\,(\Lambda^\theta \tilde{D}_x v, \Lambda^\theta v)_Q + \mathcal{O}(\|v\|^2_{H^\theta(Q)})\,. \tag{4.3}$$

Theorem 4.2. – The following identities hold for the operators $P = P(x, y; D_t, D_x, D_y)$ and $B = B(y; D_x, D_y)$ (see (3.10*b*), (3.12*b*), (3.13*b*) and (4.3))

a)

$$2\,\mathrm{Im}\,(Pu, \tilde{D}_x u)_Q = |Bu|^2_\Sigma + \langle D_1 u, u\rangle_\Sigma + 4\gamma\,\mathrm{Re}\,(D_2 u, \tilde{D}_x u)_Q + \\ + i(wD_1 u, u)_Q - i(\tilde{D}_x u, w\tilde{D}_x u) + i([\tilde{D}_x, D_1]u, u)_Q + i(\tilde{D}_x u, F_1 u)_Q \tag{4.4a}$$

where we recall from assumption (1.12) that $\tilde{D}_x u = Bu$ on Σ, and where

$$\text{symbol of } F_1 = \sum_{j=1}^{n-1} f_j(x, y)\eta_j \in S^1_{1,0}(R^n_{yx^+})\,, \qquad \text{uniformly in } x, y \in R^1_{x^+}\times R^{n-1}_y \tag{4.4b}$$

$f_j(x, y) =$ smooth, depending on coefficients of P.

b)

$$2\,\mathrm{Im}\,(Pu, \tilde{D}_x u)_Q = |Bu|^2_\Sigma + \langle D_1 u, u\rangle_\Sigma + \mathcal{O}\big(\|u\|^2_{H^1(Q)}\big)\,. \qquad \square \tag{4.5}$$

Remark 4.1. – Note from (3.14*a*) that the operator $\tilde{D}_x$ has symbol

$$\frac{1}{2}\,\frac{\partial}{\partial \xi}\,p(x, y; \tau, \xi, \eta)\,. \qquad \square$$

Theorem 4.3. – The following identities (inequalities) hold for $P = P(x, y; D_t, D_x, D_y)$ and $B = B(y; D_x, D_y)$

a)

$$(Pu, u)_Q = \|\tilde{D}_x u\|^2_Q - (D_1 u, u)_Q + (\tilde{D}_x u, wu)_Q + 2i\gamma(D_2 u, u)_Q + i\langle Bu, u\rangle_\Sigma\,, \tag{4.6a}$$

$$\mathrm{Re}\,(Pu, u)_Q = \|\tilde{D}_x u\|^2_Q - \mathrm{Re}\,(D_1 u, u)_Q + \mathrm{Re}\,(\tilde{D}_x u, wu)_Q - \mathrm{Im}\,\langle Bu, u\rangle_\Sigma\,, \tag{4.6b}$$

where from assumption (1.12) $\tilde{D}_x u = Bu$ on Σ;

$$\left(1 - \frac{\varepsilon}{2}\right)\|\tilde{D}_x u\|^2_Q \leq \mathrm{Re}\,(Pu, u)_Q + \mathrm{Re}\,(D_1 u, u)_Q + \mathrm{Im}\,\langle Bu, u\rangle_\Sigma + \frac{1}{2\varepsilon}\,C_w\|u\|^2_Q \tag{4.6c}$$

for any $\varepsilon > 0$ small;

b) (variation of (4.6))

$$\|\tilde{D}_x u\|^2_Q = \mathcal{O}\{\mathrm{Re}\,(Pu, u)_Q + \mathrm{Re}\,(D_1 u, u)_Q + \|u\|^2_Q + |Bu|^2_\Sigma\}\,. \qquad \square \tag{4.7}$$

Remark 4.2. – Note that, unlike (4.6*b*), identity (inequality) (4.7) does *not* require knowledge of u on Σ. (As we shall see in the proof below, (4.7) combines both (4.6*b*) and (4.3) for $\theta = 0$.)

The next result will be used only in section 7 when dealing with the nonhomogeneous boundary case $g \neq 0$ in (1.6*b*). To this end, we first introduce a new pseudodifferential operator, denoted by $\frac{1}{2}(\partial P/\partial\tau)$, and corresponding to the symbol $\frac{1}{2}(\partial p/\partial\tau) = -a\tau$ (from (1.7)), which is defined by

$$\left(\frac{1}{2}\frac{\partial P}{\partial\tau}\right)h = (2\pi)^{-n}\int_{R^n_{\sigma\eta}} \frac{1}{2}\frac{\partial p}{\partial\tau}(x, y, \sigma)\exp\left[-i[\sigma t + \eta\cdot y]\right][\mathcal{F}_{ty}h](\sigma, x, \eta)\,d\sigma\,d\eta$$

in agreement with (3.8).

THEOREM 4.4. – *a*) With $\frac{1}{2}(\partial P/\partial\tau)$ defined above, and $P = P(x, y; D_t, D_x, D_y)$ we have the following identity:

$$(4.8)\qquad -2\,\mathrm{Im}\left(Pu, \frac{1}{2}\frac{\partial P}{\partial\tau}u\right)_Q = 2\,\mathrm{Re}\,\langle\tilde{D}_x u, D_2 u\rangle_\Sigma - 2\gamma\,\mathrm{Im}\,\langle\tilde{D}_x u, au\rangle_\Sigma + $$
$$+ 2\,\mathrm{Im}\,(\tilde{D}_x u, [wD_2 + \tilde{K}_2]u)_Q - (u, S_2 u)_Q +$$
$$+ 2\gamma\left\{\|D_2 u\|^2_Q + \|a^{1/2}\tilde{D}_x u\|^2_Q + \gamma^2\|au\|^2_Q + \left(\frac{1}{a}Du, u\right)_Q\right\} +$$
$$+ 2\gamma\,\mathrm{Re}\,(\tilde{D}_x u, [wa + (\tilde{D}_x a)]u)_Q$$

where on Σ: $\tilde{D}_x u = Bu = g$ from (1.12) and (1.6*b*), and where

$$(4.9a)\qquad \tilde{K}_2 = [\tilde{D}_x, D_2] \in OPS^1_{1,0}(R^{1+n}_{txy}), \quad \text{uniformly in } x, y \in R^1_{x+}\times R^{n-1}_y,$$

$$(4.9b)\qquad \text{symbol of } \tilde{K}_2 = \tilde{k}(x, y)\sigma, \quad \text{smooth } \tilde{k}(x, y),$$

$$(4.10a)\qquad [wD_2 + \tilde{K}_2] \in OPS^1_{1,0}(R^{1+n}_{txy}), \quad \text{uniformly in } x, y \in R^1_{x+}\times R^{n-1}_y,$$

$$(4.10b)\qquad \text{symbol of } [wD_2 + \tilde{K}_2] = \left\{\left[\sum_{j=1}^{n-1} D_{y_j}a_{nj}(x, y)\right]a(x, y) + \tilde{k}(x, y)\right\}\sigma,$$

$$(4.11a)\qquad S_2 \in OPS^2_{1,0}(R^{n+1}_{tyx}), \quad \text{uniformly in } x \in R^1_{x+}, \quad S_2 = \text{self-adjoint on } L_2(Q),$$

$$(4.11b)\qquad \text{symbol of } S_2 = \left[\sum_{j=1}^{n-1} s_j(x, y)\eta_j\right]\sigma, \quad \text{smooth } s_j(x, y),$$

S_2 is defined explicitly in (4.64) below.

b) There exist constants $C_0, \gamma_0 > 0$ such that for all $\gamma > \gamma_0$, and for all compact sets $K \in R^n_{ty}$ we have:

$$(4.12a)\qquad -2\,\mathrm{Im}\left(Pu, \frac{1}{2}\frac{\partial P}{\partial\tau}u\right)_Q \geqslant 2\,\mathrm{Re}\,\langle\tilde{D}_x u, D_2 u\rangle_\Sigma -$$
$$- 2\gamma\,\mathrm{Im}\,\langle\tilde{D}_x u, au\rangle_\Sigma + \gamma C_0\|u\|^2_{H^1(Q)}$$

for all $u \in C_0^\infty(K) \times C^\infty(R^1_{x^+})$, where

$$\frac{1}{2}\frac{\partial P}{\partial \tau} = -D_2 + ia\gamma I\,. \tag{4.12b}$$

4.2. *Application to localized problem* (3.22).

We now collect the trace theorem and energy equalities in the form in which they will be used in section 6; i.e. as they apply to the localized problem (3.22).

COROLLARY 4.5. – The following identities hold for the localized problem (3.22):

a) (trace theorem, see (4.3))

$$\|\chi u\|^2_{H^\theta(\Sigma)} = 2\,\mathrm{Im}\,(\Lambda^\theta \tilde{D}_x \chi u, \Lambda^\theta \chi u) + \mathcal{O}(\|\chi u\|^2_{H^\theta(Q)})\,, \tag{4.13}$$

b) (see (4.5))

$$\langle D_1 \chi u, \chi u\rangle_\Sigma = -\,|g_\chi|^2_\Sigma + \mathrm{Im}\,(f_\chi, \tilde{D}_x \chi u)_Q + \mathcal{O}(\|\chi u\|^2_{H^1(Q)})\,, \tag{4.14}$$

c) (see (4.6))

$$\|\tilde{D}_x \chi u\|^2_Q = (D_1 \chi u, \chi u)_Q - (\tilde{D}_x \chi u, w\chi u)_Q - 2i\gamma(D_2 \chi u, \chi u)_Q - \\ - i\langle g_\chi, \chi u\rangle_\Sigma + (f_\chi, \chi u)_Q\,, \tag{4.15}$$

d) (see (4.12)). There exist constants $c_0, \gamma_0 > 0$ such that for all $\gamma > \gamma_0$

$$-2\,\mathrm{Im}\left(f_\chi, \frac{1}{2}\frac{\partial P}{\partial \tau}(\chi u)\right)_Q \geqslant 2\,\mathrm{Re}\,\langle g_\chi, D_2 \chi u\rangle_\Sigma - 2\gamma\langle g_\chi, a\chi u\rangle_\Sigma + \gamma c_0 \|\chi u\|^2_{H^1(Q)}\,. \tag{4.16}$$

4.3. *Proofs.*

We begin with an integration-by-part Lemma for the operator $\tilde{D}_x$ in (3.10*b*).

LEMMA 4.6. – For $u, v \in H^1(\Omega)$

$$(\tilde{D}_x u, v)_\Omega = i\langle u, v\rangle_\Gamma + (u, \tilde{D}_x v)_\Omega + (u, wv)_\Omega \tag{4.17}$$

where w is defined by (4.2*b*) and $|(u, wv)_\Omega| \leqslant \mathrm{const}_w \|u\|_\Omega \|v\|_\Omega$, see (4.2*c*).

PROOF. – We perform integration by parts in x and y_j on the definition

$$(\tilde{D}_x u, v)_\Omega = (D_x u, v)_\Omega + \sum_{j=1}^{n-1} (a_{nj} D_{y_j} u, v)_\Omega$$

and use that an element of $H^1(\Omega)$ vanishes at $x = \infty$ and $y_j = \pm\infty$. □

Integrating (4.17) in t, we obtain

COROLLARY 4.7. – For $u, v \in H^1(Q)$

$$(4.18)\qquad (\tilde{D}_x u, v)_Q = i\langle u, v\rangle_\Sigma + (u, \tilde{D}_x v)_Q + (u, wv)_Q\,, \qquad |(u, wv)_Q| \leqslant \text{const}_w \|u\|_Q \|v\|_Q\,.$$

PROOF OF THEOREM 4.1. – *a*) Select $u = v \in H^1(\Omega)$ in (4.17). *b*) Integrate (4.2*a*) in t after replacing u with $\Lambda^\theta u$ to get

$$(4.19)\qquad |\Lambda^\theta u|_\Sigma^2 = 2\,\text{Im}\,(\tilde{D}_x \Lambda^\theta u, \Lambda^\theta u)_Q + i(\Lambda^\theta u, w\Lambda^\theta u)_Q = \\ = 2\,\text{Im}\,(\Lambda^\theta \tilde{D}_x u, \Lambda^\theta u)_Q + 2\,\text{Im}\,([\tilde{D}_x, \Lambda^\theta]u, \Lambda^\theta u)_Q + i(\Lambda^\theta u, w\Lambda^\theta u)_Q$$

and (4.3) of part *b*) follows from (4.19), since

$$|\Lambda^\theta u|_\Sigma = |u|_{H^\theta(\Sigma)} \qquad (\text{from (4.1}b\text{) with } s = \theta)\,,$$

and moreover, with $\theta > 0$, the commutator in (4.19) is a fortiori $\in OPS^{1+\theta-1}_{1,0}(Q)$ and thus is continuous $H^\theta(Q) \to L_2(Q)$: see [T.1, Thm. 6.5, p. 51] where the qualification « loc » can now be discarded since $\tilde{D}_x$ (and Λ^θ) are constant in the variables x and y outside a compact set in Ω (Remark 3.2). □

PROOF OF THEOREM 4.2. – *a*) From (3.14*b*)

$$(4.20)\qquad (Pu, \tilde{D}_x u)_Q = (\tilde{D}_x^2 u, \tilde{D}_x u)_Q - (D_1 u, \tilde{D}_x u)_Q + 2i\gamma(D_2 u, \tilde{D}_x u)_Q\,.$$

The first term on the right hand side of (4.20) is computed via (4.18)

$$(\tilde{D}_x^2 u, \tilde{D}_x u)_Q = i|\tilde{D}_x u|_\Sigma^2 + (\tilde{D}_x u, \tilde{D}_x^2 u)_Q + (\tilde{D}_x u, w\tilde{D}_x u)_Q$$

and thus

$$(4.21)\qquad 2i\,\text{Im}\,(\tilde{D}_x^2 u, \tilde{D}_x u)_Q = i|\tilde{D}_x u|_\Sigma^2 + (\tilde{D}_x u, wD_x u)_Q\,.$$

Similarly for the second term on the right of (4.20); by (4.18)

$$(\tilde{D}_x u, D_1 u)_Q = i\langle u, D_1 u\rangle_\Sigma + (u, \tilde{D}_x D_1 u)_Q + (u, wD_1 u)_Q\,,$$

$$(4.22)\qquad (D_1 u, \tilde{D}_x u)_Q = \overline{(\tilde{D}_x u, D_1 u)_Q} = -i\langle D_1 u, u\rangle_\Sigma + (\tilde{D}_x D_1 u, u)_Q + (wD_1 u, u)_Q = \\ = -i\langle D_1 u, u\rangle_\Sigma + (D_1 \tilde{D}_x u, u)_Q + ([\tilde{D}_x, D_1]u, u)_Q + (wD_1 u, u)_Q\,.$$

By using [T.1, Theorem 4.2, p. 45] on the operator D_1 with real symbol d_1 given by (3.31*a*), one can readily show that

$$(4.23a)\qquad [\text{symbol of } D_1^*] = d_1 + \sum_{j=1}^{n-1} f_j(x, y)\eta_j\,.$$

Thus recalling (4.4*b*) we have

$$D_1^* = D_1 + F_1 . \tag{4.23b}$$

Inserting

$$(D_1 \tilde{D}_x u, u)_Q = (\tilde{D}_x u, D_1 u)_Q + (\tilde{D}_x u, F_1 u)_Q$$

into (4.22), we obtain

$$2i \operatorname{Im} (D_1 u, \tilde{D}_x u)_Q = (D_1 u, \tilde{D}_x u)_Q - (\tilde{D}_x u, D_1 u)_Q =$$
$$= - i\langle D_1 u, u\rangle_\Sigma + (wD_1 u, u)_Q + ([\tilde{D}_x, D_1]u, u)_Q + (\tilde{D}_x u, F_1 u)_Q .$$

Taking the imaginary part of identity (4.20) and using (4.21) and (4.24) yields (4.4), as desired.

b) We use part *a*) along with the estimates

$$|(\tilde{D}_x u, w\tilde{D}_x u)_\Omega| \leq \text{const}_w \|u\|^2_{H^1(\Omega)} , \tag{4.25}$$

$$|(wD_1 u, u)_Q| = \mathcal{O}(\|u\|^2_{H^1(Q)}) , \tag{4.26}$$

$$|([\tilde{D}_x, D_1]u, u)_Q| = \mathcal{O}(\|u\|^2_{H^1(Q)}) , \tag{4.27}$$

$$|(D_2 u, \tilde{D}_x u)_Q| = \mathcal{O}(\|u\|^2_{H^1(Q)}) , \tag{4.28}$$

$$|(\tilde{D}_x u, F_1 u)_Q| = \mathcal{O}(\|u\|^2_{H^1(Q)}) . \tag{4.29}$$

Equation (4.25) follows from D_x, D_{y_j}: continuous $H^1(\Omega) \to L_2(\Omega)$ [L-M.1, p. 85]. For (4.28) we use, in addition, that $D_2 \in OPS^1_{1,0}(Q)$, see (3.12), and then Remark 3.2 implies D_2: $H^1(Q) \to L_2(Q)$, because of the coefficients constants in x, y outside a compact set of Ω. As to (4.26) with $D_1 \in S^2_{1,0}(R^n_{ty})$ uniformly in $x \in R^1_{x+}$, see (3.13), we write: $(wD_1 u, u)_Q = (wJ^{-1}D_1 u, Ju)_Q +$ lower order terms, where J is a self adjoint isomorphism $H^1(R^n_{ty}) \to L^2(R^n_{ty})$ (e.g. J has symbol $(1 + \sigma^2 + |\eta|^2)^{1/2}$), $J \in S^1_{1,0}(R^n_{ty})$. Then $J^{-1}D_1 \in S^{-1+2}_{1,0}(R^n_{ty})$, uniformly in $x \in R^1_{x+}$ by the product theorem [T.1, p. 46] and J, $J^{-1}D_1$: $H^1(Q) \to L_2(Q)$, continuously by Remark 3.2 and Lemma 3.1 *b*), and (4.26) follows. The proof is similar for (4.27) and (4.29). □

PROOF OF THEOREM 4.3. – (4.6) Again by (3.14*b*)

$$(Pu, u)_Q = (\tilde{D}_x^2 u, u)_Q - (D_1 u, u)_Q + 2i\gamma(D_2 u, u)_Q$$

and by (4.18)

$$(\tilde{D}_x^2 u, u)_Q = i\langle \tilde{D}_x u, u\rangle_\Sigma + \|\tilde{D}_x u\|^2_Q + (\tilde{D}_x u, wu)_Q .$$

These last two equations yield (4.6*a*). Then (4.6*b*) follows using the fact that D_2 is self-adjoint in $L_2(Q)$, see (4.37) below. Finally (4.6*c*) is obtained from (4.6*b*) by applying Schwarz inequality to the term $(\tilde{D}_x u, wu)_Q$.

(4.7) We rewrite (4.6*b*) as

$$(4.30)\qquad \|\tilde{D}_x u\|_Q^2 = \operatorname{Re}(Pu, u)_Q + \operatorname{Re}(D_1 u, u)_Q - \operatorname{Re}(\tilde{D}_x u, wu)_Q + \operatorname{Im}\langle Bu, u\rangle_\Sigma \leqslant$$

$$\leqslant \operatorname{Re}(Pu, u)_Q + \operatorname{Re}(D_1 u, u)_Q + \frac{\varepsilon}{2}\|\tilde{D}_x u\|_Q^2 + \frac{1}{2\varepsilon} C_w \|u\|_Q^2 + \frac{1}{2\varepsilon}|Bu|_\Sigma^2 + \frac{\varepsilon}{2}|u|_\Sigma^2 .$$

But recalling the trace theory result (4.3) for $\theta = 0$, we obtain

$$|u|_\Sigma^2 = 2\operatorname{Im}(\tilde{D}_x u, u)_Q + \mathcal{O}(\|u\|_Q^2) \leqslant \varepsilon\|\tilde{D}_x u\|_Q^2 + \mathcal{O}_\varepsilon(\|u\|_Q^2)$$

which inserted into the right hand side of (4.30) produces

$$\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{2}\right)\|\tilde{D}_x u\|_Q^2 \leqslant \operatorname{Re}(Pu, u)_Q + \operatorname{Re}(D_1 u, u)_Q + \mathcal{O}_\varepsilon\{\|u\|_Q^2 + |Bu|_\Sigma^2\}$$

choosing $\varepsilon > 0$ sufficiently small yields (4.7). □

PROOF OF THEOREM 4.4. - First, from (1.7)

$$(4.31a)\qquad \frac{1}{2}\frac{\partial p}{\partial \tau} = -a(x, y)\tau = -a(x, y)(\sigma - i\gamma) = -a(x, y)\sigma + ia(x, y)\gamma$$

so that the pseudo differential operator $\frac{1}{2}(\partial P/\partial\tau)$ defined above Theorem 4.4 is

$$(4.31b)\qquad \frac{1}{2}\frac{\partial P}{\partial \tau} = -D_2 + ia\gamma I$$

from (4.31) and (3.12*a*), and (4.12*b*) is verified. We then compute from (3.14*b*) and (4.31)

$$(4.32)\qquad \left(Pu, \frac{1}{2}\frac{\partial P}{\partial\tau}u\right)_Q = (\tilde{D}_x^2 u - D_1 u + 2i\gamma D_2 u, -D_2 u + ia\gamma u)_Q =$$

$$= -(\tilde{D}_x^2 u, D_2 u)_Q + (D_1 u, D_2 u)_Q + 2\gamma^2(D_2 u, au)_Q -$$

$$- i\gamma\{2\|D_2 u\|_Q^2 + (\tilde{D}_x^2 u, au)_Q - (D_1 u, au)_Q\} .$$

But from (4.18) of Corollary 4.7 we have:

$$(4.33)\qquad (\tilde{D}_x^2 u, D_2 u)_Q = i\langle\tilde{D}_x u, D_2 u\rangle_\Sigma + (\tilde{D}_x u, \tilde{D}_x D_2 u)_Q + (\tilde{D}_x u, wD_2 u)_Q ,$$

$$(4.34)\qquad (\tilde{D}_x^2 u, au)_Q = i\langle\tilde{D}_x u, au\rangle_\Sigma + (\tilde{D}_x u, \tilde{D}_x(au))_Q + (\tilde{D}_x u, wau)_Q ,$$

where, from (3.10*b*), $\tilde{D}_x(au) = (\tilde{D}_x a)u + a\tilde{D}_x u$, and so

$$(4.35)\qquad (\tilde{D}_x u, \tilde{D}_x(au))_Q = (\tilde{D}_x u, (\tilde{D}_x a)u)_Q + \|a^{1/2}\tilde{D}_x u\|_Q .$$

Inserting first (4.35) into (4.34), and then the resulting (4.34) as well as (4.33) into (4.32), we arrive at

$$\left(Pu, \frac{1}{2}\frac{\partial P}{\partial \tau} u\right)_Q = -i\langle \tilde{D}_x u, D_2 u\rangle_\Sigma + \gamma\langle \tilde{D}_x u, au\rangle_\Sigma - \tag{4.36}$$
$$-(\tilde{D}_x u, \tilde{D}_x D_2 u)_Q - (\tilde{D}_x u, wD_2 u)_Q +$$
$$+(D_1 u, D_2 u)_Q + 2\gamma^2 (D_2 u, au)_Q - 2i\gamma \|D_2 u\|_Q^2 + i\gamma(D_1 u, au) -$$
$$- i\gamma\|a^{1/2}\tilde{D}_x u\|_Q^2 - i\gamma(\tilde{D}_x u, (\tilde{D}_x a)u)_Q - i\gamma(\tilde{D}_x u, w(au))_Q .$$

We now take the imaginary part of (4.36). We need the following results, to be established at the end of this proof.

CLAIM 1. – D_2 is self-adjoint on $L_2(Q)$:

$$D_2 = D_2^* \quad \text{and} \quad (D_2 u, au)_Q = \text{real}, \quad \text{Im}\,(D_2 u, au) = 0 . \quad \square \tag{4.37}$$

CLAIM 2. – With S_2 given by (4.11), we have

$$2\,\text{Im}\,(D_1 u, D_2 u)_Q = (u, S_2 u)_Q . \quad \square \tag{4.38}$$

CLAIM 3. – With $\tilde{K}_2$ given by (4.9), we have

$$\text{Im}\,(\tilde{D}_x u, \tilde{D}_x D_2 u)_Q = \text{Im}\,(\tilde{D}_x u, \tilde{K}_2 u)_Q . \quad \square \tag{4.39}$$

Thus taking twice the imaginary part of (4.36) and using (4.37)-(4.39) yields

$$-2\,\text{Im}\left(Pu, \frac{1}{2}\frac{\partial P}{\partial \tau}\right)_Q = \tag{4.40}$$
$$= 2\,\text{Re}\,\langle \tilde{D}_x u; D_2 u\rangle_\Sigma - 2\gamma\,\text{Im}\,\langle \tilde{D}_x u, au\rangle_\Sigma + (1) + (2) + (3),$$

$$(1) = 2\,\text{Im}\,(\tilde{D}_x u, [wD_2 + \tilde{K}_2]u)_Q - (u, S_2 u)_Q , \tag{4.41}$$

$$(2) = 2\gamma\,\text{Re}\,(\tilde{D}_x u, [wa + (\tilde{D}_x a)]u)_Q , \tag{4.42}$$

$$(3) = 2\gamma\{2\|D_2 u\|_Q^2 + \|a^{1/2}\tilde{D}_x u\|_Q^2 - (D_1 u, au)_Q\} . \tag{4.43}$$

But from (3.13*a*) and (3.12*a*)

$$D_1 = \frac{1}{a} D_2^2 - a\gamma^2 - \frac{1}{a^2} D \tag{4.44}$$

and thus

$$-(D_1 u, au)_Q = -\|D_2 u\|_Q^2 + \gamma^2\|au\|_Q^2 + \left(\frac{1}{a} Du, u\right)_Q . \tag{4.45}$$

Substituting (4.45) in (4.43) yields

$$(3) = 2\gamma\left\{\|D_2 u\|_Q^2 + \|a^{1/2}\tilde{D}_x u\|_Q^2 + \left(\frac{1}{a}\,Du,\,u\right)_Q\right\} + 2\gamma^3\|au\|_Q^2 . \tag{4.46}$$

Then (4.40), (4.41), (4.42) and (4.46) prove (4.8), once the three Claims above, (4.37)-(4.39), are established (below).

b) We now estimate the terms (1), (2), (3). By Schwarz inequality

$$(1) \geqslant -\|\tilde{D}_x u\|_Q^2 - \|[wD_2 + \tilde{K}_2]u\|_Q^2 - |(u,\,S_2 u)_Q| , \tag{4.47}$$

$$(2) \geqslant -\gamma C\left[\varepsilon\|\tilde{D}_x u\|_Q^2 + \frac{1}{\varepsilon}\|u\|_Q^2\right] \tag{4.48}$$

where C is a positive constant depending on the coefficients a and a_{nj} (recall (4.2*b*)). Thus we obtain from (4.46)-(4.48)

$$\begin{aligned}(1) + (2) + (3) \geqslant 2\gamma\|D_2 u\|_Q^2 - \|[wD_2 + \tilde{K}_2]u\|_Q^2 + \\ + \gamma\{2\|a^{1/2}\tilde{D}_x u\|_Q^2 - \varepsilon C\|\tilde{D}_x u\|^2\} - \|\tilde{D}_x u\|_Q^2 + 2\gamma\left(\frac{1}{a}\,Du,\,u\right)_Q + \\ + \gamma\left\{2\gamma^2\|au\|_Q^2 - \frac{1}{\varepsilon}\,C\|u\|^2\right\} - |(u,\,S_2 u)_Q| .\end{aligned} \tag{4.49}$$

Recalling now that both D_2 and $wD_2 + \tilde{K}_2$ have symbols of the type: coeff $(x, y)\sigma$, see (3.12*a*) and (4.10*b*), we select $\varepsilon > 0$ suitably small, and γ suitably large in comparision with $1/\varepsilon$, and obtain from (4.49)

$$\begin{aligned}(1) + (2) + (3) \geqslant 2\gamma\left\{c_1\|D_2 u\|_Q^2 + \left(\frac{1}{a}\,Du,\,u\right)_Q + c_2\|\tilde{D}_x u\|_Q^2\right\} + \\ + \gamma^3 c_3\|u\|_Q^2 - |(u,\,S_2 u)_Q|\end{aligned} \tag{4.50}$$

for suitable $c_1, c_2, c_3 > 0$ and suitably large γ, say $\gamma >$ som $\gamma_1 > 0$. Consider now the operator (recall $D_2 = D_2^*$ from (4.37))

$$W \equiv c_1 D_2^2 + \frac{1}{a}\,D \tag{4.51}$$

whose symbol is

$$\text{symbol of } W = c_1 d_2^2(x, y;\sigma) + \frac{1}{a}\,d(x, y;\eta) . \tag{4.52}$$

We now recall the strict hyperbolicity condition (1.10) plus positivity of a in (1.8) to claim that

$$\frac{1}{a}\,d(x, y;\eta) \geqslant C|\eta|^2 , \qquad |\eta| \text{ large} \tag{4.53}$$

while from (3.12*a*), $d_2^2(x, y, \sigma) \geq c|\sigma|^2$, so that

$$\text{Re}\{\text{symbol of } W\} \geq C\{|\sigma|^2 + |\eta|^2\}, \qquad |\sigma|, |\eta| \text{ large, uniformly in } x \in R^1_{x+}. \tag{4.54}$$

We can then apply Garding's inequality [T.1, Theorem 8.1 with $s = 0$, p. 55] on (4.54) and conclude: there are constants $C_1, C_2 > 0$ such that for any compact set K in R^n_{ty} and all $u \in C_0^\infty(K) \times C^\infty(R^1_{x+})$ we have

$$\text{Re}(Wu, u)_Q = C_1\|D_2 u\|_Q^2 + \left(\frac{1}{a} Du, u\right)_Q \geq C_2\{\|D_t u\|_Q^2 + \|D_y u\|_Q^2\} - C_1\|u\|_Q^2. \tag{4.55}$$

But from (3.10*b*)

$$\|u\|^2_{H^1(Q)} = \|D_t u\|_Q^2 + \|D_y u\|_Q^2 + \left\|\tilde{D}_x u - \sum_{j=1}^{n-1} a_{nj} D_{y_j} u\right\|_Q^2 \leq C\{\|D_t u\|_Q^2 + \|\tilde{D}_x u\|_Q^2 + \|D_y u\|_Q^2\}. \tag{4.56}$$

Thus using (4.55) and (4.56), we have that for all $u \in C_0^\infty(K) \times C^\infty(R^1_{x+})$

$$C_1\|D_2 u\|_Q^2 + \left(\frac{1}{a} Du, u\right)_Q + c_2\|\tilde{D}_x u\|_Q^2 \geq c_0\|u\|^2_{H^1(Q)} - c_1\|u\|_Q^2 \tag{4.57}$$

where $c_0 = \min\{C_2, c_2\}/C$. Using (4.57) into (4.50) we obtain for all $u \in C_0^\infty(K) \times \times C^\infty(R^1_{x+})$

$$(1) + (2) + (3) \geq 2\gamma c_0\|u\|^2_{H^1(Q)} + (\gamma^3 c_3 - 2\gamma) c_1\|u\|_Q^2 - |(u, S_2 u)_Q|. \tag{4.58}$$

If now J is the self-adjoint isomorphism

$$H^1(R^n_{ty}) \to L_2(R^n_{ty}) \qquad \text{with symbol } (1 + \sigma^2 + |\eta|^2)^{1/2},$$

we have from (4.11*a*) that $J^{-1}S_2 \in OPS^1_{1,0}(R^n_{ty})$, uniformly in $x \in R^1_{x+}$ and Lemma 3.1 *b*) yields $J^{-1}S_2$: continuous $H^1(Q) \to L_2(Q)$.

Thus by Schwarz inequality

$$|(u, S_2 u)_Q| = |(Ju, J^{-1}S_2 u)_Q| \leq C\|u\|^2_{H^1(Q)}. \tag{4.59}$$

Finally, (4.58) and (4.59) imply for all $u \in C_0^\infty(K) \times C^\infty(R^1_{x+})$

$$(1) + (2) + (3) \geq (2\gamma c_0 - C)\|u\|^2_{H^1(Q)} + (\gamma^3 C_3 - 2\gamma) c_1\|u\|_Q^2 \geq \gamma C_0\|u\|^2_{H^1(Q)} \tag{4.60}$$

for all $\gamma \geq$ suitably large $\gamma_0 > 0$, where C_0 is a suitable constant. Then (4.40) and (4.60) together prove (4.12), as desired.

It remains to prove Claims 1 through 3, (4.38)-(4.39).

CLAIM 1. – (4.37) That $D_2^* = D_2$ with

$$[\text{symbol of } D_2^*] = [\text{symbol of } D_2] = a(x, y)\,\sigma = \text{real}\,,$$

follows from [T.1, Theorem 4.2, p. 45] since $d_2(x, y; \sigma)$ in (3.12*a*) is real.

CLAIM 2. – (4.38). Recalling (4.23*b*) with F_1 as in (4.4*b*) we have

$$(4.61)\qquad (D_1 u, D_2 u)_Q = (u, D_1^* D_2 u)_Q = (u, D_1 D_2 u)_Q + (u, F_1 D_2 u)_Q = \\ = (u, D_2 D_1 u)_Q + (u, [K_{12} + F_1 D_2] u)_Q\,,$$

$$(4.62)\qquad (\text{by } (4.37)) = \overline{(D_1 u, D_2 u)_Q} + (u, S_2 u)_Q$$

where

$$(4.63)\qquad K_{12} \equiv [D_1, D_2] = D_1 D_2 - D_2 D_1\,,$$

$$(4.64)\qquad S_2 \equiv K_{12} + F_1 D_2\,.$$

From (4.62) we obtain the sought after (4.38)

$$2\,\mathrm{Im}\,(D_1 u, D_2 u)_Q = (u, S_2 u)_Q\,.$$

Moreover, from (4.63), using the product theorem [T.1, Theorem 4.4, p. 46] we readily compute

$$(4.65)\qquad [\text{symbol of } K_{12}] = \Big[\sum_{j=1}^{n-1} k_j(x, y)\,\eta_j\Big]\sigma\,,$$

$$k_j(x, y) = \text{smooth, depending on coefficients of } P\,.$$

Similarly from (4.16*a*), (4.16*b*) and (3.12*a*) and the product theorem we readily check

$$(4.66)\qquad [\text{symbol of } F_1 D_2] = \Big[\sum_{j=1}^{n-1} r_j(x, y)\,\eta_j\Big]\sigma + r_0(x, y)\,\sigma\,,$$

$$r_j(x, y) = \text{smooth, depending on coefficients of } P\,.$$

Then (4.64)-(4.66) prove the desired form (4.11*b*) for the symbol of S_2. Also, (4.38) shows that S_2 is self-adjoint on $L_2(Q)$.

CLAIM 3. – (4.39). Let

$$(4.67)\qquad \tilde{K}_2 \equiv [\tilde{D}_x, D_2] = \tilde{D}_x D_2 - D_2 \tilde{D}_x\,.$$

Then using (4.67) and $D_2 = D_2^*$,

$$(\tilde{D}_x u, \tilde{D}_x D_2 u)_Q = (\tilde{D}_x u, D_2 \tilde{D}_x u)_Q + (\tilde{D}_x u, \tilde{K}_2 u)_Q = (D_2 \tilde{D}_x u, \tilde{D}_x u) + (\tilde{D}_x u, \tilde{K}_2 u)_Q$$

$$(4.68) \qquad (\text{by } (4.67)) = \overline{(\tilde{D}_x u, \tilde{D}_x D_2 u)_Q} + (\tilde{D}_x u, \tilde{K}_2 u)_Q - (\tilde{K}_2 u, \tilde{D}_x u)_Q\,.$$

Thus (4.68) yields the desired (4.39)

$$2\,\mathrm{Im}\,(\tilde{D}_x u, \tilde{D}_x D_2 u)_Q = 2\,\mathrm{Im}\,(\tilde{D}_x u, \tilde{K}_2 u)_Q\,.$$

Moreover (4.67) and [T.1, Theorem 4.4, p. 46] yield readily from (3.10*a*) and (3.12*a*) that the symbol of $\tilde{K}_2$ has the form as in (4.9*b*).

The proof of Theorem 4.4 is now complete. □

5. – Operators of localization χ and their properties.

5.1. *Definitions and statement of properties.*

We return to the symbol $d_1(x, y; \sigma, \eta)$ of class $S^2_{1,0}(R^n_{ty})$ uniformly in $x \in R^1_{x+}$ defined by (3.13*a*), where we consider $\gamma > 0$ to be fixed. (If we wish, we may take the symmetric positive definite quadratic form $d(x, y; \eta)$ in (1.9)-(1.10), or (3.11*a*) to be in canonical form $d(x, y; \eta) = \sum_{j=1}^{n-1} \lambda_j^2(x, y)\eta_j^2$ without loss of generality; i.e. modulo a similarity transformation with an (x, y)-dependent orthogonal matrix). We recall that as a consequence of assumption (i), section 1, above (1.7*a*) on the coefficients of P and B, all symbols $d(x, y; \eta)$, $d_1(x, y; \sigma, \eta)$ etc. are constant in x and y outside a compact set $\mathcal{K}_{xy}$ of $\Omega = R^1_{x+} \times R^{n-1}_y$. As the point (x, y) varies and $\gamma > 0$ is fixed, the equation

$$(5.0) \qquad d_1(x, y; \sigma, \eta) = 0\,, \quad \text{i.e. by (3.13}a\text{): } \frac{\sigma^2}{\gamma^2} - \frac{d(x, y; \eta)}{a^3(x, y)\gamma^2} = 1$$

describes a family of hyperboloids in the space $R^1_\sigma \times R^{n-1}_\eta$ (which reduces to a fixed hyperboloid for (x, y) outside $\mathcal{K}_{xy}$), all passing through the points $\sigma = \pm\gamma$, $\eta = 0$. Henceforth, because of the symmetry in σ, we may restrict our analysis to the half-space $\sigma > 0$. Setting

$$(5.1) \qquad \inf_{\substack{x,y\\ |\eta|=1}} \frac{d(x, y; \eta)}{a^3(x, y)} \equiv m^2\,; \qquad \sup_{\substack{x,y\\ |\eta|=1}} \frac{d(x, y; \eta)}{a^3(x, y)} \equiv M^2$$

we have $m > 0$ and $M < +\infty$ by (1.8), (1.10) and assumption (i), section 1. Then, from (5.0)-(5.1)

$$(5.2) \qquad m^2|\eta|^2 \leqslant \sigma^2 = \gamma^2 + \frac{d(x, y; \eta)}{a^3(x, y)} \leqslant \gamma^2 + M^2|\eta|^2 \leqslant 2M^2|\eta|^2$$

for all η outside the η-sphere of radius γ/M centered at the origin.

Thus, for $\sigma > 0$, all points of the family of hyperboloids with $|\eta| \geqslant \gamma/M$, lie between two equilateral cones: $\sigma = m|\eta|$ and $\sigma = \sqrt{2}\,M|\eta|$, uniformly in (x, y). With this observation at hand, we now introduce a few mutually disjoint subregions (cones), which will exhaust all of $R^{2n}(+) \equiv R^1_{x+} \times R^{n-1}_y \times R^1_{\sigma+} \times R^{n-1}_\eta$. They are (see Fig. 5.1):

$$\mathcal{B}(x, y; \sigma, \eta) \equiv \left\{(x, y; \sigma, \eta) \in R^{2n}(+): \frac{m}{2}|\eta| \leqslant \sigma \leqslant 2M|\eta|\right\} \tag{5.3}$$

$$\mathcal{G}^{\mathrm{tr}}_{\mathrm{I}}(x, y; \sigma, \eta) = \left\{(x, y; \sigma, \eta) \in R^{2n}(+): 2M|\eta| < \sigma \leqslant \frac{5}{2} M|\eta|\right\} \tag{5.4}$$

$$\mathcal{G}^{\mathrm{I}}(x, y; \sigma, \eta) = \left\{(x, y; \sigma, \eta) \in R^{2n}(+): \frac{5}{2} M|\eta| < \sigma\right\} \tag{5.5}$$

$$\mathcal{G}^{\mathrm{II}}_{\mathrm{tr}}(x, y; \sigma, \eta) = \left\{(x, y; \sigma, \eta) \in R^{2n}(+): \frac{m}{4}|\eta| \leqslant \sigma < \frac{m}{2}|\eta|\right\} \tag{5.6}$$

$$\mathcal{G}^{\mathrm{II}}(x, y; \sigma, \eta) = \left\{(x, y; \sigma, \eta) \in R^{2n}(+): 0 \leqslant \sigma < \frac{m}{4}|\eta|\right\} \tag{5.7}$$

$$R^{2n}(+) = \mathcal{B} \cup \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}} \cup \mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{II}}_{\mathrm{tr}} \cup \mathcal{G}^{\mathrm{II}}$$

($\mathcal{G}$ stands for «good» region, $\mathcal{B}$ for «bad» region, with respect to the symbol d_1, see (5.0); the subscript «tr» stands for «transition», as the definition of the localizing symbols below makes it transparent).

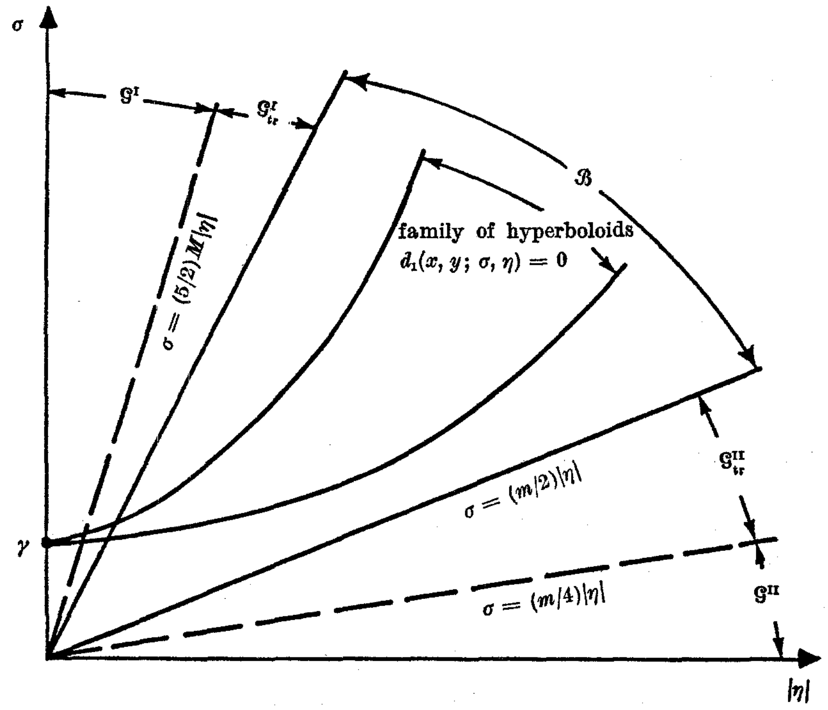

Figure 5.1. Regions $\mathcal{B}$ and $\mathcal{G}^{\mathrm{I}}, \ldots, \mathcal{G}^{\mathrm{II}}$.

For future use, we note the following two claims.

CLAIM 1. – In $\mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}}$, where $\sigma > 2M|\eta|$, we have

$$\begin{cases} d_1(x, y; \sigma, \eta) \geqslant C_1^2[\sigma^2 + |\eta|^2], & \text{for } \sigma^2 \geqslant \gamma^2 \dfrac{\max a(x, y)}{\frac{1}{2}\min a(x, y)} \\ C_1^2 = \left[\min\left\{\dfrac{1}{8};\ \dfrac{M^2}{2}\right\}\right] \min a(x, y)\,. \end{cases} \tag{5.8}$$

CLAIM 2. – In $\mathcal{G}^{\mathrm{II}} \cup \mathcal{G}^{\mathrm{II}}_{\mathrm{tr}}$, where $\sigma < (m/2)|\eta|$, we have

$$-d_1(x, y; \sigma, \eta) \geqslant C_2^2[\sigma^2 + |\eta|^2]\,, \qquad C_2^2 = [\min a(x, y)]\cdot \min\left\{\frac{3}{8}m^2,\ \frac{3}{2}\right\}. \tag{5.9}$$

PROOF OF CLAIM 1. – By (3.13a) and (5.1) (also (1.8))

$$d_1(x, y; \sigma, \eta) \geqslant a(x, y)\sigma^2 - a(x, y)\gamma^2 - a(x, y) M^2 |\eta|^2 \geqslant$$

$$\geqslant a(x, y)[1 - \tfrac{1}{4}]\sigma^2 - a(x, y)\gamma^2 \geqslant \tfrac{1}{4}[\min a(x, y)]\sigma^2,$$

$$d_1(x, y; \sigma, \eta) \geqslant M^2[\min a(x, y)]\,|\eta|^2\,.$$

Summing up the last two inequalities yields (5.8). □

PROOF OF CLAIM 2. – By (3.13a) and (5.1)

$$-d_1(x, y; \sigma, \eta) = \frac{d(x, y; \eta)}{a^2(x, y)} - a(x, y)\sigma^2 + a(x, y)\gamma^2 \geqslant$$

$$\geqslant m^2 a(x, y)|\eta|^2 - \frac{m^2}{4} a(x, y)|\eta|^2 + a(x, y)\gamma^2 \geqslant$$

$$\text{(by (1.8))} \qquad \geqslant \frac{3}{4}|\eta|^2 m^2 \min a(x, y)$$

$$-d_1(x, y; \sigma, \eta) \geqslant 3\sigma^2 \min a(x, y)\,.$$

Summing up the last two inequalities yields (5.9). □

We now introduce symbols of localization of the above regions. Let $\chi(x, y; \sigma, \eta)$ —distinguished by an appropriate superscript—be C^∞-functions in all variables, monotone, such that

$$\chi^{\mathcal{B}}(x, y; \sigma, \eta) = \begin{cases} 1 & \text{on } \mathcal{B}\,, \\ 0 & \text{on } \mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{II}}\,, \end{cases} \tag{5.10}$$

$$\chi^{\mathrm{I}}(x, y; \sigma, \eta) = \begin{cases} 1 & \text{on } \mathcal{G}^{\mathrm{I}}, \\ 0 & \text{on } \mathcal{B} \cup \mathcal{G}^{\mathrm{II}}_{\mathrm{tr}} \cup \mathcal{G}^{\mathrm{II}}, \end{cases} \tag{5.11}$$

$$\chi^{\mathrm{II}}(x, y; \sigma, \eta) = \begin{cases} 1 & \text{on } \mathcal{G}^{\mathrm{II}}, \\ 0 & \text{on } \mathcal{B} \cup \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}} \cup \mathcal{G}^{\mathrm{I}}, \end{cases} \tag{5.12}$$

and

$$\chi^{\mathcal{B}}(x, y; \sigma, \eta) + \chi^{\mathrm{I}}(x, y; \sigma, \eta) + \chi^{\mathrm{II}}(x, y; \sigma, \eta) \equiv 1 \quad \text{in } R^{2n}(+), \tag{5.13a}$$

$$\chi^{\mathcal{G}} = \chi^{\mathrm{I}} + \chi^{\mathrm{II}}. \tag{5.13b}$$

Moreover, $\chi^{\mathcal{B}}$ and χ^{I} are defined by homogeneity of order zero in the transition region $\mathcal{G}^{\mathrm{I}}_{\mathrm{tr}}$ (they are first defined on the unit sphere of $R^1_\sigma \times R^{n-1}_\eta$ for fixed (x, y) and then extended by homogeneity of order zero, i.e. by constancy on each ray in $R^n_{\sigma\eta}$). Likewise, $\chi^{\mathcal{B}}$ and χ^{II} are defined by homogeneity of order zero in the transition region $\mathcal{G}^{\mathrm{II}}_{\mathrm{rI}}$. Thus [T.1, p. 37]

(5.14) $\chi^{\mathcal{B}}$, χ^{I}, χ are homogeneous symbols of order zero in (σ, η), i.e. of class $S^0(R^{n+1}_{tyx}) \in S^0_{1,0}(R^{n+1}_{tyx})$ uniformly in $x \in R^1_{x+}$.

Finally, we need to divide further the region $\mathcal{B}$ into three mutually disjoint subregions, which will exhaust all of $\mathcal{B}$. Let r be a number which for the time being we take to be strictly between 1 and 2: $1 < r < 2$ (r will be identified in section 6 to be $r = 8/5$) [1]. Define

$$\mathcal{B}^+_r(x, y; \sigma, \eta) = \{(x, y, \sigma, \eta) \in \mathcal{B}: 2\sigma^r < d_1(x, y; \sigma, \eta)\}, \tag{5.15}$$

$$\mathcal{B}^-_r(x, y; \sigma, \eta) = \{(x, y, \sigma, \eta) \in \mathcal{B}: d_1(x, y; \sigma, \eta) \leqslant \sigma^r\}, \tag{5.16}$$

$$\mathcal{B}_{\mathrm{tr},r}(x, y; \sigma, \eta) = \{(x, y, \sigma, \eta) \in \mathcal{B}: \sigma^r < d_1(x, y; \sigma, \eta) \leqslant 2\sigma^r\}. \tag{5.17}$$

Then, $\mathcal{B} = \mathcal{B}^+_r \cup \mathcal{B}^-_r \cup \mathcal{B}_{\mathrm{tr},r}$ (tr = « transition »). Let $\chi(x, y; \sigma, \eta)$—distinguished by an appropriate superscript—be C^∞-functions in all variables, monotone, non-homogeneous such that (they will be precisely defined in the proof of Lemma 5.1 below):

$$\chi^{\mathcal{B}^+_r}(x, y; \sigma, \eta) = \begin{cases} 1 & \text{on } \mathcal{B}^+_r, \\ 0 & \text{on } \mathcal{B}^-_r \cup \mathcal{G}^{\mathrm{II}}_{\mathrm{tr}} \cup \mathcal{G}^{\mathrm{II}} \cup \mathcal{G}^{\mathrm{I}}, \\ \chi^{\mathcal{B}} & \text{on } \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}} \text{ (homogeneous of order zero in } \sigma, \eta), \end{cases} \tag{5.18}$$

$$\chi^{\mathcal{B}^-_r}(x, y; \sigma, \eta) = \begin{cases} 1 & \text{on } \mathcal{B}^-_r, \\ 0 & \text{on } \mathcal{B}^+_r \cup \mathcal{B}^{\mathrm{I}}_{\mathrm{tr}} \cup \mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{II}}, \\ \chi^{\mathcal{B}} & \text{on } \mathcal{G}^{\mathrm{II}}_{\mathrm{tr}} \text{ (homogeneous of order zero in } \sigma, \eta), \end{cases} \tag{5.19}$$

[1] The subsequent reading is simplified if one thinks of r as $r = 8/5$.

and, moreover,

$$\chi^{\mathcal{B}_r^+}(x, y; \sigma, \eta) + \chi^{\mathcal{B}_r^-}(x, y; \sigma, \eta) \equiv \chi^{\mathcal{B}}(x, y; \sigma, \eta) \quad \text{on } \mathcal{B}\,. \tag{5.20a}$$

Thus, by (5.13)

$$(\chi^{\mathcal{B}_r^+} + \chi^{\mathcal{B}_r^-} + \chi^{\mathrm{I}} + \chi^{\mathrm{II}})(x, y; \sigma, \eta) \equiv 1 \quad \text{in } R^{2n}(+)\,.$$

The symbols of localization defined in (5.11), (5.12), (5.18), (5.19) are those for which we shall consider the corresponding localized problem defined by (3.22). As seen in (3.22), crucial to the analysis of (3.22) will be the determination of the character or action of the commutator operators $[P, \boldsymbol{\chi}]$ and $[B, \boldsymbol{\chi}]$, where $\boldsymbol{\chi}$ is the pseudo-differential operator defined by (3.21) with the symbol χ there being now any of the symbols in (5.11), (5.12), (5.18), (5.19). The analysis is simplest for $\chi = \chi^{\mathrm{I}}$ and $\chi = \chi^{\mathrm{II}}$ since these symbols are homogeneous (of order zero) i.e. of class $S^0(Q) \subset S^0_{1,0}(Q)$. The differential operators P and B have also, of course, homogeneous symbols of order 2 and 1, respectively.

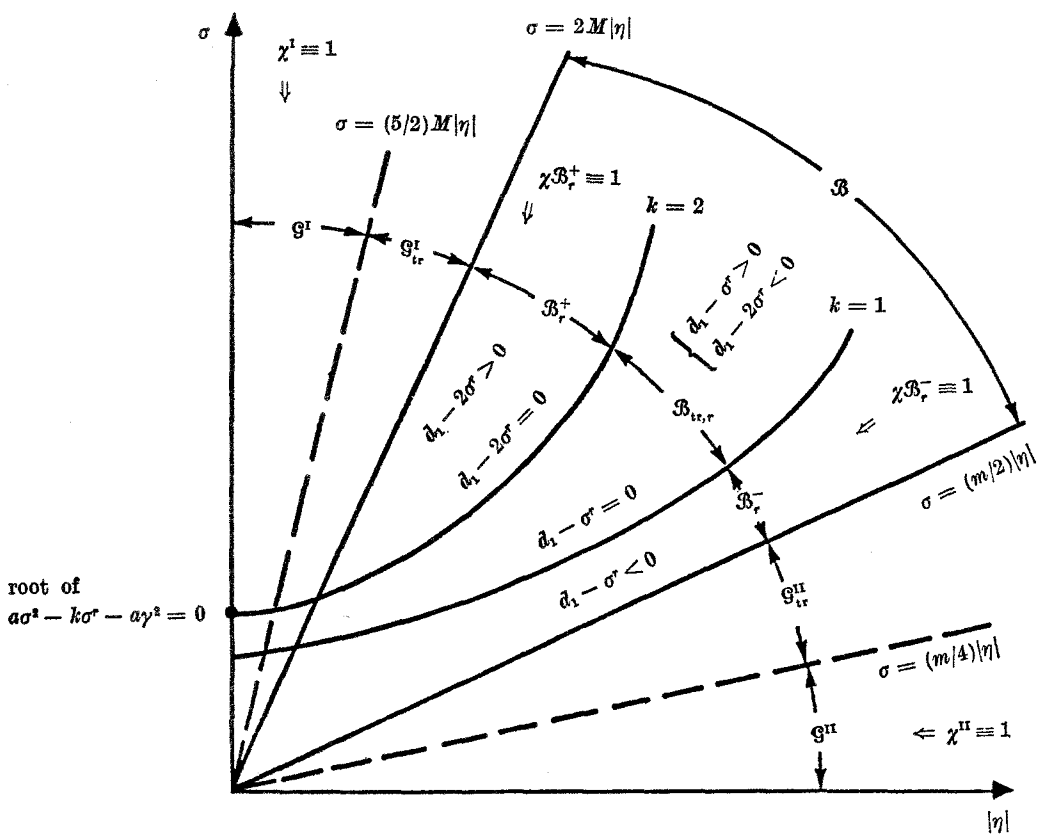

Figure 5.2. Regions $\mathcal{B}_r^+$, $\mathcal{B}_{tr,r}$ and $\mathcal{B}_r^-$.

LEMMA 5.0. – With $P = P(x, y; D_t, D_x, D_y) \in OPS^2_{1,0}(Q)$, $B|_{x=0} \in OPS^1_{1,0}(\Sigma)$, and $\chi^i \in S^0_{1,0}(Q)$, $i = \mathrm{I}, \mathrm{II}$ we have:

$$[P, \chi^i] \in OPS^1_{1,0}(Q)\,, \tag{5.21a}$$

$$[B, \chi^i]\,|_{x=0} \in OPS^0_{1,0}(\Sigma)\,, \tag{5.22a}$$

Hence

$$[P, \chi^i]: \quad \text{continuous } H^s(Q) \to H^{s-1}(Q)\,, \tag{5.21b}$$

$$[B, \chi^i]\,|_{x=0}: \quad \text{continuous } H^s(\Sigma) \to H^s(\Sigma)\,, \tag{5.22b}$$

$$\text{moreover } \chi^i: \text{ continuous } H^s(Q) \to H^s(Q)\,. \tag{5.22c}$$

PROOF. – Conclusions (5.21a), (5.22a) on commutator in the homogeneous case are standard (or apply Lemma 3.2). Then Eqts. (5.21b)-(5.22b) follow via [T.1, Thm. 6.5, p. 51] as in Remark 3.2, since P, B, χ^i are constant in (x, y) outside a compact set $\mathcal{K}_{xy}$ of Ω. ☐

However, the symbols $\chi^{\mathcal{B}_r^\pm}$ are non-homogeneous (and of order zero) and the question regarding the character or action of the corresponding commutators with P and B becomes much more delicate. In this respect, the following lemma is fundamental.

LEMMA 5.1. – In the notation of (5.15)-(5.19) with $1 < r < 2$, and recalling Definition 3.2 below (3.15) and Remark 3.1, we can define—in fact, constructively in the proof below—symbols of localization $\chi^{\mathcal{B}_r^+}$ and $\chi^{\mathcal{B}_r^-}$ so that

$$|D^\beta_{\substack{x\\y}} D^\alpha_{\substack{\sigma\\\eta}} \chi^{\mathcal{B}_r^\pm}(x, y; \sigma, \eta) \leqslant C_{\alpha,\beta}(|\sigma| + |\eta|)^{-|\alpha|(r-1)+|\beta|(2-r)} \qquad (x, y, \sigma, \eta) \in \mathcal{B}_{\mathrm{tr},r} \text{ as } |\sigma| \sim |\eta| \to \infty\,, \tag{5.23}$$

where $C_{\alpha,\beta}$ is independent of $x, y \in R^1_{x^+} \times R^{n-1}_y$. Recalling from (5.18)-(5.19) that $\chi^{\mathcal{B}_r^\pm}$ are homogeneous of order zero in $\mathcal{G}^{\mathrm{I}}_{\mathrm{tr}}$ or $\mathcal{G}^{\mathrm{II}}_{\mathrm{tr}}$, respectively, and identically zero or one elsewhere, we conclude from (5.23) that

$$\begin{cases} \chi^{\mathcal{B}_r^\pm}(x, y; \sigma, \eta) \in S^0_{r-1,\,2-r}(R^{n+1}_{tyx})\,, & \text{uniformly in } x \in R^1_{x^+}\,, \\ \chi^{\mathcal{B}_r^\pm} \in OPS^0_{r-1,\,2-r}(R^{n+1}_{tyx})\,, & \text{uniformly in } x \in R^1_{x^+}\,. \end{cases} \tag{5.24}$$

COROLLARY 5.2. – For $3/2 < r < 2$, we have

$$\chi^{\mathcal{B}_r^\pm}: \quad \text{continuous } H^s(Q) \to H^s(Q)\,,\ 0 \leqslant s \leqslant 1\,, \tag{5.25a}$$

$$\chi^{\mathcal{B}_r^\pm}|_{x=0}: \text{ continuous } H^s(\Sigma) \to H^s(\Sigma)\,,\ 0 \leqslant s \leqslant 1\,. \tag{5.25b}$$

PROOF OF COROLLARY 5.2. – Use (5.24) and apply Lemma 3.1 with

$$r-1>2-r. \qquad \square$$

5.2. *Proof of Lemma* 5.1.

We shall write the proof for the symbol $\chi^{\mathcal{B}_r^+}$ which, for simplicity, will be indicated as χ^r. The proof for $\chi^{\mathcal{B}_r^-}$ is similar.

STEP 1. – Definitions (5.15) and (5.17) suggest introducing a family of smooth surfaces $\mathcal{S}_k$ in the «tube» $\mathcal{B}_{\mathrm{tr},r}$, which will be parametrized by a real parameter k, $1\leqslant k\leqslant 2$. These surfaces are defined by (see (3.13a))

$$(5.26) \qquad \mathcal{S}_k\colon\ d_1(x,y;\sigma,\eta)\equiv k\sigma^r, \qquad \text{or} \qquad a(x,y)(\sigma^2-\gamma^2)-k\sigma^r=\frac{d(x,y;\eta)}{a^2(x,y)}.$$

Conversely, for any point $(x,y,\sigma,\eta)\in\mathcal{B}_{\mathrm{tr},r}$, there is only one surfaces $\mathcal{S}_k$ passing through it, namely the surface corresponding to the parameter (comprised between 1 and 2)

$$(5.27) \qquad k(x,y,\sigma,\eta)=\frac{a(x,y)(\sigma^2-\gamma^2)-d(x,y;\eta)/a^2(x,y)}{\sigma^r}.$$

Thus $\mathcal{B}_{\mathrm{tr},r}=\bigcup_k \mathcal{S}_k$, $1\leqslant k\leqslant 2$.

CONSTRUCTIVE DEFINITION OF $\chi^r=\chi^{\mathcal{B}_r^+}(x,y;\sigma,\eta)$ IN $R^{2n}(+)$. – See Fig. 5.3. With η_1 being the first coordinate of η, we first fix a *reference point* (x_0,y_0) and select a sufficiently large $\sigma_0>0$ so that the *reference level hyperplane* $\sigma=\sigma_0$ intersects on the *reference coordinate plane* (σ,η_1) both curves

$$(5.28) \qquad 0=d_1(x_0,y_0;\sigma,\eta=[\eta_1,0,\ldots,0])-k\sigma^r= \text{(by (5.26) and (1.19))}$$
$$=a(x_0,y_0)(\sigma^2-\gamma^2)-k\sigma^r-\{a_{11}(x_0,y_0)-a_{n1}^2(x_0,y_0)\}\eta_1^2,$$

for $k=1$ and $k=2$, in two points $\eta_{1,l}=\eta_{1,\mathrm{left}}$ and $\eta_{1,\mathrm{r}}=\eta_{1,\mathrm{right}}$. The segment on $\sigma=\sigma_0$, $\eta=[\eta_1,0\ldots0]$ comprised between these two intersection points is

$$(5.29a) \qquad \eta_{1,l}\leqslant\eta_1(k,x_0,y_0,\sigma_0)\leqslant\eta_{1,\mathrm{r}}, \qquad 1\leqslant k\leqslant 2,$$

$$(5.29b) \qquad \eta_{1,l}\equiv\eta_1(k=2,x_0,y_0,\sigma_0); \qquad \eta_{1,\mathrm{r}}=\eta_1(k=1,x_0,y_0,\sigma_0),$$

when k runs over $1\leqslant k\leqslant 2$, where by (5.28)

$$(5.30) \qquad \eta_1^2(k,x_0,y_0,\sigma_0)=\frac{a(x_0,y_0)(\sigma_0^2-\gamma^2)-k\sigma_0^r}{a_{11}(x_0,y_0)-a_{n_1}^2(x_0,y_0)},$$

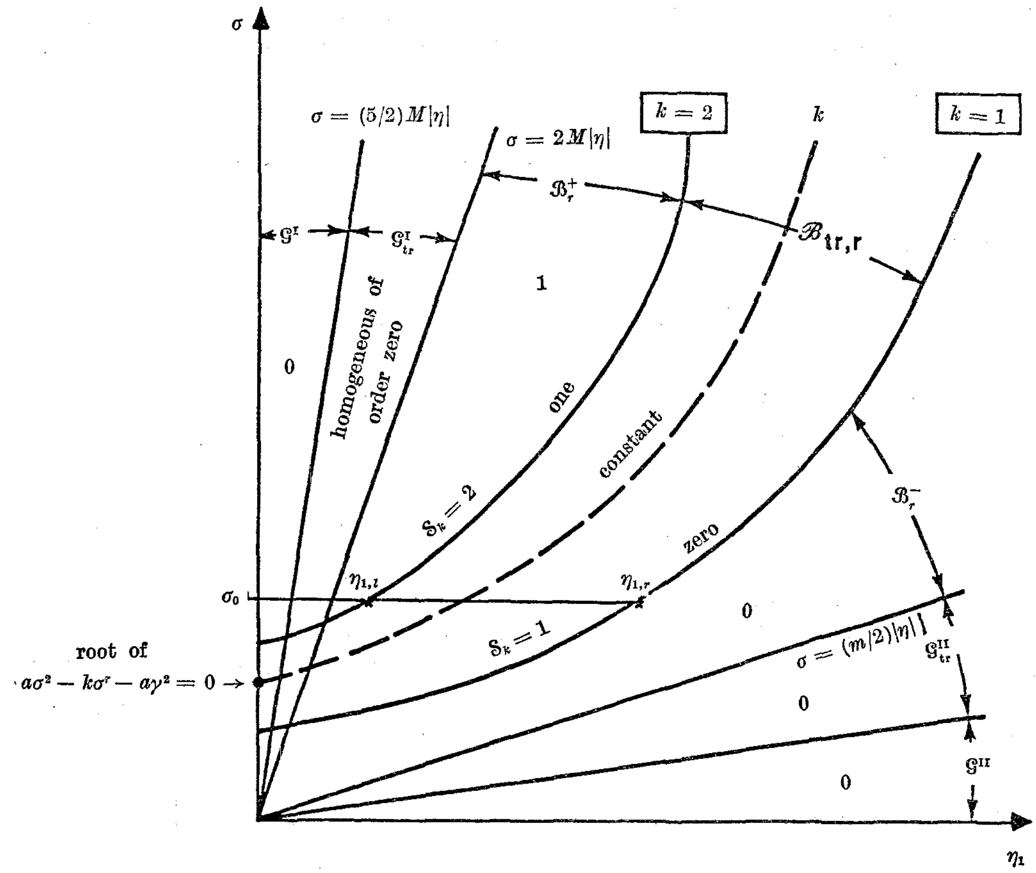

Figure 5.3. Definition of $\chi^r = \chi^{\mathcal{B}_r^+}$.

Let now $F_h(s)$ be a C^∞-function of the real scalar variable s defined so that

$$F_h(s) = \begin{cases} 1 & s \leqslant 0\,, \\ 0 & s \geqslant h > 0\,, \end{cases} \tag{5.31}$$

and monotonically decreasing on $0 \leqslant s \leqslant h$. With $F_h(s)$ at hand, we now define the symbol $\chi^r(x, y; \sigma, \eta)$ over all of $\mathcal{B}_{tr,r}$ through the following steps:

(i) First define χ^r on the *reference segment* (5.29)

$$\{(x_0, y_0; \sigma_0, \eta = [\eta_1, 0, \ldots, 0]),\ \eta_{1,l} < \eta_1 < \eta_{1,r}\} \tag{5.32}$$

by « transplanting » the decreasing part of the graph of $F_h(s)$ suitably « dilated or compressed » so that the point $s = 0$ of the graph falls on the point $(x_0, y_0, \sigma_0, \eta = [\eta_{1,l}, 0, \ldots, 0])$, while the point $s = h$ of the graph falls on the point $(x_0, y_0, \sigma_0, \eta = [\eta_{1,r}, 0, \ldots, 0])$. This means that we set

$$\chi^r(x_0, y_0; \sigma_0, \eta = [\eta_1, 0, \ldots, 0]) \equiv F_h(\eta_1 - \eta_{1,l}), \qquad \eta_{1,l} < \eta_1 < \eta_{1,r}$$

where in (5.33) we have by (5.29*b*)

$$h = \eta_{1,r} - \eta_{1,l} = \eta_1(k = 1, x_0, y_0, \sigma_0) - \eta_1(k = 2, x_0, y_0, \sigma_0). \tag{5.34}$$

(ii) Next, we extend the definition of $\chi^r = \chi^{\mathcal{B}_r^+}$ over the entire « tube » $\mathcal{B}_{\text{tr},r} = \bigcup_{1 \leq k \leq 2} S_k$ by parametrization, as follows: we impose that the symbol χ^r assumes the same constant value on each surface S_k, for each fixed k, $1 \leq k \leq 2$ (such value is therefore equal to the value that χ^r takes on at the intersection point of S_k with the reference segment (5.29)). Analytically this means the following: given any point $(x, y, \sigma, \eta) \in \mathcal{B}_{\text{tr},r}$, we first determine via (5.27) the particular value of the parameter $k(x, y, \sigma, \eta)$ (between 1 and 2), whose corresponding surface $S_{k(x,y,\sigma,\eta)}$ passes through it. Next, we consider via (5.30) the coordinate $\eta_1(k(x, y, \sigma, \eta), x_0, y_0, \sigma_0)$ on the *reference segment* (5.29), which has the same parameter as the original point (x, y, σ, η) and finally set

$$\chi^r(x, y, \sigma, \eta) = \chi^r\Big(x_0, y_0, \sigma_0, \big[\eta_1(k(x, y, \sigma, \eta), x_0, y_0, \sigma_0), 0, \ldots, 0\big]\Big), \tag{5.35a}$$

$$(\text{by } (5.33)) = F_h\Big(\eta_1(k(x, y, \sigma, \eta), x_0, y_0, \sigma_0) - \eta_{1,l}\Big), \qquad (x, y, \sigma, \eta) \in \mathcal{B}_{\text{tr},r}, \tag{5.35b}$$

where h is given by (5.34). Equation (5.35) defines χ^r in $\mathcal{B}_{\text{tr},r}$.

From (5.18) we have, moreover, that

$$\chi^r = \chi^{\mathcal{B}_r^+} = \begin{cases} 1 & \text{on } \mathcal{B}_r^+, \\ 0 & \text{on } \mathcal{B}_r^- \cup \mathcal{G}_{\text{tr}}^{\text{II}} \cup \mathcal{G}^{\text{II}} \cup \mathcal{G}^{\text{I}}. \end{cases} \tag{5.36}$$

As to the definition of χ^r over $\mathcal{G}_{\text{tr}}^{\text{I}}$, we proceed in agreement with (5.18) and the paragraph preceding (5.14); i.e. by homogeneity of order zero in σ, η: we first define χ^r smoothly decreasing from the value one on the cone $(x, y, \sigma = 2M|\eta|, \eta)$ —for (x, y) fixed—to the value zero on the cone $(x, y, \sigma = \frac{3}{2}M|\eta|, \eta)$ along, say, a unit sphere and then extend to all of $\mathcal{G}_{\text{tr}}^{\text{I}}$ by homogeneity or order zero (constancy) in σ, η.

The definition of $\chi^r = \chi^{\mathcal{B}_r^+}$ on all of $R^{2n}(+)$ is complete, as an C^∞-function on all of its variables.

STEP 2. – We now prove (5.23) in the case $\beta = 0$, $|\alpha| = 1$; i.e. we must estimate $\partial\chi^r/\partial\eta_j$ and $\partial\chi^r/\partial\sigma$ on $\mathcal{B}_{\text{tr},r}$, where $\sigma \sim |\eta|$. From (5.35*b*), by chain rule

$$\frac{\partial\chi^r}{\partial\eta_j} = \left(\frac{dF_h}{ds}\right)\left(\frac{\partial\eta_1}{\partial\eta_j}\left(k(x, y, \sigma, \eta), x_0, y_0, \sigma_0\right)\right) =$$

(from (5.30) with $k = k(x, y, \sigma, \eta)$)

$$= \frac{dF_h}{ds}\left[\frac{-\sigma_0^r}{2\eta_1(k, x_0, y_0, \sigma_0)[a_{11}(x_0, y_0) - a_{n1}^2(x_0, y_0)]}\right]\frac{\partial k}{\partial\eta_j}(x, y, \sigma, \eta) = \tag{5.37}$$

$$= \mathcal{O}\left(\frac{\partial k}{\partial\eta_j}(x, y, \sigma, \eta)\right) \tag{5.38}$$

with $\mathcal{O}$ denoting as usual an upper bound for the absolute value with a constant independent of $x, y \in \Omega$. From (5.27) and (1.9)

$$\frac{\partial k}{\partial\eta_j}(x, y, \sigma, \eta) = -\frac{1}{\sigma^r}\frac{1}{a^2(x, y)}\frac{\partial d}{\partial\eta_j}(x, y; \eta) = \tag{5.39}$$

$$= -\frac{1}{\sigma^r}\left[\sum_{\substack{i=1\\ i\neq j}}^{n-1} a_{ij}\eta_i + 2a_{jj}\eta_j - 2\left(\sum_{i=1}^{n-1} a_{ni}\eta_i\right)a_{nj}\right] =$$

$$= \frac{1}{\sigma^r}\mathcal{O}(|\eta|) = \mathcal{O}(\sigma^{1-r}) \qquad x, y \in \Omega;\ (x, y, \sigma, \eta) \in \mathcal{B}_{\text{tr},r}$$

the constant on $\mathcal{O}$ being independent of $x, y \in \Omega$ (by assumption (i), section 1), since in the region $\mathcal{B}_{\text{tr},r}$ we have $\sigma \sim |\eta|$. Returning to (5.38), we conclude

$$\frac{\partial x^r}{\partial\eta_j}(x, y; \sigma, \eta) = \mathcal{O}(\sigma^{1-r})\,; \qquad (x, y, \sigma, \eta) \in \mathcal{B}_{\text{tr},r}; \qquad x, y \in \Omega \tag{5.40}$$

which is (5.23) in the present case. In the same way, we compute

$$\frac{\partial x^r}{\partial\sigma}(x, y, \sigma, \eta) = \mathcal{O}\left(\frac{\partial k}{\partial\sigma}(x, y, \sigma, \eta)\right) = \tag{5.41}$$

(from (5.27))

$$= \left((2 - r)\sigma^{1-r}a(x, y) + r\left[a(x, y)\gamma^2 + \frac{d(x, y, \eta)}{a^2(x, y)}\right]\sigma^{r(r+1)}\right) =$$

$$= \mathcal{O}(\sigma^{1-r}) \qquad \text{for } (x, y, \sigma, \eta) \in \mathcal{B}_{\text{tr},r},\ \sigma \geqslant 1,\ x, y \in \Omega$$

which is (5.23) in the present case. The case $\beta = 0$, $|\alpha| = 1$ is settled.

STEP 3. – We now analyze the case $\alpha = 0$, $|\beta| = 1$ by estimating $\partial\chi^r/\partial x$ and $\partial\chi^r/\partial y_j$. From (5.35*b*) and (5.30) with $k = k(x, y, \sigma, \eta)$, we obtain as before (see (5.38))

$$\frac{\partial \chi^r}{\partial x} = \frac{dF_h}{ds}\frac{\partial \eta_1}{\partial x}\left(k(x, y, \sigma, \eta), x_0, y_0, \sigma_0\right) = \mathcal{O}\left(\frac{\partial k}{\partial x}(x, y, \sigma; \eta)\right) =$$

(from (5.27)) $$= \mathcal{O}(\sigma^{2-r}) + \mathcal{O}\left(\frac{1}{\sigma^r}\right) + \mathcal{O}\left(\frac{1}{\sigma^r}\frac{\partial}{\partial x}\frac{d(x, y; \eta)}{a^2(x, y)}\right) =$$

(from (1.9)) $$= \mathcal{O}(\sigma^{2-r}) + \mathcal{O}\left(\frac{1}{\sigma^r}\right) + \mathcal{O}\left(\frac{1}{\sigma^r}|\eta|^2\right).$$

Thus, using $\sigma \sim |\eta|$ in $\mathcal{B}_{\mathrm{tr},r}$, and $\sigma \geqslant 1$ we obtain as $\sigma \sim |\eta| \to \infty$

$$\left.\begin{array}{l}\dfrac{\partial \chi^r}{\partial x}(x, y, \sigma, \eta) \\ \dfrac{\partial}{\partial y_j}\chi^r(x, y, \sigma, \eta)\end{array}\right\} = \mathcal{O}(\sigma^{2-r}), \quad (x, y, \sigma, \eta) \in \mathcal{B}_{\mathrm{tr},r} \quad (x, y) \in \Omega \tag{5.42}$$

which is (5.23) in the present case.

Similarly, one can check the general case. □

5.3. *The commutator* $[P, \boldsymbol{\chi}^{\mathcal{B}_r^\pm}]$.

The goal of the present subsection is to prove the following result, which will be fundamental in the analysis in section 6. Here $P = P(x, y; D_t, D_x, D_y)$ as in (1.2).

THEOREM 5.3. – Let [2] $3/2 < r < 2$. Then

a)

$$[P, \boldsymbol{\chi}^{\mathcal{B}_r^\pm}]D_t^{r-2} \text{ continuous } H^1(Q) \to L_2(Q), \tag{5.43}$$

b) (recall $\Lambda = \Lambda_{t,y}$ from (4.1*a*))

$$[P, \boldsymbol{\chi}^{\mathcal{B}_r^\pm}]\Lambda^{r-2} \text{ continuous } H^1(Q) \to L_2(Q). \tag{5.44}$$

PROOF. – We prove first part *a*).

STEP 1. – Writing $\boldsymbol{\chi}^r$ for $\boldsymbol{\chi}^{\mathcal{B}_r^\pm}$ throughout and recalling (3.14*b*) and (3.10*b*), we have

$$[P, \boldsymbol{\chi}^r] = [D_x^2, \boldsymbol{\chi}^r] + \left[\left(\sum_{j=1}^{n-1} a_{nj}D_{y_j}\right)D_x, \boldsymbol{\chi}^r\right] + \left[D_x\left(\sum_{j=1}^{n-1} a_{nj}D_{y_j}\right), \boldsymbol{\chi}^r\right] + \\ + \left[\left(\sum_{j=1}^{n-1} a_{nj}D_{y_j}\right)^2, \boldsymbol{\chi}^r\right] - [D_1 - 2i\gamma D_2, \boldsymbol{\chi}^r]. \tag{5.45}$$

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[2] As alread pointed out, r will be identified in section 6 as $r = 8/5$, so that $r - 2 = -2/5$.

STEP 2 (*Analysis of* $[D_x^2, \chi^r]$: *statements*). – We compute by chain rule

$$[D_x^2, \chi^r]u = D_x^2(\chi^r u) - \chi^r(D_x^2 u) = (D_x^2 \chi^r)u + 2(D_x \chi^r)(D_x u) . \tag{5.46}$$

Hence, since D_t^{r-2} commutes with D_x

$$[D_x^2, \chi^r]D_t^{r-2}u = (D_x^2 \chi^r)D_t^{r-2}u + 2(D_x \chi^r)D_t^{r-2}(D_x u) . \tag{5.47}$$

With $u \in H^1(Q)$, we have $D_x u \in L_2(Q)$ in (5.47). The following two lemmas are then seen to be needed.

LEMMA 5.4. – Let $3/2 < r < 2$. Then, with reference to Definition 3.2, below (3.15) we have

$$(D_x^2 \chi^r)D_t^{r-2} \in OPS^{2-r}_{r-1,2-r}(R^{n+1}_{tyx}) , \qquad \text{uniformly in } x \in R^1_{x+} . \tag{5.48}$$

LEMMA 5.5. – Let $3/2 < r < 2$. Then, with reference to Definition 3.2, we have

$$(D_x \chi^r)D_t^{r-2} \in OPS^{0}_{r-1,2-r}(R^{n+1}_{tyx}) , \qquad \text{uniformly in } x \in R^1_{x+} . \tag{5.49}$$

COROLLARY 5.6 (to Lemma 5.4). – Let $3/2 < r < 2$. Then

$$(D_x^2 \chi^r)D_t^{r-2}: \text{ continuous } H^1(Q) \to L_2(R_x; H^{r-1}(R^n_{ty})) . \qquad \square \tag{5.50}$$

COROLLARY 5.7 (to Lemma 5.5). – Let $3/2 < r < 2$. Then

$$(D_x \chi^r)D_t^{r-2}: \quad \text{continuous } L_2(Q) \to L_2(Q) . \qquad \square \tag{5.51}$$

COROLLARY 5.8. – Let $3/2 < r < 2$. Then

$$[D_x^2, \chi^r]D_t^{r-2}: \quad \text{continuous } H^1(Q) \to L_2(Q) . \qquad \square \tag{5.52}$$

STEP 3. – *Analysis of* $[D_x^2, \chi^r]$: *proofs.*

PROOF OF LEMMA 5.4. – From (3.21) taking $\gamma = 0$ w.l.o.g.

$$\text{symbol of } (D_x^2 \chi^r) = D_x^2 \chi^r ; \qquad \text{symbol of } D_t^{r-2} = \sigma^{r-2} . \tag{5.53}$$

The essence of the proof is that:

$$D_x^2 \chi^r \in S^{2(2-r)}_{r-1,2-r}(R^{n+1}_{tyx}) , \qquad \text{uniformly in } x \in R^1_{x+} , \tag{5.54}$$

$$\sigma^{r-2} \in S^{r-2}_{1,0}(R^n_{ty}) , \qquad \text{uniformly in } x \in R^1_{x+} . \tag{5.55}$$

Then, $D_x^2\chi^r$ having compact support in Ω (by assumption (i), section 1), the product theorem as in [T.1, Thm. 4.4, p. 46] applies and gives the desired conclusion (5.48) since: $2(2-r)+r-2=2-r$, $\min\{r-1,1\}=r-1$, $\max\{2-r,0\}=2-r$. To see (5.54), it suffices to consider $D_x^2\chi^r$ in the region $\mathcal{B}_{\text{tr},r}$ where (5.23) of Lemma 5.1 holds, $\chi^r=\chi^{\mathcal{B}_r^\pm}$ (for $\chi^{\mathcal{B}_r^\pm}$ is either homogeneous of order zero in $\mathcal{G}_{\text{tr}}^{\text{I}}$ or $\mathcal{G}_{\text{tr}}^{\text{II}}$, respectively, or identically zero or one elsewhere). Then, by replacing $|\beta|$ with $|\beta|+2$ in (5.23), it follows that

$$(5.56)\qquad |D_{\substack{x\\y}}^{\beta} D_{\substack{\sigma\\\eta}}^{\alpha} D_x^2\chi^r(x,y,\sigma,\eta)\leqslant C_{\alpha,\beta}(|\sigma|+|\eta|)^{2(2-r)-|\alpha|(r-1)+|\beta|(2-r)},$$
$$(x,y,\sigma,\eta)\in\mathcal{B}_{\text{tr},r}\,,\quad \sigma\sim|\eta|\to\infty\,,$$

with $C_{\alpha,\beta}$ independent of $(x,y)\in\Omega$; i.e. (5.54) is obtained, via Definition 3.2, Remark 3.1, below (3.15).

(Instead if appealing to the product theorem, one can use

$$(5.57)\qquad \text{symbol of } \{(D_x^2\chi^r)D_t^{r-2}\}=(D_x^2\chi^r)\sigma^{r-2}$$

which follows from the asymptotic expansion [T.1, (4.5), p. 46]). □

PROOF OF LEMMA 5.5. – It is similar to that of Lemma 5.4. Now

$$(5.58)\qquad \text{symbol of } (D_x\chi^r)=D_x\chi^r$$

so that, from (5.23) with $|\beta|$ replaced by $|\beta|+1$, we obtain

$$(5.59)\qquad |D_{\substack{x\\y}}^{\beta} D_{\substack{\sigma\\\eta}}^{\alpha} D_x\chi^r(x,y,\sigma,\eta)|\leqslant C_{\alpha,\beta}(|\sigma|+|\eta|)^{2-r-|\alpha|(r-1)+|\beta|(2-r)},$$
$$(x,y,\sigma,\eta)\in\mathcal{B}_{\text{tr},r}\,,\quad \sigma\sim|\eta|\to\infty\,,$$

with $C_{\alpha,\beta}$ independent of $(x,y)\in\Omega$. This means

$$(5.60)\qquad D_x\chi^r\in S_{r-1,2-r}^{2-r}(R_{ty}^n)\,,\quad \text{uniformly in } x\in R_{x+}^1\,.$$

Then, (5.60) and (5.55) imply the desired conclusion (5.49) via the product theorem [T.1, Thm. 4.4] since $2-r+r-2=0$. □

PROOF OF COROLLARY 5.6. – Apply Lemma 5.4 and Lemma 3.1, Eq. (3.17*c*). □

PROOF OF COROLLARY 5.7. – Apply Lemma 5.5 and Lemma 3.1, Eq. (3.17*a*) with $s=0$. □

PROOF OF COROLLARY 5.8. – Use identity (5.47), Corollary 5.6 and Corollary 5.7. □

STEP 4. – *Analysis of* $[a_{nj}D_{y_j}D_x, \chi^r]$: *statements.* By direct computations

$$[a_{nj}D_{y_j}D_x, \chi^r]u = a_{nj}D_{y_j}D_x(\chi^r u) - \chi^r a_{nj}D_{y_j}D_x u = \\ = a_{nj}(D_{y_j}D_x\chi^r)u + a_{nj}(D_x\chi^r)(D_{y_j}u) + a_{nj}(D_{y_j}\chi^r)(D_x u) + \\ + a_{nj}\chi^r D_{y_j}(D_x u) - \chi^r a_{nj}D_{y_j}(D_x u)\,. \tag{5.61}$$

Thus, since D_t^{r-2} commutes with both D_x and D_{y_j}

$$[a_{nj}D_{y_j}D_x, \chi^r]D_t^{r-2}u = a_{nj}(D_{y_j}D_x\chi^r)D_t^{r-2}u + a_{nj}(D_x\chi^r)D_t^{r-2}(D_{y_j}u) + \\ + a_{nj}(D_{y_j}\chi^r)D_t^{r-2}(D_x u) + [a_{nj}, \chi^r]D_{y_j}D_t^{r-2}(D_x u)\,. \tag{5.62}$$

With $u \in H^1(Q)$, the following lemmas are then needed for the terms in (5.62).

LEMMA 5.9. – Let $3/2 < r < 2$. Then

$$a)\ a_{nj}(D_{y_j}D_x\chi^r)D_t^{r-2}: \quad \text{continuous } H^1(Q) \to L_2(Q)\,, \tag{5.63}$$

$$b)\ a_{nj}(D_x\chi^r)D_t^{r-2}: \quad \text{continuous } L_2(Q) \to L_2(Q)\,, \tag{5.64}$$

$$c)\ a_{nj}(D_{y_j}\chi^r)D_t^{r-2}: \quad \text{continuous } L_2(Q) \to L_2(Q)\,. \quad \square \tag{5.65}$$

LEMMA 5.10. – Let $3/2 < r < 2$. Then

$$[a_{nj}, \chi^r]D_{y_j}D_t^{r-2}: \quad \text{continuous } L_2(Q) \to L_2(Q)\,. \quad \square \tag{5.66}$$

COROLLARY 5.11 (to Lemmas 5.9 and 5.10). – Let $3/2 < r < 2$. Then

$$[a_{nj}D_{y_j}D_x, \chi^r]D_t^{r-2}: \quad \text{continuous } H^1(Q) \to L_2(Q)\,. \quad \square \tag{5.67}$$

STEP 5. – *Analysis of* $[a_{nj}D_{y_j}D_x, \chi^r]$: *proofs.*

PROOF OF LEMMA 5.9. – Assertion (5.63) follows as in the arguments leading to Corollary 5.6 for $(D_x^2\chi^r)D_t^{r-2}$. Also, assertions (5.64)-(5.65) follow as in the arguments leading to Corollary 5.7 for $(D_x\chi^r)D_t^{r-2}$. $\square$

PROOF OF LEMMA 5.10. – The essence of the proof is the following argument. Since

$$a_{n_j} \in S^0_{1,0}(R^n_{ty}) \quad \text{and} \quad \chi^r \in S^0_{r-1,2-r}(R^{n+1}_{tyx})\,, \quad \text{uniformly in } x \in R^1_{x+} \tag{5.68}$$

the second statement in (5.68) being Lemma 5.1, then (the commutator) Lemma 3.2 implies that

$$[a_{n_j}, \chi^r] \in OPS^{1-r}_{r-1,2-r}(R^{n+1}_{tyx})\,, \quad \text{uniformly in } x \in R^1_{x+}\,. \tag{5.69}$$

This is so since in this case (in the notation of Lemma 3.2), we have

$$\varrho' - \delta'' = 1 - (2 - r) = r - 1 , \quad \varrho'' - \delta' = r - 1 - 0 = r - 1 ,$$

so that $m_1 = r - 1$ and the order is $0 + 0 - m_1 = 1 - r$. (Assumption $\delta'' = 2 - \varrho < \min\{1, r - 1\}$ holds for $3/2 < r < 2$). But in the crucial region $\mathcal{B}_{\mathrm{tr},r}$ where $\sigma \sim |\eta|$, the two operators D_t and D_{y_j} have «the same behavior» in the variables t and y. This means that

$$D_{y_j} D_t^{r-2} \in OPS_{1,0}^{r-1}(R^{n+1}_{tyx}) , \quad \text{uniformly in } x \in R^1_{x+} . \tag{5.70}$$

Finally, from (5.69), (5.70), the product theorem [T.1, Thm 4.4, p. 46] gives

$$[a_{n_j}, \chi^r] D_{y_j} D_t^{r-2} \in OPS^0_{r-1,\,2-r}(R^{n+1}_{tyx}) , \quad \text{uniformly in } x \in R^1_{x+} \tag{5.71}$$

since $1 - r + r - 1 = 0$, $\min\{r - 1, 1\} = r - 1$, $\max[2 - r, 0] = 2 - r$. Once (5.71) is proved, then the desired conclusion (5.66) follows by applying Lemma 3.1, Eq. (3.17) with $s = 0$.

The detailed proof of (5.71) is based on noticing that the asymptotic expansion for the symbol of a product gives [T.1, (4.5), p. 46]

$$\text{symbol of } \{a_{n_j} \chi^r D_{y_j} D_t^{r-2}\} = a_{nj} \chi^r \eta_j \sigma^{r-2} , \tag{5.72}$$

$$\text{symbol of } \{\chi^r a_{nj} D_{y_j} D_t^{r-2}\} \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} \{D^\alpha_{\substack{\eta\\ \xi}} \chi^r\} \{D^\alpha_{\substack{y\\ x}} a_{nj}\} \eta_j \sigma^{r-2} \tag{5.73}$$

so that in the region $\mathcal{B}_{\mathrm{tr},r}$ where $\sigma \sim |\eta|$, we have:

$$\text{symbol of } \{[a_{nj}, \chi^r] D_{y_j} D_t^{r-2}\} \sim \sum_{\alpha \geq 1} \frac{i^{|\alpha|}}{\alpha!} \{D^\alpha_\eta \chi^r\} \{D^\alpha_y a_{nj}\} \sigma^{r-1} - (D^1_\eta \chi^r\} \sigma^{r-1} . \tag{5.74}$$

From (5.74), using Lemma 5.1, one obtains by considering the Eorst case $\alpha = 1$

$$\left| D^\beta_{\substack{y\\ t}} D^{\alpha'}_{\substack{\eta\\ \sigma}} \{\text{symbol of } \{[a_{n_j}, \chi^r] D_{y_j} D_t^{r-2}\}\} \right| \leq C_{\alpha',\beta} (|\sigma| + |\eta|)^{-|\alpha'|(r-1) + |\beta|(2-r)} , \tag{5.75}$$
$$(x, y, \sigma, \eta) \in \mathcal{B}_{\mathrm{tr},r} , \quad \sigma \sim |\eta| \to \infty ,$$

$C_{\alpha',\beta}$ independent of $(x, y) \in \Omega$, and (5.71) follows. □

PROOF OF COROLLARY 5.11. - Use identity (5.62) and Lemmas 5.9 and 5.10. □

STEP 6. - *Analysis of* $[D_x(a_{nj} D_{y_j}), \chi^r]$. We reduce this term to the term considered in Step 4, modulo a first order (lower order) commutator in all variables

$$D_x(a_{nj} D_{y_j}) = (a_{nj} D_{y_j}) D_x + [D_x, a_{nj} D_{y_j}] . \tag{5.76}$$

Thus, by Corollary 5.11, we have

LEMMA 5.12. – Let $3/2 < r < 2$. Then

$$[D_x(a_{nj}D_{y_j}), \chi^r]D_t^{r-2}: \quad \text{continuous } H^1(Q) \to L_2(Q) . \tag{5.77}$$

STEP 7. – *Analysis of remaining terms* $[(a_{nj}D_{y_j})(a_{n_i}D_{y_i}), \chi^r]$, $[D_1, \chi^r]$, *and* $[D_2, \chi^r]$. Recalling identity (5.45), Corollary 5.8, Corollary 5.11 and Lemma 5.12, we see that the proof of part *a*) of Theorem 5.3 is complete, as soon as we establish the following lemmas

LEMMA 5.13. – Let $3/2 < r < 2$. Then

$$\left.\begin{matrix}[(a_{nj}D_{y_j})(a_{nj}D_{y_i}), \chi^r]D_t^{r-2} \\ [D_1, \chi^r]D_t^{r-2}\end{matrix}\right\} \in OPS^1_{r-1,2-r}(R^{n+1}_{tyx}) , \quad \text{uniformly in } x \in R^1_{x^+} . \tag{5.78}$$

Hence, by Lemma 3.1, Eq. (3.17*b*)

$$\left.\begin{matrix}([a_{nj}D_{y_j})(a_{ni}D_{y_i}), \chi^r]D_t^{r-2} \\ [D_1, \chi^r]D_t^{r-2}\end{matrix}\right\}: \quad \text{continuous } H^1(Q) \to L_2(Q) . \quad \square \tag{5.79}$$

LEMMA 5.14. – Let $3/2 < r < 2$. Then

$$[D_2, \chi^r]D_t^{r-2} \in OPS^0_{r-1,2-r}(R^{n+1}_{tyx}) , \quad \text{uniformly in } x \in R^1_{x^+} . \tag{5.80}$$

Hence, by Lemma 3.1, Eq. (3.17*a*) with $s = 1$

$$[D_2, \chi^r]D_t^{r-2}: \quad \text{continuous } H^1(Q) \to H^1(Q) . \quad \square \tag{5.81}$$

PROOF OF LEMMA 5.13. – Recalling Lemma 5.1 and (3.13*a*), we have

$$\left.\begin{matrix}(a_{nj}D_{y_j})(a_{ni}D_{y_i}) \\ D_1\end{matrix}\right\} \in OPS^2_{1,0}(R^{n+1}_{tyx}) , \quad \text{uniformly in } x \in R^1_{x^+} , \tag{5.82}$$

$$\chi^r \in S^0_{r-1,2-r}(R^{n+1}_{tyx}) , \qquad \text{uniformly in } x \in R^1_{x^+} .$$

Thus, (the commutator) Lemma 3.2 implies

$$\left.\begin{matrix}[(a_{nj}D_{y_j})(a_{ni}D_{y_i}), \chi^r] \\ [D_1, \chi^r]\end{matrix}\right\} \in OPS^{3-r}_{r-1,2-r}(R^{n+1}_{tyx}) , \quad \text{uniformly in } x \in R^1_{x^+} \tag{5.83}$$

since, in the notation of Lemma 3.2, we have

$$\varrho' - \delta'' = 1 - (2-r) = r - 1\,, \qquad \varrho'' - \delta' = r - 1 - 0\,,$$

i.e. $m_1 = r - 1$, and the order is then $2 + 0 - m_1 = 3 - r$. (The assumption $\delta'' = 2 - r < \varrho = r - 1$ holds for $3/2 < r < 2$). The product theorem [T.1, Thm. 4.4 p. 46] between the operators in (5.83) and $D_t^{r-2} \in OPS_{1,0}^{r-2}(R_{tyx}^{n+1})$, uniformly in $x \in R_x^1$, yields (5.78), as desired. □

PROOF OF LEMMA 5.14. – Similarly, since $D_2 \in OPS_{1,0}^1(R_{tyx}^n)$, uniformly in $x \in R_{x+}^1$ (see (3.12*a*)), the commutator Lemma 3.2 implies

(5.84) $$[D_2, \chi^r] \in OPS_{r-1,2-r}^{1-(r-1)}(R_{tyx}^{n+1})\,, \quad \text{uniformly in } x \in R_{x+}^1$$

since $m_1 = \varrho' - \delta'' = \varrho'' - \delta' = r - 1$ and the product theorem yields (5.80). □

Part *a*) of Theorem 5.3 is proved. Part *b*) then is an immediate consequence of part *a*), in view of the definition (4.1*a*) of Λ_{ty} (in the crucial region $\mathcal{B}_{tr,r}$, we have that D_t^{r-2} and D_y^{r-2} behave likewise, $\sigma \sim |\eta|$). □

5.4. *The commutator* $[B, \chi^{\mathcal{B}_r^\pm}]$.

Let $B = B(y; D_x, D_y)$ as in (1.4). The object of this subsection is to prove

THEOREM 5.15. – Let $1 < r < 2$. Then

a)

(5.85) $$[B, \chi^{\mathcal{B}_r^\pm}]\,|_{x=0} \in OPS_{r-1,2-r}^{2-r}(\Sigma)\,,$$

b) more generally, for any real k

(5.86) $$[B, \chi^{\mathcal{B}_r^\pm}]\,|_{x=0} D_t^k \in OPS_{r-1,2-r}^{2-r+k}(\Sigma)\,,$$

c) let $3/2 < r < 2$ (so that $r - 1 > 2 - r$). Then, with $\Lambda = \Lambda_{ty}$ as in (4.1*a*)

(5.87*a*) $\Lambda^k [B, \chi^{\mathcal{B}_r^\pm}]\,|_{x=0}$

(5.87*b*) $[B, \chi^{\mathcal{B}_r^\pm}]\,|_{x=0} D_t^k$

$$\left.\begin{matrix}\Lambda^k [B, \chi^{\mathcal{B}_r^\pm}]\,|_{x=0} \\ [B, \chi^{\mathcal{B}_r^\pm}]\,|_{x=0} D_t^k\end{matrix}\right\}: \quad \text{continuous } H^s(\Sigma) \to H^{s-(2-r+k)}(\Sigma)\,. \qquad \square$$

PROOF. – The essence of the proof is that $B|_{x=0} \in OPS_{1,0}^1(\Sigma)$ and $\chi^{\mathcal{B}_r^\pm}|_{x=0} \in OPS_{r-1,2-r}^0(\Sigma)$ by Lemma 5.1. Then, (the commutator) Lemma 3.2 implies at once (5.85), since in the notation of Lemma 3.2, the order is $1 + 0 - m_1 = 1 - (r-1) = 2 - r$, for $m_1 = \varrho' - \delta'' = 1 - (2-r) = \varrho'' - \delta' = r - 1$. The product theorem [T.1, Thm. 4.4, p. 46] with $D_t^k \in OPS_{1,0}^k(\Sigma)$ then yields (5.86). (Note that the operator in (5.85) is constant in y outside a compact set). Lemma 3.1, Eq. (3.17c) —applied this time on Σ—yields (5.87) from (5.86). About Λ^k we recall that Λ^k and D_t^k belong to the same class in $\mathcal{B}$ where $\sigma \sim |\eta|$.

A detailed computation, based on asymptotic expansions of symbols, [T.1, (4.5), p. 46] can be given to show (5.86) explicitly. One has, writing $\chi^r \equiv \chi^{\mathcal{B}_r^\pm}$

$$\{\text{symbol of } B\chi^r|_{x=0}\} = b\chi^r + 1\cdot D_x^1 \chi^r + (D_\eta^{|\alpha|=1} b)(D_y^{|\alpha|=1}\chi^r) \quad \text{at } x = 0\,, \tag{5.88}$$

$$\{\text{symbol of } \chi^r B|_{x=0}\} \sim \tag{5.89}$$
$$\sim \chi^r b + (D_\eta^{|\alpha|=1} b)(D_y^{|\alpha|=1}\chi^r) + \sum_{\alpha>1} \frac{i^{|\alpha|}}{\alpha!}\{D_\eta^\alpha \chi^r\}\{D_y^\alpha b\} \quad \text{at } x = 0$$

where $D_{\eta_j}^1 b = b_j(y)$ and $D_y^\alpha b = \sum_{j=1}^{n-1} (D_y^\alpha b_j(y))\eta_j$ from (3.15). After subtracting (5.89) from (5.88) one obtains an explicit expansion

$$D_{\substack{t\\x\\y}}^\beta D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'}\{\text{symbol of } \{[B,\chi^r]|_{x=0} D_t^k\}\} \sim \tag{5.90}$$
$$\sim \{D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'} D_{\substack{t\\x\\y}}^\beta (D_x^1\chi^r)\sigma^k + (D_{\substack{t\\x\\y}}^\beta D_\eta^{|\alpha|=1} b)(D_y^{|\alpha|=1} D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'}\chi^r)\sigma^k + (D_\eta^{|\alpha|=1} b)(D_{\substack{t\\y\\x}}^\beta D_y^{|\alpha|=1} D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'}\chi^r)\sigma^k +$$
$$+ \sigma^k \sum_{\alpha>1}\frac{i^{|\alpha|}}{\alpha!}\Bigg\{(D_{\substack{t\\x\\y}}^\beta D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'} D_\eta^\alpha \chi^r)\left(\sum_{j=1}^{n-1}(D_y^\alpha b_j(y))\eta_j\right) + (D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'} D_\eta^\alpha\chi^r)\left(\sum_{j=1}^{n-1}(D_t^\beta D_y^\alpha b_j(y))\eta_j\right) +$$
$$+ (D_{\substack{t\\x\\y}}^\beta D_\eta^\alpha \chi^r) D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'}\left(\sum_{j=1}^{n-1}(D_y^\alpha b_j(y))\eta_j\right) + (D_\eta^\alpha\chi^r)\left(D_{\substack{\sigma\\\xi\\\eta}}^{\alpha'}\sum_{j=1}^{n-1}(D_{\substack{t\\x\\y}}^\beta D_y^\alpha b_j(y))\eta_j\right)\Bigg\} \quad \text{at } x = 0\,.$$

Using (5.23) in Lemma 5.1 on the terms of (5.90) yields the upperbound

$$C_{\alpha',\beta}(\sigma + |\eta|)^{(2-r+k)-|\alpha'|(r-1)+|\beta|(2-r)} \quad \text{for } (x=0, y, \sigma, \eta) \in \mathcal{B}_{\text{tr},r}$$

$\sigma \sim |\eta| \to \infty$ as desired, and (5.86) follows. □

5.5. *The operator* $D_1\chi^{\mathcal{B}_r^-}$.

The goal of the present subsection is to prove the following result.

THEOREM 5.16. - Let $3/2 < r < 2$. With reference to (3.13) and (5.16), (5.19), we have

a)

$$D_1\chi^{\mathcal{B}_r^-} \in OPS_{r-1,2-r}^r(R_{tyx}^{n+1})\,, \quad \text{uniformly in } x \in R_{x^+}^1\,. \tag{5.91a}$$

Hence

$$D_1\chi^{\mathcal{B}_r^-}: \quad \text{continuous } H^1(Q) \to L_2(R^1_{x^+}; H^{1-r}(R^n_{ty})) . \tag{5.91b}$$

b) With reference to (4.1*a*), we have for $\alpha \geqslant 0$

$$\Lambda^\alpha D_1\chi^{\mathcal{B}_r^-} \in OPS^{r+\alpha}_{r-1,2-r}(R^{n+1}_{tyx}) , \qquad \text{uniformly in } x \in R^1_{x^+} . \tag{5.92a}$$

Hence

$$\Lambda^\alpha D_1\chi^{\mathcal{B}_r^-}: \quad \text{continuous } H^1(Q) \to L_2(R^1_{x^+}; H^{1-r-\alpha}(R^n_{ty})) . \tag{5.92b}$$

PROOF. – It suffices to prove (5.91*a*) and then invoke the product theorem [T.1, Thm. 4.4, p. 46] with $\delta'' = 2 - r < \min\{1, r-1\} = r - 1$ for (5.92*a*); and Lemma 3.1 *c*) with $r - 1 > 2 - r$ for (5.91*b*)-(5.92*b*).

PROOF OF (5.91*a*). – The asymptotic expansion for the symbol of $D_1\chi^{\mathcal{B}_r^-}$ is [T.1, (4.5), p. 46]

$$\begin{aligned}\{\text{symbol of } (D_1\chi^{\mathcal{B}_r^-})\} &\sim \sum_{|\alpha|\geqslant 0} \frac{i^{|\alpha|}}{\alpha!} \{D^\alpha_\eta d_1(x, y, \sigma, \eta)\}\{D^\alpha_y \chi^{\mathcal{B}_r^-}(x, y, \sigma, \eta)\} = \\ &= d_1\chi^{\mathcal{B}_r^-} + i\{D^1_\eta d_1\}\{D^1_y\chi^{\mathcal{B}_r^-}\} - \frac{1}{2!}\{D^2_\eta d_1\}\{D^2_y\chi^{\mathcal{B}_r^-}\}\end{aligned} \tag{5.93}$$

since d_1 is a second order symbol in η ((3.13), (3.11)). Recalling from (5.19) that $\chi^{\mathcal{B}_r^-}$ vanishes on $\mathcal{B}_r^+$ and recalling from (5.17)-(5.18) that the growth of d_1 is $\mathcal{O}(\sigma^r)$ on $\mathcal{B}_r^- \cup \mathcal{B}_{\mathrm{tr},r}$, we obtain the first of the estimates below, while the other two estimates follow from (5.23) of Lemma 5.1 along with $\chi^{\mathcal{B}_r^-} \equiv 1$ on $\mathcal{B}_r^-$:

$$\left.\begin{aligned} d_1\chi^{\mathcal{B}_r^-} &= \mathcal{O}(\sigma^r) \\ \{D^1_{\eta_j} d_1\}\{D^1_{y_j}\chi^{\mathcal{B}_r^-}\} &= \mathcal{O}(|\eta|\,|\eta|^{2-r}) \\ \{D^2_{\eta_j} d_1\}\{D^2_{y_j}\chi^{\mathcal{B}_r^-}\} &= \mathcal{O}(|\eta|^{2(2-r)}) \end{aligned}\right\} \text{ in } \mathcal{B} \text{ where } \sigma \sim |\eta| \to +\infty .$$

Moreover, for $3/2 < r < 2$, we have $3 - r < r$, and $2(2 - r) < r$. Thus

$$\{\text{symbol of } (D_1\chi^{\mathcal{B}_r^-})\} = \mathcal{O}(|\eta|^r) \quad \text{in } \mathcal{B} .$$

From (5.93) applying $D^\beta_y D^{\alpha'}_\eta$ we obtain similarly

$$D^\beta_y D^{\alpha'}_\eta\{\text{symbol of } (D_1\chi^{\mathcal{B}_r^-})\} = \mathcal{O}(|\eta|^{r-|\alpha'|(r-1)+|\beta|(2-r)}) \quad \text{in } \mathcal{B}$$

and (5.91*a*) follows. □

6. – Completion of the proof of Theorem 1.2.

We return to the partition of unity relation (5.20*b*)

$$\chi^{\mathcal{B}_r^+}(x,y;\sigma,\eta)+\chi^{\mathcal{B}_r^-}(x,y;\sigma,\eta)+\chi^{\mathrm{I}}(x,y;\sigma,\eta)+\chi^{\mathrm{II}}(x,y;\sigma,\eta)\equiv 1 \qquad (x,y,\sigma,\eta)\in R^{2n}(+)\,, \tag{6.1}$$

so that after multiplying (6.1) by $\hat{u}(\sigma, x, \eta)$ and applying (3.8) with $v = u$, we obtain

$$u(t,x,y)=\boldsymbol{\chi}^{\mathcal{B}_r^+}u(t,x,y)+\boldsymbol{\chi}^{\mathcal{B}_r^-}u(t,x,y)+\boldsymbol{\chi}^{\mathrm{I}}u(t,x,y)+\boldsymbol{\chi}^{\mathrm{II}}u(t,x,y)\,. \tag{6.2}$$

We shall then seek the desired estimate (in the desired norm based on Σ) of each term on the right hand side of (6.2) separately, by analyzing each localized problem (3.22), with $\boldsymbol{\chi}$ there being each of the operators in (6.2).

6.1. *Regularity of* $\boldsymbol{\chi}^i u$, $i = \mathrm{I}, \mathrm{II}$.

An estimate of the last two terms in (6.2), involving the «good» regions $\mathcal{G}^{\mathrm{I}}$ and $\mathcal{G}^{\mathrm{II}}$, is readily obtained—in a norm, in fact, higher than our final result. This is contained in the following:

THEOREM 6.1. – Let $3/2 < r < 2$. With reference to the localized problem (3.22) with $g = 0$, we have $\boldsymbol{\chi}^{\mathrm{I}}u$ and $\boldsymbol{\chi}^{\mathrm{II}}u \in H^1(\Sigma)$; more precisely

$$|\boldsymbol{\chi}^{\mathrm{I}}u|^2_{H^1(\Sigma)}+|\boldsymbol{\chi}^{\mathrm{II}}u|^2_{H^1(\Sigma)}=\mathcal{O}(\|f\|^2_{L_2(Q)})\,. \tag{6.3}$$

PROOF. – We use Corollary 4.5 *b*), Eq. (4.14), with $\chi = \chi^i$, $i = \mathrm{I}, \mathrm{II}$, where f_χ and g_χ are given by (3.22*c*)-(3.22*d*), and $g = 0$. We obtain

$$\langle D_1\boldsymbol{\chi}^i u, \boldsymbol{\chi}^i u\rangle_\Sigma = -\,|[B,\boldsymbol{\chi}^i]u|_{x=0}|^2_\Sigma+\mathcal{O}(\|\boldsymbol{\chi}^i u\|^2_{H^1(Q)})+\operatorname{Im}(\boldsymbol{\chi}^i f+[P,\boldsymbol{\chi}^i]u,\tilde{D}_x\boldsymbol{\chi}^i u)_Q\,. \tag{6.4}$$

By Lemma 5.0, Eq. (5.21*b*) with $s = 1$ and the a-priori regularity of Lemma 1.1, we have

$$\|[P,\boldsymbol{\chi}^i]u\|_{L_2(Q)}=\mathcal{O}(\|u\|_{H^1(Q)})=\mathcal{O}(\|f\|_{L_2(Q)})\,. \tag{6.5}$$

Similarly, by Lemma 5.0, Eq. (5.22*b*) with $s = 0$, standard trace theory, and Lemma 1.1 on a-priori regularity

$$|[B,\boldsymbol{\chi}^i]u|_{x=0}|^2_{L_2(\Sigma)}=\mathcal{O}(|u|_{x=0}|_{L_2(\Sigma)})=\mathcal{O}(\|u\|_{H^1(Q)})=\mathcal{O}(\|f\|_{L_2(Q)})\,. \tag{6.6}$$

Also, by Lemma 5.0, Eq. (5.22c) and (3.10b), we have

$$\|\tilde{D}_x \chi^i u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)}).$$

This, along with (6.3)-(6.5), yields

$$\langle D_1 \chi^i u, \chi^i u\rangle_\Sigma = \mathcal{O}(\|f\|_Q^2), \qquad i = \mathrm{I}, \mathrm{II}. \tag{6.7}$$

We now recall Claims 1 and 2 in (5.8), (5.9)

$$d_1(x, y; \sigma, \eta) \geq C_1^2[\sigma^2 + |\eta|^2] \quad \text{on } \mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}}, \tag{6.8}$$

$$-d_1(x, y; \sigma, \eta) \geq C_2^2[\sigma^2 + |\eta|^2] \quad \text{on } \mathcal{G}^{\mathrm{II}} \cup \mathcal{G}^{\mathrm{II}}_{\mathrm{tr}}, \tag{6.9}$$

outside a finite sphere of the half (x, y, σ, η)-space $R^{2n}(+)$. By (6.8) we can then define a real symbol $d_{1,\mathrm{ext}}(x, y; \sigma, \eta)$ belonging to the same class $S^2_{1,0}$ as the symbol $d_1(x, y; \sigma, \eta)$ in (3.13), such that

$$d_{1,\mathrm{ext}}(x, y; \sigma, \eta) \equiv d_1(x, y; \sigma, \eta) \quad \text{on } \mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}}, \tag{6.10a}$$

$$d_{1,\mathrm{ext}}(x, y; \sigma, \eta) \geq c[\sigma^2 + |\eta|^2] \quad \text{outside a finite sphere of } R^{2n}(+). \tag{6.10b}$$

Thus, Gärding's inequality for symbols [T.1, Thm. 8.1, with $s = 0$, p. 55] gives

$$\langle D_{1,\mathrm{ext}} \chi^{\mathrm{I}} u, \chi^{\mathrm{I}} u\rangle_\Sigma = \mathrm{Re}\,\langle D_{1,\mathrm{ext}} \chi^{\mathrm{I}} u, \chi^{\mathrm{I}} u\rangle \geq C_0\|\chi^{\mathrm{I}} u\|^2_{H^1(\Sigma)} - C_1|\chi^{\mathrm{I}} u|^2_{L_2(\Sigma)} \tag{6.11}$$

where $D_{1,\mathrm{ext}} \in OPS^2_{1,0}$ is the pseudo-differential operator corresponding to the symbol $d_{1,\mathrm{ext}}$ via (3.8). But, from the product theorem [T.1, Eq. (4.5), p. 46] one sees that

$$\text{support of symbol of } (D_{1,\mathrm{ext}} \chi^{\mathrm{I}}) \subset \text{support of symbol of } \chi^{\mathrm{I}} = \\ = \text{support of } \chi^{\mathrm{I}} = \mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{I}}_{\mathrm{tr}}. \tag{6.12}$$

Thus, the definition (3.8) and (6.10a) give

$$D_{1,\mathrm{ext}} \chi^{\mathrm{I}} u = D_1 \chi^{\mathrm{I}} u. \tag{6.13}$$

Using (6.13) in (6.11)

$$|\langle D_1 \chi^i u, \chi^i u\rangle_\Sigma| \geq C_0\|\chi^i u\|^2_{H^1(\Sigma)} - C_1|\chi^i u|^2_{L_2(\Sigma)} \tag{6.14}$$

for $i = \mathrm{I}$. The validity of (6.14) for $i = \mathrm{II}$ follows in a similar manner from (6.9). Recalling (6.6) in (6.14)

$$\|\chi^i u\|^2_{H^1(\Sigma)} = \mathcal{O}(\|f\|^2_{L_2(Q)}) + \mathcal{O}(|\chi^{\mathrm{I}} u|^2_{L_2(\Sigma)}) = \mathcal{O}(\|f\|)^2_{L_2(Q)} \tag{6.15}$$

where in the last step we have applied eq. (5.14), standard trace theory, and the a-priori regularity of Lemma 1.1. Equation (6.15) gives (6.2). □

6.2. *Regularity of* $\chi^{\mathcal{B}_r^+} u$.

We next obtain an estimate of the term $\chi^{\mathcal{B}_r^+} u$ (see (5.15)) in (6.1).

THEOREM 6.2. – Let $3/2 < r < 2$. With reference to the localized problem (3.22) with $g = 0$, we have that $\chi^{\mathcal{B}_r^+} u \in H^{r-1}(\Sigma)$; more precisely

$$\|\chi^{\mathcal{B}_r^+} u\|_{H^{r-1}(\Sigma)} = \mathcal{O}(\|f\|_{L_2(Q)}) . \tag{6.16}$$

PROOF. – We apply the operator $D_t^{r/2-1}$ (where $r/2 - 1 < 0$, we shall eventually identity $r = 8/5$, whereby then $r/2 - 1 = -2/5$) to problem (3.22) with $\chi = \chi^{\mathcal{B}_r^-}$. Since $D_t^{r/2-1}$ commutes with the time invariant coefficient operators P, B, χ, $[P, \chi]$, $[B, \chi]$, (a property we shall use freely below), we obtain

$$P(\chi^{\mathcal{B}_r^+} D_t^{r/2-1} u) = \chi^{\mathcal{B}_r^+} D_t^{r/2-1} f + [P, \chi^{\mathcal{B}_r^+}] D_t^{r/2-1} u \quad \text{on } Q , \tag{6.17a}$$

$$B(\chi^{\mathcal{B}_r^+} D_t^{r/2-1} u) = [B, \chi^{\mathcal{B}_r^+}] D_t^{r/2-1} u|_{x=0} \quad \text{on } \Sigma . \tag{6.17b}$$

Applying Corollary 4.5 *b*), Eq. (4.14), with u replaced by $D_t^{r/2-1} u$ to the solution $\chi^{\mathcal{B}_r^+} D_t^{r/2-1} u$ of (6.17), we obtain (g_χ and f_χ are defined in (3.22*c*)-(3.22*d*))

$$\begin{aligned}\langle D_1 \chi^{\mathcal{B}_r^+} D_t^{r/2-1} u, \chi^{\mathcal{B}_r^+} D_t^{r/2-1} u \rangle_\Sigma = &- |[B, \chi^{\mathcal{B}_r^+}] D_t^{r/2-1} u|_{x=0}|^2_{L_2(\Sigma)} + \\ &+ \operatorname{Im}([P, \chi^{\mathcal{B}_r^+}] D_t^{r-2} u, \tilde{D}_x \chi^{\mathcal{B}_r^+} u)_Q + \operatorname{Im}(D_t^{r/2-1} \chi^{\mathcal{B}_r^+} f, D_t^{r/2-1} \tilde{D}_x \chi^{\mathcal{B}_r^+} u)_Q + \\ &+ \mathcal{O}(\|D_t^{r/2-1} \chi^{\mathcal{B}_r^+} u\|_{H^1(Q)}) .\end{aligned} \tag{6.18}$$

CLAIM. – We have

$$\langle D_1 \chi^{\mathcal{B}_r^+} D_t^{r/2-1} u, \chi^{\mathcal{B}_r^+} D_t^{r/2-1} u \rangle_\Sigma = \mathcal{O}(\|f\|^2_{L_2(Q)}) . \tag{6.19}$$

PROOF OF CLAIM. – We analyze the four terms on the right of (6.18).

First term. With $u \in H^1(Q)$ (a priori regularity of Lemma 1.1) we have $u|_{x=0} \in H^{1/2}(\Sigma)$. Theorem 5.15*c*, Eq. (5.87*b*), applies with $s = \frac{1}{2}$ and $k = r/2 - 1$, so that $\frac{1}{4} < s + r/2 - 1 < \frac{1}{2}$. Hence

$$|[B, \chi^{\mathcal{B}_r^+}] D_t^{r/2-1} u|_{x=0}|_{L_2(\Sigma)} = \mathcal{O}(\|f\|_{L_2(Q)}) . \tag{6.20}$$

Second term. By Theorem 5.3 *a*), Eq. (5.43) and a-priori regularity (Lemma 1.1)

$$\|[P, \chi^{\mathcal{B}_r^+}] D_t^{r-2} u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)}) \tag{6.21}$$

while by Corollary 5.2, (5.25*a*), and (3.10*b*),

$$\|\tilde{D}_x \chi^{\mathcal{B}_r^+} u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)}) . \tag{6.22}$$

Equations (6.20)-(6.22), combined with the obvious analysis of the last two terms on the right of (6.18), produce the claim. □

We return to (6.19) and recall definitions (5.15), (5.17) for the real symbol d_1:

$$d_1(x, y, \sigma, \eta) > \sigma^r \quad \text{on } \mathcal{B}_r^+ \cup \mathcal{B}_{\text{tr},r}, \text{ where } \sigma \sim |\eta| . \tag{6.23}$$

Thus, we are in a situation similar to the one in (6.7), (6.8). Proceeding as in going from (6.8) to (6.14), we likewise apply Gärding$_x$s inequality for symbols [T.1, p. 46] and obtain from (6.23)

$$\langle D_1 \chi^{\mathcal{B}_r^+} D_t^{r/2-1} u, \chi^{\mathcal{B}_r^+} D_t^{r/2-1} u\rangle_\Sigma \geqslant C_0 \|\chi^{\mathcal{B}_r^+} D_t^{r/2-1} u\|_{H^{r/2}(\Sigma)} - C_1 |\chi^{\mathcal{B}_r^+} D_t^{r/2-1} u|^2_{L_2(\Sigma)} . \tag{6.24}$$

Thus, using (6.19) in (6.26) implies, since $D_t^{r/2-1}$ commutes with $\chi^{\mathcal{B}_r^+}$

$$\|D_t^{r/2-1} \chi^{\mathcal{B}_r^+} u\|^2_{H^{r/2}(\Sigma)} = \mathcal{O}(\|f\|^2_{L_2(Q)}) + \mathcal{O}(|D_t^{r/2-1} \chi^{\mathcal{B}_r^+} u|^2_{L_2(\Sigma)}) = \mathcal{O}(\|f\|^2_{L_2(Q)}) \tag{6.25}$$

where in the last step we used $r/2 - 1 < 0$ and $\chi^{\mathcal{B}_r^+} u|_{x=0} \in H^{1/2}(\Sigma)$ by Corollary 5.2, Eq. (5.25*b*), and Lemma 1.1 (a-priori regularity). Moreover, since $\sigma \sim |\eta|$ in the region $\mathcal{B}_r^+$, see (5.3), we have that the operators $D_{y_j}^{r/2-1}\chi^{\mathcal{B}_r^+}$ and $D_t^{r/2-1}\chi^{\mathcal{B}_r^+}$ belong to the same symbol class. Thus from (6.25), we obtain likewise

$$\|D_{y_j}^{r/2-1} \chi^{\mathcal{B}_r^+} u\|_{H^{r/2}(\Sigma)} = \mathcal{O}(\|f\|_{L_2(Q)}) . \tag{6.26}$$

Equations (6.25)-(6.26) together give the desired estimate

$$\|D_t^{r-1} \chi^{\mathcal{B}_r^+} u\|^2_{L_2(\Sigma)} + \|D_{y_j}^{r-1} \chi^{\mathcal{B}_r^+} u\|^2_{L_2(\Sigma)} = \mathcal{O}(\|f\|^2_{L_2(Q)})$$

i.e. (6.16). Lemma 6.2 is proved. □

6.3. *Regularity of* $\chi^{\mathcal{B}_r^-} u$.

Finally, we estimate the term $\chi^{\mathcal{B}_r^-} u$ (see (5.16)) in (6.1). To this end, we collect some preliminary results needed in the proof of the theorem below.

LEMMA 6.3. - Let $3/2 < r < 2$. With reference to (4.1*a*), (3.10), (3.13) we have for $0 \leqslant \theta \leqslant 1$

a)

(6.28a) $$[\Lambda^\theta, \tilde{D}_x] \in OPS^{\theta}_{1,0}(R^{n+1}_{tyx}), \quad \text{uniformly in } x \in R^1_{x+},$$

b)

(6.29a) $$[P, \Lambda^\theta] \in OPS^{\theta+1}_{1,0}(R^{n+1}_{tyx}), \quad \text{uniformly in } x \in R^1_{x+},$$

c)

(6.30a) $$[D_1, \Lambda^\theta] \in OPS^{\theta+1}_{1,0}(R^{n+1}_{tyx}), \quad \text{uniformly in } x \in R^1_{x+}.$$

Hence:

(6.28b) $$[\Lambda^\theta, \tilde{D}_x] \quad \text{continuous } H^1(Q) \to L_2(R^1_{x+}; H^{1-\theta}(R^n_{ty})),$$

(6.29b) $$\Lambda^q[P, \Lambda^\theta]: \quad \text{continuous } H^1(Q) \to L_2(R^1_{x+}; H^{1-(\theta+1)-q}(R^n_{ty})), \quad q \leqslant -\theta,$$

(6.30b) $$\Lambda^q[D_1, \Lambda^\theta]: \quad \text{continuous } H^1(Q) \to L_2(R^1_{x+}; H^{1-(\theta+1)-q}(R^n_{ty})), \quad q \leqslant -\theta.$$

PROOF. – The proofs of cases a), b), c) are similar, and are based on the asymptotic expansions of symbols [T.1, (4.5), p. 46]. A main point in claiming uniformity with respect to $x \in R^1_{x+}$ is that the symbol of the commutator in each of the three cases does not depend on ξ.

a)

(6.31) $$\{\text{symbol of } (\Lambda^\theta \tilde{D}_x)\} \sim (\gamma^2 + \sigma^2 + |\eta|^2)^{\theta/2}\left(\xi + \sum_j^{n-1} a_{nj}\eta_j\right) + \\ + \sum_{|\alpha|\geqslant 1} \frac{i^{|\alpha|}}{\alpha!}\{D^\alpha_{\substack{\sigma\\ \xi\\ \eta}}(\gamma^2 + \sigma^2 + |\eta|^2)^{\theta/2}\}\left\{D^\alpha_{\substack{t\\ x\\ y}}\left(\xi + \sum_j^{n-1} a_{nj}\eta_j\right)\right\},$$

(6.32) $$\{\text{symbol of } \tilde{D}_x \Lambda^\theta\} = \left(\xi + \sum_j^{n-1} a_{nj}\eta_j\right)(\gamma^2 + \sigma^2 + |\eta^2|)^{\theta/2},$$

so that subtracting (6.32) from (6.31) and noticing that in (6.31) only the terms corresponding to derivatives in η and y are active

(6.33) $$\{\text{symbol of } [\Lambda^\theta, \tilde{D}_x]\} \sim \sum_{|\alpha|\geqslant 1} \frac{i^{|\alpha|}}{\alpha!}\{D^\alpha_\eta(\gamma^2 + \sigma^2 + |\eta|^2)^{\theta/2}\}\left\{\sum_j^{n-1}(D^\alpha_y a_{nj})\eta_j\right\}$$

and the symbol of the commutator is independent of ξ. From (6.33), one obtains easily (6.28a), according to Definition 3.2. The proof of (6.29a), (6.30a) are similar. Then, Lemma 3.1 c) applies and yields, respectively, (6.28b), (6.29b), (6.30b), the last two after applying the product theorem [T.1, p. 46]. □

THEOREM 6.4. – Let $3/2 < r < 2$. With reference to the localized problem (3.22) with $g = 0$, we have $\chi^{\mathcal{B}_r^-} u \in H^{1-r/4}(\Sigma)$; more precisely

(6.34) $$\|\chi^{\mathcal{B}_r^-} u\|_{H^{1-r/4}(\Sigma)} = \mathcal{O}(\|f\|_{L_2(Q)}). \quad □$$

PROOF. – The proof makes use of both part *a*), Eq. (4.13), and part *c*), Eq. (4.15), of Corollary 4.5, as they apply to problem (3.22) with $\chi = \chi^{\mathcal{B}_r^-}$ and $g = 0$. In fact, from the present version of eq. (4.13)

$$|\chi^{\mathcal{B}_r^-} u|^2_{H^\theta(\Sigma)} = 2\,\mathrm{Im}\,(\Lambda^\theta \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^\theta \chi^{\mathcal{B}_r^-} u)_Q + \mathcal{O}(\|\chi^{\mathcal{B}_r^-} u\|^2_{H^\theta(Q)}) \tag{6.35}$$

for now: $0 < \theta \leqslant 1$

where u, hence $\chi^{\mathcal{B}_r^-} u$, belong to $H^1(Q)$, (Lemma 1.1 and Corollary 5.2), we see that we obtain

$$\chi^{\mathcal{B}_r^-} u \in H^\theta(\Sigma)\,, \qquad \text{indeed } \|\chi^{\mathcal{B}_r^-} u\|_{H^\theta(\Sigma)} = \mathcal{O}(\|f\|_{L_2(Q)}) \tag{6.36}$$

provided we show that for such θ

$$(\Lambda^\theta \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^\theta \chi^{\mathcal{B}_r^-} u)_Q = \mathcal{O}(\|f\|^2_{L_2(Q)}) \tag{6.37}$$

with $\Lambda^\theta = \Lambda^\theta_{ty}$ defined in (4.1*a*). To this end ([3]), we first note that for $0 < \alpha \leqslant 1$ and $u \in H^1(Q)$, hence $\chi^{\mathcal{B}_r^-} u \in H^1(Q)$, then by Lemma 6.3, Eq. (6.28),

$$[\Lambda^\alpha, \tilde{D}_x]\chi^{\mathcal{B}_r^-} u \in L_2(R^1_{x+}; H^{1-\alpha}(R^n_{ty})) \in L_2(Q)\,. \tag{6.38a}$$

In fact

$$\|[\Lambda^\alpha, \tilde{D}_x]\chi^{\mathcal{B}_r^-} u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)})\,. \tag{6.38b}$$

Next, we write

$$\begin{aligned} \Lambda^\alpha \tilde{D}_x \chi^{\mathcal{B}_r^-} u &= \tilde{D}_x \Lambda^\alpha \chi^{\mathcal{B}_r^-} u + [\Lambda^\alpha, \tilde{D}_x]\chi^{\mathcal{B}_r^-} u \\ (\text{by } (6.38b)) &= \tilde{D}_x \Lambda^\alpha \chi^{\mathcal{B}_r^-} u + \mathcal{O}(\|f\|_{L_2(Q)})\,, \quad 0 < \alpha \leqslant 1\,. \end{aligned} \tag{6.39}$$

Thus

$$(\Lambda^\theta \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^\theta \chi^{\mathcal{B}_r^-} u)_Q = \int_{R^1_{x+}} (\Lambda^\theta \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^\theta \chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx \tag{6.40}$$

(by self-adjointness of Λ on $L_2(R^n_{ty})$)

$$= \int_{R^1_{x+}} (\Lambda^{2\theta-1} \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda \chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx$$

$$(\text{by } (6.39)) = \int_{R^1_{x+}} (\tilde{D}_x \Lambda^{2\theta-1} \chi^{\mathcal{B}_r^-} u, \Lambda \chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx + \mathcal{O}(\|f\|^2_{L_2(Q)})$$

([3]) Instead of working with Λ^θ, we could work only with D^θ_t (which commutes with the various operators !!). Then, we use that $D^\theta_{y_j}\chi^{\mathcal{B}_r^-}$ and $D^\theta_t\chi^{\mathcal{B}_r^-}$ belong to the same class symbol, since $\sigma \sim |\eta|$ in $\mathcal{B}_r^-$; this approach was followed in Theorem 6.2.

since $\|\Lambda\chi^{\mathcal{B}_r^-}u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)})$. Thus, the desired equation (6.37) is established, *as soon as we prove that*

$$\|\tilde{D}_x\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)}), \quad \tfrac{1}{2}<\theta\leqslant 1 \tag{6.41}$$

since for $0\leqslant\theta\leqslant\frac{1}{2}$, Eq. (6.37) is plainly true. To this end, we return to problem (3.22) with $\chi = \chi^{\mathcal{B}_r^-}$ and apply $\Lambda^{2\theta-1}$ to it, in the interesting case $0<2\theta-1\leqslant 1$. We thus obtain the following problem (P and B as in (3.22))

$$P(\Lambda^{2\theta-1}\chi u) = \Lambda^{2\theta-1}f_\chi + [P, \Lambda^{2\theta-1}]\chi u \quad \text{in } Q, \tag{6.42a}$$

$$B(\Lambda^{2\theta-1}\chi u) = \Lambda^{2\theta-1}g_\chi + [B, \Lambda^{2\theta-1}]\chi u|_{x=0} \quad \text{in } \Sigma, \tag{6.42b}$$

$$f_\chi = \chi f + [P, \chi]u, \tag{6.42c}$$

$$g_\chi = \chi g|_{x=0} + [B, \chi]u|_{x=0}, \tag{6.42d}$$

in the solution $\Lambda^{2\theta-1}\chi u$, $\chi = \chi^{\mathcal{B}_r^-}$, where we are presently taking $g=0$. We now use Corollary 4.5 *c*), Eq. (5.15), as it applies to problem (6.42), *in particular* with χu in (4.15) replaced now by $\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u$. We obtain

$$\|\tilde{D}_x\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u\|^2_{L_2(Q)} = (1)+(2)+(3)+(4), \tag{6.43}$$

$$(1) = (\Lambda^{2\theta-1}f_{\chi^{\mathcal{B}_r^-}} + [P, \Lambda^{2\theta-1}]\chi^{\mathcal{B}_r^-}u, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q = (1a)+(1b)+(1c), \quad \text{by (6.42}c), \tag{6.44}$$

$$(1a) = (\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}f, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q, \tag{6.45a}$$

$$(1b) = (\Lambda^{2\theta-1}[P, \chi^{\mathcal{B}_r^-}]u, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q, \tag{6.45b}$$

$$(1c) = ([P, \Lambda^{2\theta-1}]\chi^{\mathcal{B}_r^-}u, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q, \tag{6.45c}$$

$$-(2) = (2a) + 2i\gamma(2b), \tag{6.46}$$

$$(2a) = (\tilde{D}_x\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u, w\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q \quad (w \text{ as in (4.2}b)), \tag{6.47a}$$

$$(2b) = (D_2\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q, \tag{6.47b}$$

$$-i(3) = (3a)+(3b) \; (\text{using } [B, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}] = [B, \Lambda^{2\theta-1}]\chi^{\mathcal{B}_r^-} + \Lambda^{2\theta-1}[B, \chi^{\mathcal{B}_r^-}]), \tag{6.48}$$

$$(3a) = \langle \Lambda^{2\theta-1}[B, \chi^{\mathcal{B}_r^-}]u|_{x=0}, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u|_{x=0}\rangle_\Sigma, \tag{6.49a}$$

$$(3b) = \langle B, \Lambda^{2\theta-1}]\chi^{\mathcal{B}_r^-}u|_{x=0}, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}|_{x=0}\rangle_\Sigma, \tag{6.49b}$$

$$(4) = (D_1\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u, \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-}u)_Q. \tag{6.50}$$

We shall examine individually each of the terms on the right of (6.43).

First term (1). From (6.45a), since Λ is self-adjoint on $L_2(R^n_{ty})$

$$(1a) = \int_{R^1_{x+}} (\chi^{\mathcal{B}_r^-} f, \Lambda^{4\theta-2}\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx = \mathcal{O}(\|f\|^2_{L_2(Q)}), \tag{6.51a}$$

provided $4\theta - 2 \leq 1$, i.e. *provided*

$$\theta \leq 3/4 \tag{6.51b}$$

by Corollary 5.2 with $u \in H^1(Q)$.
Similarly, from (6.45b)

$$(1b) = \int_{R^1_{x+}} (\Lambda^{r-2}[P, \chi^{\mathcal{B}_r^-}]u, \Lambda^{2-r+4\theta-2}\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx = \mathcal{O}(\|f\|^2_{L_2(Q)}), \tag{6.52}$$

provided $2 - r + 4\theta - 2 \leq 1$, i.e. *provided*

$$\theta \leq \frac{1}{4} + \frac{r}{4} \tag{6.53}$$

by Theorem 5.3 b), Eq. (5.44), and Corollary 5.2 with $u \in H^1(Q)$.
From (6.45c)

$$(1c) = \int_{R^1_x} (\Lambda^{2\theta-2}[P, \Lambda^{2\theta-1}]\chi^{\mathcal{B}_r^-} u, \Lambda\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx = \mathcal{O}(\|f\|^2_{L_2(Q)}), \tag{6.54}$$

provided, by Lemma 6.3, Eq. (6.29b), we have

$$1 - (2\theta - 1 + 1) - (2\theta - 2) = 3 - 4\theta \geq 0, \quad \text{i.e. provided (6.51}b\text{) holds}.$$

Combining (6.51), (6.52), (6.54) we have

$$(1) = \mathcal{O}(\|f\|^2_{L_2(Q)}) \quad \text{provided} \begin{cases} \theta \leq 3/4, \\ \theta \leq 1/4 + r/4. \end{cases} \tag{6.55}$$

Second term (2). From (6.47a)

$$(2a) = \int_{R^1_{x+}} (\tilde{D}_x \Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-} u, w\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx = \int_{R^1_{x+}} (\Lambda^{2\theta-2}\tilde{D}_x\Lambda^{2\theta-1}\chi^{\mathcal{B}_r^-} u, w\Lambda\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx$$

(using (6.39), (5.25a) with $s = 1$, and (4.1b))

$$= \int_{R^1_{x+}} (\Lambda^{4\theta-3}\tilde{D}_x\chi^{\mathcal{B}_r^-} u, w\Lambda\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx + \mathcal{O}(\|f\|^2_{L_2(Q)}) = \mathcal{O}(\|f\|^2_{L_2(Q)}) \tag{6.56}$$

provided $4\theta - 3 \leqslant 0$, i.e. *provided* (6.51*b*) holds (both $\tilde{D}_x \chi^{\mathcal{B}_r^-} u$ and $w\Lambda\chi^{\mathcal{B}_r^-} u \in L_2(Q)$).
Similarly, from (6.47*b*)

$$(6.57) \qquad (2b) = \int_{R^1_{x+}} (\Lambda^{2\theta-2} D_2 \Lambda^{2\theta-1} \chi^{\mathcal{B}_r^-} u, \Lambda\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx = \mathcal{O}(\|f\|^2_{L_2(Q)})$$

provided $2\theta - 2 + 2\theta - 1 + 1 \leqslant 1$ (recall (3.12)), i.e. *provided* (6.51*b*) holds.
Combining (6.56)-(5.57), we obtain

$$(6.58) \qquad (2) = \mathcal{O}(\|f\|^2_{L_2(Q)}) \quad \text{provided } \theta \leqslant 3/4 .$$

Third term (3). From (6.49*a*), since Λ is self-adjoint on $L_2(\Sigma)$

$$(6.59) \qquad (3a) = \langle \Lambda^{2\theta-1+2\theta-3/2} [B, \chi^{\mathcal{B}_r^-}] u|_{x=0}, \Lambda^{1/2} \chi^{\mathcal{B}_r^-} u|_{x=0} \rangle = \mathcal{O}(\|f\|^2_{L_2(Q)})$$

provided

$$(6.60) \qquad \theta \leqslant 1/4 + r/4 .$$

This is so since, with $u \in H^1(Q)$, we have $\Lambda^{1/2} \chi^{\mathcal{B}_r^-} u|_{x=0} \in L_2(\Sigma)$ by Corollary 5.2 and, moreover, $[B, \chi^{\mathcal{B}_r^-}] u|_{x=0} \in H^{1/2-(2-r)}(\Sigma)$ by Theorem 5.15, Eq. (5.87) with $s = 1/2$; thus, we must require for the left hand side term in the Σ-inner product of (3*a*) to be in $L_2(\Sigma)$ that: $1/2 - (2-r) + 4\theta - 5/2 \geqslant 0$, i.e. (6.60). Similarly from (6.49*b*)

$$(6.61) \qquad ((3b) = \langle \Lambda^{2\theta-3/2} [B, \Lambda^{2\theta-1}] \chi^{\mathcal{B}_r^-} u|_{x=0}, \Lambda^{1/2} \chi^{\mathcal{B}_r^-}|_{x=0} \rangle_\Sigma = \mathcal{O}(\|f\|^2_{L_2(Q)})$$

provided $\theta \leqslant 3/4$. This is so, since plainly $[B, \Lambda^{2\theta-1}]|_{x=0} \in OPS^{2\theta-1}_{1,0}(\Sigma)$ so that $u|_{x=0} \in H^{1/2}(\Sigma)$ and $[B, \Lambda^{2\theta-1}] \chi^{\mathcal{B}_r^-} u|_{x=0} \in H^{1/2-(2\theta-1)}(\Sigma)$ and the left hand side term in the Σ-inner product of (3*b*) is in $L_2(\Sigma)$, provided $1/2 - (2\theta - 1) - (2\theta - 3/2) \geqslant 0$, i.e. provided $\theta \leqslant 3/4$. Combining (6.59) and (6.61), we conclude that

$$(6.62) \qquad (3) = \mathcal{O}(\|f\|^2_{L_2(Q)}) \quad \text{provided} \begin{cases} \theta \leqslant 3/4 , \\ \theta \leqslant 1/4 + r/4 . \end{cases}$$

Fourth term (4). From (6.50),

$$(6.53) \qquad (4) = \int_{R^1_{x+}} (\Lambda^{2\theta-2} D_1 \Lambda^{2\theta-1} \chi^{\mathcal{B}_r^-} u, \Lambda\chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx .$$

But

$$(6.64) \qquad \Lambda^{2\theta-2} D_1 \Lambda^{2\theta-1} \chi^{\mathcal{B}_r^-} u = \Lambda^{2\theta-2} \Lambda^{2\theta-1} D_1 \chi^{\mathcal{B}_r^-} u + \Lambda^{2\theta-2} [D_1, \Lambda^{2\theta-1}] \chi^{\mathcal{B}_r^-} u = \\ = \Lambda^{4\theta-3} D_1 \chi^{\mathcal{B}_r^-} u + \mathcal{O}(\|f\|^2_{L_2(Q)})$$

by applying Lemma 6.3, Eq. (6.30*b*), Corollary 5.2 and $u \in H^1(Q)$ (Lemma 1.1), provided $1-(2\theta-1+1)-(2\theta-2)\geqslant 0$, i.e. $\theta\leqslant 3/4$. Moreover, recalling Theorem 5.16, Eq. (5.92*b*), with $u\in H^1(Q)$, we have

$$\Lambda^{4\theta-3} D_1 \chi^{\mathcal{B}_r^-} u \in L_2(R^1_{x^+}, H^{1-r-(4\theta-3)})(R^n_{ty}))$$

and we then require $1-r-(4\theta-3)\geqslant 0$, i.e.

$$\theta \leqslant 1 - r/4 \tag{6.64}$$

to obtain

$$\|\Lambda^{4\theta-3} D_1 \chi^{\mathcal{B}_r^-} u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)})\,. \tag{6.65}$$

Putting together (6.63)-(6.65), we get

$$(4) = \mathcal{O}(\|f\|^2_{L_2(Q)})\,, \qquad \text{provided} \begin{cases} \theta\leqslant 1-r/4\,, \\ \theta \leqslant 3/4\,. \end{cases} \tag{6.66}$$

We can finally conclude. We return to (6.43) using (6.55), (6.58), (6.62), and (6.66). We thus obtain

$$\|\tilde{D}_x \Lambda^{2\theta-1} \chi^{\mathcal{B}_r^-} u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L_2(Q)}) \tag{6.67}$$

provided

$$\begin{cases} \theta \leqslant 3/4 \\ \theta \leqslant 1/4 + r/4 \qquad \text{with } 3/2 < r < 2\,. \\ \theta \leqslant 1 - r/4 \end{cases} \tag{6.68}$$

But, for $3/2 < r < 2$, we have $1-r/4 < 1/4 + r/4 < 3/4$. Thus we have

$$\|\tilde{D}_x \Lambda^{2\theta-1} \chi^{\mathcal{B}_r^-} u\|_{L_2(Q)} = \mathcal{O}(\|f\|_{L^2(Q)}) \qquad \text{for all } \theta \leqslant 1 - r/4\,. \tag{6.69}$$

Returning to (6.36)-(6.37) and the statement above (6.41), we conclude by virtue of (6.69) that

$$(6.36) \text{ holds for all } \theta \leqslant 1 - r/4\,. \tag{6.70}$$

The highest regularity of $\chi^{\mathcal{B}_r^-} u$ is then obtained by choosing $\theta = 1 - r/4$. Theorem 6.4 is proved. □

6.4. *Final step in the proof of Theorem* 1.2.

Theorems 6.1, 6.2, and 6.4 provide the regularity of the various components of the partition of unity decomposition in (6.1). Intersecting the segments $\{1 - r/4,$

$3/2 < r < 2\}$ and $\{r-1, 3/2 < r < 2\}$, we obtain (as already announced)

$$r = 8/5$$

in which case $r - 1 = 1 - r/4 = 3/5$ is the optimal value which provides the highest regularity to $\chi^{\mathcal{B}_r^+} u$ and $\chi^{\mathcal{B}_r^-} u$ on Σ *simultaneously*; i.e.

$$\text{for } r = 8/5 \Rightarrow \chi^{\mathcal{B}_r^+} u\,, \quad \chi^{\mathcal{B}_r^-} u \in H^{3/5}(\Sigma)\,.$$

The proof of Theorem 1.2 is complete!! □

7. – Completion of the proof of Theorem 1.3: $f = 0$ and $g \in L_2(Q)$

ORIENTATION. – The a-priori *interior* regularity result, Lemma 1.1*b*), claims that: if $u_0 = u_1 = f = 0$ and $g \in L_2(\Sigma)$, then a-fortiori the solution u of the corresponding non-homogeneous problem (1.6) satisfies $u \in H^{1/2}(Q)$. Authorized by this, we shall then assume as a-priori information the *interior* regularity $u \in H^q(Q)$, $1/2 \leqslant q \leqslant 3/5$ for the solution. (Actually, only $u \in H^q(R_t^1; R_{xy}^n)$ or $D_t^q u \in L_2(Q)$ will suffice). As a consequence, we shall prove the following *trace* regularity, that $u|_\Sigma \in H^{q-2/5}(\Sigma)$. This is the content of Section 7.1. Next, in Section 7.2, such *trace* theory result will then be used to improve the original *interior* regularity of the solution to read that, in fact, the solution u satisfies $u \in H^{q+(3/10-q/2)}(Q) = H^{q/2+3/10}(Q)$, $1/2 \leqslant q \leqslant 3/5$, and this, in turn, induces a corresponding improvement of the *trace* regularity expressed by $u|_\Sigma \in H^{q-2/5+(3/10-q/2)}(\Sigma) = H^{q/2-1/10}(\Sigma)$. Finally, in Section 7.3, we shall then carry out the ensuing «boost-strap» argument, starting with $q_0 = 1/2$, to conclude simultaneously that, in fact, $u \in H^{3/5-\varepsilon}(Q) = H^{1/2+1/10-\varepsilon}(Q)$ and $u|_\Sigma \in H^{1/5-\varepsilon}(\Sigma)$, $\forall \varepsilon > 0$.

7.1. *From the a-priori information $D_t^q u \in L_2(Q)$ in the interior and $u|_\Sigma \in L_2(\Sigma)$ on the boundary to the trace regularity $u|_\Sigma \in H^{q-2/5}(\Sigma)$.*

The main goal of the present section is to prove the following theorem.

THEOREM 7.1. – Assume that the corresponding solution of problem (1.6) with $u_0 = u_1 = f = 0$ satisfies

$$g \in L_2(\Sigma) \to D_t^q u \in L_2(Q) \qquad \text{continuously}$$

$$[\text{or } u \in H^q(R_t^1; L_2(R_{xy}^n))] \tag{7.1}$$

$$1/2 \leqslant q \leqslant 3/5$$

(a fact a fortiori true at least for $q = 1/2$ at this stage, by Lemma 1.1*b*).

Then, $u|_\Sigma \in H^{q-2/5}(\Sigma)$; more precisely

$$|u|_\Sigma|_{H^{q-2/5}(\Sigma)} \leqslant C|g|_\Sigma$$

with constant C independent of q $(1/2 \leqslant q \leqslant 3/5)$. A fortiori, the map $g \in L_2(\Sigma) \to \to u|_\Sigma \in L_2(\Sigma)$ is compact. □

REMARK 7.1. – Throughout this entire section, we shall also explicitly use the following a-priori information that

$$g \in L_2(\Sigma) \to u|_\Sigma \in L_2(\Sigma) \tag{7.3}$$

for problem (1.6) with $u_0 = u_1 = f = 0$. This result was apparently unknown until 1984. (It does *not* follow by the known result of the time [L-M.1, Vol. II, p. 122] on *interior* regularity: $u \in L_2(0, T; H^{1/2}(\Omega))$ (improved to $u \in C([0, T]; H^{1/2}(\Omega))$ [L-T.2]) via trace theory. In May 1984, two independent proofs were given, one by J. L. LIONS [L.1] and one by the authors, during an exchange of correspondence. J. L. Lions' proof uses a Laplace transform technique. The authors' proof is based on the following three steps (with $u_0 = u_1 = f = 0$):

(*a*) re-proof of Myatake's result [M.1]

$$(*) \qquad \begin{array}{c} g \in L_2(0, T; H^{1/2}(\Gamma)) \Rightarrow u \in C([0, T]; H^1(\Omega)) \\ \Downarrow \text{ trace theory} \\ u|_\Sigma \in L_2(0, T; H^{1/2}(\Gamma)) \,; \end{array}$$

(*b*) (consequence of step (*a*) by transposition)

$$(**) \qquad g \in L_2(0, T; H^{-1/2}(\Gamma)) \Rightarrow u|_\Sigma \in L_2(0, T; H^{-1/2}(\Gamma)) \,;$$

(*c*) interpolation between (*) and (**). □

PROOF OF THEOREM 7.1. – As in Section 6, our approach is based on analyzing separately the regularity of each component of the partition of unity decomposition (5.13) or (6.0), i.e., $\chi^{\mathcal{G}} u|_\Sigma$, $\chi^{\mathcal{B}_r^+} u|_\Sigma$ and $\chi^{\mathcal{B}_r^-} u|_\Sigma$. This will be done in subsections 7.1.2, 7.1.3, and 7.1.4 respectively. First, however, in subsection 7.1.1 we need a preliminary result which claims that, under assumption (7.1), not only do we have $D_t D_t^{q-1} u \in L_2(Q)$, but also $D_x D_t^{q-1}$ and $D_y D_t^{q-1} u \in L_2(Q)$.

7.1.1. *A preliminary improvement in the interior regularity in x and y*: $D_t^{q-1}u \in H^1(Q)$, $1/2 \leqslant q \leqslant 1$.

The main result of the present subsection is

PROPOSITION 7.2. – Under assumption (7.1) there exists a constant $\gamma_0 > 0$ such that for all $k > \gamma_0$, we have

$$\|D_t^{q-1}u\|_{H^1(Q)} \leqslant C_\gamma |g|_\Sigma \tag{7.4}$$

with constant C_γ depending on γ but not on q, $1/2 \leqslant q \leqslant 1$.

PROOF OF PROPOSITION 7.2. – In addition to $D_t D_t^{q-1} u \in L_2(Q)$ which is true by assumption, we must likewise prove that

$$D_y D_t^{q-1} u \in L_2(Q)\,, \tag{7.5}$$

$$D_x D_t^{q-1} u \in L_2(Q)\,, \tag{7.6}$$

xontinuously with respect to $g \in L_2(\Sigma)$. This will be done below by splitting u as $u = \chi^{\mathcal{G}} u + \chi^{\mathcal{B}} u$. We begin with a preliminary lemma on $D_y D_t^{q-1} \chi^{\mathcal{B}} u$.

LEMMA 7.3. – Under assumption (7.1) we have

$$\|D_y D_t^{q-1} \chi^{\mathcal{B}} u\|_Q \leqslant c|g|_\Sigma \tag{7.7}$$

with constant c independent of q, $1/2 \leqslant q \leqslant 1$.

PROOF OF LEMMA 7.3. – The idea is that the operators $D_{y_j}\chi^{\mathcal{B}}$ and $D_t \chi^{\mathcal{B}}$ belong belong to the same class, since $\sigma \sim |\eta|$ in $\mathcal{B} \cup \mathcal{G}_{tr}^{I} \cup \mathcal{G}_{tr}^{II} = \operatorname{supp} \chi^{\mathcal{B}} \supset \operatorname{supp}$ [symbol of $D_{y_j}\chi^{\mathcal{B}}$], supp [symbol of $D_t \chi^{\mathcal{B}}$]. This can be checked as usual via [T.1, Theorem 4.4, p. 46] and (5.14). For this reason we have

$$D_y D_t^{-1} \chi^{\mathcal{B}} \in OPS^0_{1,0}(R^{n+1}_{ty x})\,, \quad \text{uniformly in } x \in R^1_{x+} \tag{7.8a}$$

and hence by Lemma 3.1 *a*)

$$D_y D_t^{-1} \chi^{\mathcal{B}}\colon \text{ continuous } H^s(Q) \to H^s(Q)\,. \tag{7.8b}$$

Then (7.7) follows from the assumption (7.1) and from (7.8*b*) with $s = 0$. □

The crucial part of Proposition 7.2 is given by the following result.

PROPOSITION 7.4. – Under assumption (7.1) there exists a constant $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$, we have

$$\|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q \leq C_\gamma |g|_\Sigma \tag{7.9a}$$

and hence

$$\|D_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q \leq C_\gamma |g|_\Sigma \tag{7.9b}$$

where C_γ is a constant depending on γ but independent of q, $1/2 \leq q \leq 1$.

PROOF OF PROPOSITION 7.4. – Inequality (7.9*b*) follows easily form (7.9*a*) via the definition (3.10*b*) and inequality (7.7). To prove inequality (7.9*a*) we need the following three lemmas.

LEMMA 7.5. – Under assumption (7.1), for any $\varepsilon > 0$ sufficiently small, we have

$$\|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 \leq C_\varepsilon |g|_\Sigma^2 + \frac{\varepsilon C}{1 - \varepsilon C} \|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2 \tag{7.10}$$

with constants C_ε and C independent of q, $1/2 \leq q \leq 1$. To gain information on $\chi^{\mathcal{G}} u$ in the interior, we begin with its trace on Σ.

LEMMA 7.6. – Under assumption (7.1), we have

$$|D_t^{q-1} \chi^{\mathcal{G}} u|_{H^1(\Sigma)}^2 \leq C\{|g|_\Sigma^2 + \|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2 + \|D_t^{q-1} \chi^{\mathcal{B}} u\|_{H^1(Q)}^2\} \tag{7.11}$$

with constant C independent of q, $1/2 \leq q \leq 1$.

We now obtain information on $\chi^{\mathcal{G}} u$ in the interior.

LEMMA 7.7. – Under assumption (7.1), there exists a constant $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ we have

$$\|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2 \leq C_\gamma\{|g|_\Sigma^2 + \|D_t^{q-1} \chi^{\mathcal{B}} u\|_{H^1(Q)}^2\}$$

with constant C_γ depending on γ but independent of q, $1/2 \leq q \leq 1$.

Assuming for the time being the validity of the above three lemmas, we may now readily prove Proposition 7.4. In fact, if we insert (7.12) into the right of (7.10) we obtain

$$\|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 \leq C_\varepsilon |g|_\Sigma^2 + \frac{\varepsilon C}{1 - \varepsilon C} C_\gamma \{|g|_\Sigma^2 + \|D_t^{q-1} \chi^{\mathcal{B}} u\|_{H^1(Q)}^2\} \tag{7.13}$$

where

$$\|D_t^{q-1} \chi^{\mathcal{B}} u\|_{H^1(Q)}^2 = \|D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 + \|D_t^q \chi^{\mathcal{B}} u\|_Q^2 + \|D_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 + \|D_y D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2$$

(by (7.1), (5.14), (3.10*b*), and (7.7))

$$\leqslant C|g|_{\Sigma}^{2} + \|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 . \tag{7.14}$$

Hence, inserting (7.14) into the right side of (7.13), we obtain

$$\left\{1 - \frac{\varepsilon C}{1-\varepsilon C} C_\gamma\right\} \|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 \leqslant C_{\varepsilon\gamma} |g|_{\Sigma}^2 . \tag{7.15}$$

Selecting now ε suitably small with respect to γ in (7.15), we obtain (7.9*a*) and Proposition 7.4 is proved, as soon as we establish the above three Lemmas 7.5, 7.6, and 7.7. However, before so, we draw some corollaries.

COROLLARY 7.8 (to Lemma 7.3 and Proposition 7.4). – Under assumption (7.1), there exists a constant $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$, we have

$$\|D_t^{q-1} \chi^{\mathcal{B}} u\|_{H^1(Q)} \leqslant C_\gamma |g|_{\Sigma} \tag{7.16}$$

with constant C_γ depending on γ but independent of q, $1/2 \leqslant q \leqslant 1$.

PROOF OF COROLLARY 7.8. – Combine Lemma 7.3 and Proposition 7.4. □

COROLLARY 7.9 (to Lemma 7.7 and Corollary 7.8). – Under assumption (7.1), there exists a constant $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ we have

$$\|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)} \leqslant C_\gamma |g|_{\Sigma} . \tag{7.17}$$

with constant C_γ depending on γ but independent of q, $1/2 \leqslant q \leqslant 1$.

PROOF OF COROLLARY 7.9. – Combine Lemma 7.7 and Corollary 7.8. □

Continuing with the proof of (7.4) of Proposition 7.2, we see that $u = \chi^{\mathcal{B}} u + \chi^{\mathcal{G}} u$ combined with (7.16) of Corollary 7.8 and (7.17) of Corollary 7.9 provide the desired conclusion. Thus Proposition 7.2 is proved as soon as we establish Lemmas 7.5, 7.6, and 7.7.

PROOF OF LEMMA 7.5. We shall invoke inequality (4.7) (or else inequality (4.6*c*)) of Theorem 4.3 for problem (1.6):

$$\|\tilde{D}_x u\|_Q^2 = \mathcal{O}\{\operatorname{Re}(Pu, u)_Q + \operatorname{Re}(D_1 u, u)_Q + \|u\|_Q^2 + |Bu|_{\Sigma}^2\} . \tag{7.18}$$

If now we apply D_t^{q-1} to the localized problem (3.22) and use the property that D_t commutes with all other time-independent operators in (3.22), we then obtain for $f = 0$:

$$(7.19)\quad \begin{cases} P(D_t^{q-1}\chi u) = [P, \chi]D_t^{q-1}u\,, & \text{in } Q\,, \; -\infty < t < \infty\,,\\ B(D_t^{q-1}\chi u) = \chi D_t^{q-1}g|_{x=0} + [B, \chi]D_t^{q-1}u|_{x=0}\,, & \text{in } \Gamma\,, \; -\infty < t < \infty\,,\end{cases}$$

which we shall use now for $\chi = \chi^{\mathcal{B}}$. Applying the version of (7.18) which corresponds to problem (7.19) in the solution $D_t^{q-1}\chi^{\mathcal{B}}u$ (instead of u), we obtain:

$$(7.20)\quad \|\tilde{D}_x D_t^{q-1}\chi^{\mathcal{B}}u\|_Q^2 = O\{(1) + (2) + (3) + (4)\}\,,$$

$$(7.21)\quad (1) = \operatorname{Re}\left([P, \chi^{\mathcal{B}}]D_t^{q-1}u,\, D_t^{q-1}\chi^{\mathcal{B}}u\right)_Q\,,$$

$$(7.22)\quad (2) = \operatorname{Re}\left(D_1 D_t^{q-1}\chi^{\mathcal{B}}u,\, D_t^{q-1}\chi^{\mathcal{B}}u\right)_Q\,,$$

$$(7.23)\quad (3) = \|D_t^{q-1}\chi^{\mathcal{B}}u\|_Q^2 \leqslant C|g|_\Sigma^2\,, \qquad C \text{ independent of } q\,,$$

$$(7.24)\quad (4) = |(4_1) + (4_2)|_\Sigma^2 \leqslant 2\{|(4_1)|_\Sigma^2 + |(4_2)|_\Sigma^2\}\,,$$

$$(7.25)\quad (4_1) = \chi^{\mathcal{B}}D_t^{q-1}g|_{x=0}\,,$$

$$(7.26)\quad (4_2) \quad [B, \chi^{\mathcal{B}}]D_t^{q-1}u|_{x=0}\,.$$

But the operators $\chi^{\mathcal{B}}$ and $\chi^{\mathcal{G}}$ (corresponding to the homogeneous symbols $\chi^{\mathcal{B}}$ and $\chi^{\mathcal{G}}$ see (5.14)(, belong a fortiori to the class $OPS^0_{1,0}(R^n_{t y})$, uniformly in $x \in R$E, so that Lemma 3.1 *a*) with $s = 0$ applies. Moreover,

$$(7.27)\quad [P, \chi^{\mathcal{B}}],\, [P, \chi^{\mathcal{G}}] \in OPS^1_{1,0}(Q)\,,$$

$$(7.28)\quad [B, \chi^{\mathcal{B}}],\, [B, \chi^{\mathcal{G}}] \in OPS^1_{1,0}(\Sigma)\,,$$

by the commutator Lemma 3.2 (which is standard for homogeneous symbols).

Term (1). Using (7.27) and $u = \chi^{\mathcal{B}}u + \chi^{\mathcal{G}}u$ we have from (7.21) via Schwarz inequality

$$(7.29)\quad |(1)| \leqslant 2C\|D_t^{q-1}u\|_{H^1(Q)}\|D_t^{q-1}\chi^{\mathcal{B}}u\|_Q \leqslant C\left\{\frac{1}{\varepsilon}\|D_t^{q-1}\chi^{\mathcal{B}}u\|_Q^2 + \varepsilon\|D_t^{q-1}u\|_{H^1(Q)}^2\right\} \leqslant$$

$$\text{(by (7.1))} \quad \leqslant \frac{C}{\varepsilon}|g|_\Sigma^2 + 2\varepsilon C\|D_t^{q-1}\chi^{\mathcal{B}}u\|_{H^1(Q)}^2 + 2\varepsilon C\|D_t^{q-1}\chi^{\mathcal{G}}u\|_{H^1(Q)}^2\,.$$

Term (2). We rewrite (7.22) as

$$(7.30)\quad (2) = \operatorname{Re}\left(D_1 D_t^{-2}\chi^{\mathcal{B}}D_t^q u,\, D_t^q\chi^{\mathcal{B}}u\right)_Q\,.$$

We now use that

$$(7.31)\quad D_1 D_t^{-2}\chi^{\mathcal{B}} \in OPS^0(R^n_{t y})\,, \qquad \text{uniformly in } x \in R^1_{x+}$$

by the product theorem [T.1, Theorem 4.4, p. 46], since

$$\text{supp [symbol of } D_1 D_t^{-2} \chi^{\mathcal{B}}] \subset \sup \chi^{\mathcal{B}} = \mathcal{B} \cup \mathcal{G}_{tr}^{I} \cup \mathcal{G}_{tr}^{II},$$

a region where $\sigma \sim |\eta|$: Then (7.31) and Lemma 3.1 *a*) with $s = 0$ give, together with (7.1)

$$|(2)| \leqslant C|g|_\Sigma^2, \quad C \text{ independent of } q,\ 1/2 \leqslant q \leqslant 1. \tag{7.32}$$

Term (3). Is handled by (7.23).

Term (4). Plainly from (7.25) and (5.14*a*) since $q - 1 < 0$.

$$|(4_1)| \leqslant C|g|_\Sigma, \quad C \text{ independent of } q,\ 1/2 \leqslant q \leqslant 1 \tag{7.33a}$$

while by using (7.28) and the boundary regularity (7.3) along with $q - 1 \leqslant 0$, we have from (7.26)

$$|(4_2)| \leqslant C|g|_\Sigma, \quad C \text{ independent of } q,\ 1/2 \leqslant q \leqslant 1. \tag{7.33b}$$

Combining (7.20) with (7.29), (7.32), (7.23), (7.24), (7.33*a*)-(7.33*b*), we obtain

$$\|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 \leqslant C_\varepsilon |g|_\Sigma^2 + \varepsilon C \|D_t^{q-1} \chi^{\mathcal{B}} u\|_{H^1(Q)}^2 + \varepsilon C \|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2$$

(recalling (7.14))

$$\leqslant C_\varepsilon |g|_\Sigma^2 + \varepsilon C \|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q^2 + \varepsilon C \|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2. \tag{7.34}$$

Thus (7.34) gives

$$(1 - \varepsilon C)\|\tilde{D}_x D_t^{q-1} \chi^{\mathcal{B}} u\|_Q \leqslant C_\varepsilon |g|_\Sigma^2 + \varepsilon C \|D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2 \tag{7.35}$$

from which (7.10) follows and Lemma 7.5 is proved. □

PROOF OF LEMMA 7.6. – We shall invoke identity (4.5) of Theorem 4.2 for problem (1.6), as applied to the localized problem (7.19) with $\chi = \chi^{\mathcal{G}}$ in the unknown $D_t^{q-1} \chi^{\mathcal{G}} u$. We obtain

$$\langle D_1 D_t^{q-1} \chi^{\mathcal{G}} u, D_t^{q-1} \chi^{\mathcal{G}} u \rangle_\Sigma = [1] + [2] + [3], \tag{7.36}$$

$$-[1] = |[1_1] + [1_2]|_\Sigma^2 \leqslant 2\{|[1_1]|_\Sigma^2 + |[1_2]|_\Sigma^2\}, \tag{7.37}$$

$$[1_1] \quad = \chi^{\mathcal{G}} D_t^{q-1} g|_{x=0}, \tag{7.38a}$$

$$[1_2] \quad = [B, \chi^{\mathcal{G}}] D_t^{q-1} u|_{x=0}, \tag{7.38b}$$

$$[2] \quad = \operatorname{Im}([P, \chi^{\mathcal{G}}] D_t^{q-1} u, \tilde{D}_x D_t^{q-1} \chi^{\mathcal{G}} u)_Q, \tag{7.39}$$

$$[3] \quad = \mathcal{O}(D_t^{q-1} \chi^{\mathcal{G}} u\|_{H^1(Q)}^2). \tag{7.40}$$

Term [1]. Since $q-1\leqslant 0$, we plainly have from (7.38*a*) and (5.14)

$$|[1_1]|_\Sigma \leqslant C|g|_\Sigma\,, \quad C \text{ independent of } q\,,\ 1/2\leqslant q\leqslant 1\,, \tag{7.41a}$$

while recalling (7.28) and the boundary regularity (7.3) we likewise obtain from (7.38*b*)

$$|[1_2]|_\Sigma \leqslant C|g|_\Sigma\,, \quad C \text{ independent of } q\,,\ 1/2\leqslant q\leqslant 1\,. \tag{7.41b}$$

Term [2]. By (7.27) and $u=\chi^{\mathfrak{B}}u+\chi^{\mathfrak{G}}u$, we obtain from (7.39) via Schwarz inequality

$$\begin{aligned}|[2]| &\leqslant \|D_t^{q-1}u\|_{H^1(Q)}\|\tilde{D}_x D_t^{q-1}\chi^{\mathfrak{G}}u\|_Q \leqslant \tfrac{1}{2}\{\|D_t^{q-1}u\|^2_{H^1(Q)}+\|\tilde{D}_x D_t^{q-1}\chi^{\mathfrak{G}}u\|^2_Q\}\leqslant \\ &\leqslant \|D_t^{q-1}\chi^{\mathfrak{B}}u\|^2_{H^1(Q)}+\|D_t^{q-1}\chi^{\mathfrak{G}}u\|^2_{H^1(Q)}+\tfrac{1}{2}\|\tilde{D}_x D_t^{q-1}\chi^{\mathfrak{G}}u\|^2_Q\,.\end{aligned} \tag{7.42}$$

Recalling (3.10) we obtain from (7.42)

$$|[2]|\leqslant C\{\|D_t^{q-1}\chi^{\mathfrak{B}}u\|^2_{H^1(Q)}+\|D_t^{q-1}\chi^{\mathfrak{G}}u\|^2_{H^1(Q)}\} \tag{7.43}$$

with C independent of q, $1/2\leqslant q\leqslant 1$. Combining now (7.36) with (7.37), (7.41*a*)-(7.41*b*), (7.43) and (7.40), we obtain

$$\langle D_1 D_t^{q-1}\chi^{\mathfrak{G}}u, D_t^{q-1}\chi^{\mathfrak{G}}u\rangle_\Sigma = \mathcal{O}\{|g|^2_\Sigma+\|D_t^{q-1}\chi^{\mathfrak{B}}u\|^2_{H^1(Q)}+\|D_t^{q-1}\chi^{\mathfrak{G}}u\|^2_{H^1(Q)}\} \tag{7.44}$$

with constant in $\mathcal{O}$ independent of q. Thus, we are in (7.44) in the same technical situation encountered in (6.7), or (6.19). Proceeding as in going from (6.8) to (6.14) we likewise find by use of Garding inequality [T.1, Theorem 8.1 with $s=0$, p. 55]

$$|\langle D_1 D_t^{q-1}\chi^{\mathfrak{G}}u, D_t^{q-1}\chi^{\mathfrak{G}}u\rangle_\Sigma| \geqslant C_0|D_t^{q-1}\chi^{\mathfrak{G}}u|^2_{H^1(\Sigma)}-C_1|D_t^{q-1}\chi^{\mathfrak{G}}u|^2_\Sigma\,. \tag{7.45}$$

Using (7.45) in (7.44) along with $q-1<0$ and the boundary regularity (7.3) in the last term on the right of (7.45), we obtain (7.11) as desired. Lemma 7.6 is proved. □

PROOF OF LEMMA 7.7. – We shall invoke, and for the first time in fact, identity (4.12*a*) (with $\tilde{D}_x = B$ on Σ, by (1.4)) which for problem (1,6) with $f\equiv 0$ we rewrite as follows: there exist constants C_0, $\gamma_0>0$ such that for all $\gamma>\gamma_0$ we have

$$\begin{aligned}\gamma C_0\|u\|^2_{H^1(Q)} &\leqslant -2\,\mathrm{Im}\,(Pu, -D_2u+ia\gamma u)_Q - \\ &\quad -2\,\mathrm{Re}\,\langle Bu, D_2u\rangle_\Sigma + 2\gamma\,\mathrm{Im}\,\langle Bu, au\rangle_\Sigma\,,\end{aligned} \tag{7.46}$$

We shall now use the version of (7.46) which corresponds to the localized problem (7.19) with $\chi = \chi^{\mathcal{G}}$ in the solution $D_t^{q-1}\chi^{\mathcal{G}}u$. We obtain

$$\gamma C_0 \|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)} \leqslant (1) + (2) + (3) + (3) + (4) + (5)\,, \tag{7.47}$$

$$(1) = 2\,\mathrm{Im}\,([P, \chi^{\mathcal{G}}]D_t^{q-1}u, D_2 D_t^{q-1}\chi^{\mathcal{G}}u)_Q\,, \tag{7.48}$$

$$(2) = 2\,\mathrm{Re}\,([P, \chi^{\mathcal{G}}]D_t^{q-1}u, a\gamma D_t^{q-1}\chi^{\mathcal{G}}u)_Q\,, \tag{7.49}$$

$$(3) = -2\,\mathrm{Re}\,\langle \chi^{\mathcal{G}}D_t^{q-1}g|_{x=0}, D_2 D_t^{q-1}\chi^{\mathcal{G}}u\rangle_\Sigma\,, \tag{7.50}$$

$$(4) = -2\,\mathrm{Re}\,\langle [B, \chi^{\mathcal{G}}]D_t^{q-1}u|_{x=0}, D_2 D_t^{q-1}\chi^{\mathcal{G}}u\rangle_\Sigma\,, \tag{7.51}$$

$$(5) = 2\gamma\,\mathrm{Im}\,\langle \chi^{\mathcal{G}}D_t^{q-1}g|_{x=0} + [B, \chi^{\mathcal{G}}]D_t^{q-1}u|_{x=0}, aD_t^{q-1}\chi^{\mathcal{G}}u\rangle_\Sigma\,. \tag{7.52}$$

We next analyze the higher order terms (1), (3), (4) in (7.47).

Term (1). Using (7.27), $u = \chi^{\mathcal{B}}u + \chi^{\mathcal{G}}u$ and (3.12*a*), we obtain from (7.48).

$$\begin{aligned}|(1)| &\leqslant 2C\|D_t^{q-1}u\|_{H^1(Q)}\|D_t^{q-1}\chi^{\mathcal{G}}u\|_{H^1(Q)} \leqslant \\ &\leqslant C\{\|D_t^{q-1}u\|^2_{H^1(Q)} + \|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)}\} \leqslant C\{\|D_t^{q-1}\chi^{\mathcal{B}}u\|^2_{H^1(Q)} + \|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)}\}\end{aligned} \tag{7.53}$$

with C independent of q, $1/2 \leqslant q \leqslant 1$.

Term (3). Using $q-1 \leqslant 0$ and $D_2|_{x=0} \in OPS^1_{1,0}(\Sigma)$ (from (3.12*a*)), we obtain from (7.50)

$$|(3)| \leqslant 2C|g|_\Sigma\, |D_t^{q-1}\chi^{\mathcal{G}}u|_{H^1(\Sigma)}$$

and recalling (7.11) of Lemma 7.6

$$|(3) \leqslant C\{|g|^2_\Sigma + \|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)} + \|D_t^{q-1}\chi^{\mathcal{B}}u\|^2_{H^1(Q)}\} \tag{7.54}$$

with C independent of q, $1/2 \leqslant q \leqslant 1$.

Term (4). Similarly, recalling (7.28), $q-1<0$, the boundary regularity (7..3) as well as (7.11) of Lemma 7.6, we have from (7.51)

$$|(4)| \leqslant 2C|u|_{x=0}|_\Sigma\, |D_t^{q-1}\chi^{\mathcal{G}}u|_{H^1(\Sigma)} \leqslant C(\{|g|^2_\Sigma + \|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)} + \|D_t^{q-1}\chi^{\mathcal{B}}u\|^2_{H^1(Q)}\} \tag{7.55}$$

with C independent of q, $1/2 \leqslant q \leqslant 1$.

We now handle the lower order terms (2) and (5) which, however, depend on γ.

From (7.49) we have using (7.27) and $q-1\leqslant 0$

$$(7.56)\qquad |(2)|\leqslant 2C|D_t^{q-1}u\|_{H^1(Q)}\gamma\|D_t^{q-1}\chi^{\mathcal{G}}u\|_Q\leqslant$$
$$(\text{by }(7.1))\qquad \leqslant 2C\|D_t^{q-1}(\chi^{\mathcal{G}}u+\chi^{\mathcal{B}}u)\|_{H^1(Q)}\gamma|g|_\Sigma\leqslant$$
$$\leqslant C\{\|D_t^{q-1}(\chi^{\mathcal{G}}u+\chi^{\mathcal{B}}u)\|^2_{H^1(Q)}+\gamma^2|g|^2_\Sigma\leqslant$$
$$\leqslant 2C\|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)}+2C\|D_t^{q-1}\chi^{\mathcal{B}}u\|^2_{H^1(Q)}+C\gamma^2|g|^2_\Sigma .$$

Similarly from (7.52), recalling (7.28), the boundary regularity (7.3) and $q-1\leqslant 0$, we obtain

$$(7.57)\qquad |(5)|\leqslant C\gamma|g|_\Sigma|g|_\Sigma ,\qquad C \text{ independent of } q .$$

Finally, combining (7.47) with (7.53)-(7.57), we obtain

$$(7.58)\qquad \gamma C_0\|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)}\leqslant C\|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)}+C\|D_t^{q-1}\chi^{\mathcal{B}}u\|^2_{H^1(Q)}+C(1+\gamma+\gamma^2)|g|^2_\Sigma$$

with constant C independent of q, $1/2\leqslant q\leqslant 1$ and also independent of γ.

Selecting γ sufficiently large in (7.58) so that $\gamma C_0-C>0$, we finally obtain from (7.58)

$$(7.59)\qquad (\gamma C_0-C)\|D_t^{q-1}\chi^{\mathcal{G}}u\|^2_{H^1(Q)}\leqslant C\|D_t^{q-1}\chi^{\mathcal{B}}u\|^2_{H^1(Q)}+C(1+\gamma+\gamma^2)|g|^2_\Sigma$$

from which (7.12) follows as desired. Lemma 7.7 is proved. □

Having established Lemmas 7.5, 7.6, and 7.7 we have completed the proof of Proposition 7.2. □

7.1.2. *Regularity of the trace* $\chi^{\mathcal{G}}u|_\Sigma$: $\chi^{\mathcal{G}}u|_\Sigma\in H^q(\Sigma)$.

The following is the main result of the present subsection.

THEOREM 7.10. – Under assumption (7.1), there is a constant $\gamma_0>0$ such that if $\gamma>\gamma_0$ we then have $\chi^{\mathcal{G}}u|_\Sigma\in H^1(\Sigma)$: more precisely

$$(7.60)\qquad |\chi^{\mathcal{G}}u|_\Sigma|_{H^q(\Sigma)}\leqslant C_\gamma|g|_\Sigma$$

for a constant C_γ depending on γ but independent of q, $1/2\leqslant q\leqslant 1$. □

PROOF OF THEOREM 7.10. – We begin with a corollary to Lemma 7.6, Corollary 7.8 and Corollary 7.9.

COROLLARY 7.11. – Under assumption (7.1), there exists a constant $\gamma_0 > 0$ such that if $\gamma > \gamma_0$ then

$$|D_t^{q-1} \chi^{\mathcal{G}} u|_{H^1(\Sigma)} \leqslant C_\gamma |g|_\Sigma$$

with C_γ depending on γ but independent of q, $1/2 \leqslant q \leqslant 1$. □

PROOF OF COROLLARY 7.11. – We return to (7.11) of Lemma 7.6 and we use (7.16) and (7.17) of Corollaries 7.8 and 7.9 respectively, thus obtaining (7.61). □

Continuing with the proof of Proposition 7.10, we see that (7.61) implies

$$g \in L_2(\Sigma) \to \begin{cases} D_t^q \chi^{\mathcal{G}} u \in L_2(\Sigma) \\ D_t^{q-1} D_y \chi^{\mathcal{G}} u \in L_2(\Sigma) \end{cases} \quad \text{continuously}\,. \tag{7.62}$$

Hence

$$\chi^{\mathcal{G}} u \in L_2(R_y^{n-1}; H^q(R_t^1)) \tag{7.63}$$

$$D_y \chi^{\mathcal{G}} u \in L_2(R_y^{n-1}; H^{q-1}(R_t^1)) \tag{7.64}$$

continuously in $g \in L_2(\Sigma)$,

and by interpolation [L-M.1]

$$D_y^\theta \chi^{\mathcal{G}} u \in L_2(R_y^{n-1}, H^{q-\theta}(R_t^1))\,, \qquad 0 \leqslant \theta \leqslant 1 \tag{7.65}$$

since $q(1-\theta) + (q-1)\theta = q - \theta$. Choosing $\theta = q \leqslant 1$ in (7.65) yields

$$D_y^q \chi^{\mathcal{G}} u \in L_2(\Sigma)\,, \quad \text{continuously in } g \in L_2(\Sigma)\,. \tag{7.66}$$

Combining (7.62) with (7.66), we obtain $\chi^{\mathcal{G}} u \in H^q(\Sigma)$ continuously in $g \in L_2(\Sigma)$; i.e., (7.60). □

7.1.3. *Regularity of the trace* $\chi^{\mathcal{B}_r^+} u|_\Sigma$: $\chi^{\mathcal{B}_r^+} u|_\Sigma \in H^{q-(2-r)}(\Sigma)$, $1/2 \leqslant q \leqslant r/2$.

As in Section 6, we split $\chi^{\mathcal{B}}$ into $\chi^{\mathcal{B}_r^+}$ and $\chi^{\mathcal{B}_r^-}$ as in (5.20*a*) and analyze $\chi^{\mathcal{B}_r^+} u$ and $\chi^{\mathcal{B}_r^-} u$ separately. This subsection is devoted to $\chi^{\mathcal{B}_r^+} u$. The main result of the present subsection is

THEOREM 7.12. – Under assumption (7.1) for problem (1.6) with $u_0 = u_1 = f = 0$, restricted to $1/2 \leqslant q \leqslant r/2$, $3/2 < r < 2$, as in Section 5 [4], we have $\chi^{\mathcal{B}_r^+} u|_\Sigma \in$

---

[4] As in Section 6, we shall identify r in subsection 7.1.5 to be $r = 8/5$, so that $\frac{1}{2} \leqslant q \leqslant 4/5$; see (7.110).

$\in H^{q-(2-r)}(\Sigma)$; more precisely

$$|\chi^{\mathcal{B}_r^+} u|_\Sigma|_{H^{q-(2-r)}(\Sigma)} \leqslant C_r |g|_\Sigma \tag{7.67}$$

with constant C_r independent of q, $1/2 \leqslant q \leqslant r/2$. □

PROOF OF THEOREM 7.12. - Motivated by Sections 5.3, 5.4 and 7.1.1, we apply $D_t^{q-1+r/2-1} = D_t^{q+r/2-2}$ to the localized problem (3.22) with $\chi = \chi^{\mathcal{B}_r^+}$ and obtain for $f \equiv 0$:

$$\begin{cases} P(D_t^{q+r/2-2}\chi^{\mathcal{B}_r^+} u) = [P, \chi^{\mathcal{B}_r^+}] D_t^{q+r/2-2} u\,, & \text{in } \Omega\,,\ -\infty < t < \infty\,, \\ B(D_t^{q+r/2-2}\chi^{\mathcal{B}_r^+} u) = \chi^{\mathcal{B}_r^+} D_t^{q+r/2-2} g|_{x=0} + [B, \chi^{\mathcal{B}_r^+}] D_t^{q+r/2-2} u|_{x=0}\,, & \\ & \text{in } \Gamma\,,\ -\infty < t < \infty\,. \end{cases} \tag{7.68}$$

Problem (7.68) will be the basic localized problem for this present subsection, as problem (7.19) was the basic localized problem for Subsection 7.1.1. We apply to (7.68) the corresponding version of identity (4.5) of Theorem 4.2 *b*) in the solution $D_t^{q+r/2-2}\chi^{\mathcal{B}_r^+} u$. We obtain

$$\langle D_1 D_t^{q+r/2-2}\chi^{\mathcal{B}_r^-} u, D_t^{q+r/2-2}\chi^{\mathcal{B}_r^+} u\rangle = (1) + (2) + (3)\,, \tag{7.69}$$

$$(1) \quad = 2\,\mathrm{Im}\,\big([P, \chi^{\mathcal{B}_r^+}] D_t^{q+r/2-2} u, \tilde{D}_x D_t^{q+r/2-2}\chi^{\mathcal{B}_r^+} u\big)\,, \tag{7.70}$$

$$-(2) = |(2_1) + (2_2)|_\Sigma^2 \leqslant 2\{|(2_1)|_\Sigma^2 + |(2_2)|_\Sigma^2\}\,, \tag{7.71}$$

$$(2_1) \quad = \chi^{\mathcal{B}_r^+} D_t^{q+r/2-2} g|_{x=0}\,, \tag{7.72a}$$

$$(2_2) \quad = [B, \chi^{\mathcal{B}_r^+}] D_t^{q+r/2-2} u|_{x=0}\,, \tag{7.72b}$$

$$(3) \quad = \mathcal{O}\big(\|D_t^{q+r/2-2}\chi^{\mathcal{B}_r^+} u\|_{H^1(Q)}^2\big)\,. \tag{7.73}$$

Term (1). We rewrite (7.70) as

$$(1) = 2\,\mathrm{Im}\,\big([P, \chi^{\mathcal{B}_{ri}^+}] D_t^{r-2} D_t^{q-1} u, \tilde{D}_x D_t^{q-1}\chi^{\mathcal{B}_r^+} u\big)_Q\,. \tag{7.74}$$

We now invoke (5.43) of Theorem 5.3 as well as (7.4) of Proposition 7.2 on the left hand side term of the inner product in (7.74); while we invoke (7.9*a*) of Proposition 7.4 on the right hand side term. We obtain

$$|(1)| \leqslant C|g|_\Sigma^2\,, \qquad C \text{ independent of } q\,. \tag{7.75}$$

Term 2. From (7.72*a*) we obtain, recalling (5.25*b*) and $q + r/2 - 2 \leqslant 0$:

$$|(2_1)|_\Sigma \leqslant C|g|_\Sigma\,, \qquad C \text{ independent of } q\,. \tag{7.76}$$

As to (7.72*b*) we rewrite it as

$$(2_2) = [B, \chi^{\mathcal{B}_r^+}] D_t^{r-2} D_t^{q-r/2} u|_{x=0} \tag{7.77a}$$

and invoke (5.87) of Theorem 5.15 with $k = r-2$, $s = 0$, as well as the assumption $q \leqslant r/2$ and (7.3). We obtain

$$|(2_2)|_\Sigma \leqslant C_r |g|_\Sigma , \qquad C_r \text{ independent of } q,\ 1/2 \leqslant q \leqslant r/2 . \tag{7.77b}$$

Hence, from (7.76), (7.77*a*)-(7.77*b*) and (7.71)

$$|(2)| \leqslant C_r |g|_\Sigma^2 , \qquad C_r \text{ independent of } q,\ 1/2 \leqslant q \leqslant r/2 . \tag{7.78}$$

Term (3). We obtain from (7.73)

$$|(3)| \leqslant C \| D_t^{r/2-1} D_t^{q-1} \chi^{\mathcal{B}_r^+} u \|^2_{H^1(Q)} \leqslant C |g|_\Sigma^2 \tag{7.79}$$

C independent of q, $1/2 \leqslant q \leqslant r/2$, by recalling $r/2 - 1 < 0$, (5.25*a*) and (7,4) of Proposition 7.2.

Combining now (7.69) with (7.75), (7.78) and (7.79) we obtain

$$|\langle D_1 D_t^{q+r/2-2} \chi^{\mathcal{B}_r^+} u,\ D_t^{q+r/2-2} \chi^{\mathcal{B}_r^+} u \rangle| \leqslant C_r |g|_\Sigma^2 . \tag{7.80}$$

Thus we are in (7.80) in the same technical situation encountered in (6.7), or in (6.19), or in (7.44). Starting from (7.80) and proceeding as in going from (6.23) to (6.25) we likewise find by use of (6.23) and of Garding inequality that

$$C_0 |D_t^{q+r/2-2} \chi^{\mathcal{B}_r^+} u|^2_{H^{r/2}(\Sigma)} \leqslant |\langle D_1 D_t^{q+r/2-2} \chi^{\mathcal{B}_r^+} u,\ D_t^{q+r/2-2} \chi^{\mathcal{B}_r^+} u \rangle| + \\ + C_1 |D_t^{q+r/2-2} \chi^{\mathcal{B}_r^+} u|_\Sigma^2 \leqslant C_r |g|_\Sigma^2 \tag{7.81}$$

where in the last step we have used (7.80) as well as (7.3), $q + r/2 - 2 \leqslant 0$ and (5.25*b*). Equation (7.81) implies

$$g \in L_2(\Sigma) \to D_t^{q+r-2} \chi^{\mathcal{B}_r^+} u|_\Sigma \in L_2(\Sigma) \qquad \text{continuously} . \tag{7.82}$$

But the operators $D_t^{q+r-2} \chi^{\mathcal{B}_r^+}$ and $D_{y_j}^{q+r-2} \chi^{\mathcal{B}_r^+}$ belong to the same class, since $\sigma \sim |\eta|$ in $\mathcal{B}_r^+ \cup \mathcal{B}_{\mathrm{tr},r} \cup \mathcal{G}_{\mathrm{tr}}^{\mathrm{I}} = \operatorname{supp} \chi^{\mathcal{B}_r^+}$. Thus we likewise obtain from (7.82)

$$g \in L_2(\Sigma)' \to D_y^{q+r-2} \chi^{\mathcal{B}_r^+} u|_\Sigma \in L_2(\Sigma)' \qquad \text{continuously} . \tag{7.83}$$

Combining (7.82) and (7.83), we obtain (7.67) as desired. The proof of Theorem 7.12 is complete. □

7.1.4. *Regularity of the trace* $\chi^{\mathcal{B}_r^-} u|_\Sigma$: $\chi^{\mathcal{B}_r^-} u|_\Sigma \in H^{q-r/4}(\Sigma)$, $1/2 \leqslant q \leqslant r-1$.

This subsection is devoted to $\chi^{\mathcal{B}_r^-} u$. The main result of the present subsection is

THEOREM 7.13. – Under assumption (7.1) for problem (1.6) with $u_0 = u_1 = f \equiv 0$, restricted to $1/2 \leqslant q \leqslant r-1$, $3/2 < r < 2$ as in Section 5 (5), we have $\chi^{\mathcal{B}_r^-} u|_\Sigma \in H^{q-r/4}(\Sigma)$; more precisely

$$|\chi^{\mathcal{B}_r^-} u|_\Sigma|_{H^{q-r/4}(\Sigma)} \leqslant C_r |g|_\Sigma \tag{7.84}$$

with constant C_r independent of q, $1/2 \leqslant q \leqslant r-1$. □

PROOF OF THEOREM 7.13. – The proof is divided in two steps.

In Step 1, we provide an *interior* regularity result for $\tilde{D}_x \chi^{\mathcal{B}_r^-} u$, which is then used in Step 2 in combination with the trace Theorem 4.1 to yield the desired *trace* theory estimate given by (7.84).

STEP 1. – It is represented by the following

LEMMA 7.14. – Under the assumption of Theorem 7.13 we have

$$\tilde{D}_x D_t^{q-1+(1-r/2)} \chi^{\mathcal{B}_r^-} u = \tilde{D}_x D_t^{q-r/2} \chi^{\mathcal{B}_r^-} u \in L_2(Q);$$

more precisely

$$\|\tilde{D}_x D_t^{q-r/2} \chi^{\mathcal{B}_r^-} u\|_Q \leqslant C_r |g|_\Sigma \tag{7.85}$$

with constant C_r independent of q, $1/2 \leqslant q \leqslant r-1$. □

REMARK 7.2. – One should compare (7.85) involving $\chi^{\mathcal{B}_r^-} u$ with (7.4) (or (7.9)) involving $\chi^{\mathcal{B}} u$; namely (7.85) provides a gain of $0 < 1 - r/2 < 1/4$ in t of the regularity of $\tilde{D}_x \chi^{\mathcal{B}_r^-} u$ over the regularity of $\tilde{D}_x \chi^{\mathcal{B}} u$ in (7.9). □

PROOF OF LEMMA 7.14. – To begin with, we consider the localized problem

$$\begin{cases} P(D_t^{q-r/2} \chi^{\mathcal{B}_r^-} u) = [P, \chi^{\mathcal{B}_r^-}] D_t^{q-r/2} u\,, & \text{in } \Omega\,, \ -\infty < t < \infty\,, \\ B(D_t^{q-r/2} \chi^{\mathcal{B}_r^-} u) = \chi^{\mathcal{B}_r^-} D_t^{q-r/2} g|_{x=0} + [B, \chi^{\mathcal{B}_r^-}] D_t^{q-r/2} u|_{x=0}\,, & \\ & \text{in } \Gamma\,, \ -\infty < t < \infty\,, \end{cases} \tag{7.86}$$

(5) As in Section 6, we shall identify r in subsection 7.1.5 to be $r = 8/5$, so that $\frac{1}{2} \leqslant q \leqslant 3/5$; see (7.110).

which is obtained by applying $D_t^{q-r/2}$ to the localized problem (3.22) with $f\equiv 0$ and $\chi=\chi^{\mathcal{B}_r^-}$. Next, we write the version if inequality (4.6*c*) of Theorem 4.3 for problem (1.6) which corresponds to the localized problem (7.86) in the solution $D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u$. We obtain (compare with the proof of Lemma 7.5):

$$\left(1-\frac{\varepsilon}{2}\right)\|\tilde{D}_x D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u\|_Q^2\leqslant(1)+(2)+(3)+(4), \tag{7.87}$$

$$(1)=\operatorname{Re}\left([P,\chi^{\mathcal{B}_r^-}]D_t^{q-r/2}u,\ D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u\right)_Q, \tag{7.88}$$

$$(2)=\operatorname{Re}\left(D_1 D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u,\ D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u\right)_Q, \tag{7.89}$$

$$(3)=\frac{1}{2\varepsilon}C_w\|D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u\|_Q^2, \tag{7.90}$$

$$(4)=(4_1)+(4_2), \tag{7.91}$$

$$(4_1)=\operatorname{Im}\langle\chi^{\mathcal{B}_r^-}D_t^{q-r/2}g|_{x=0},\ D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u\rangle_\Sigma, \tag{7.92a}$$

$$(4_2)=\operatorname{Im}\langle[B,\chi^{\mathcal{B}_r^-}]D_t^{q-r/2}u|_{x=0},\ D_t^{q-r/2}\chi^{\mathcal{B}_r^-}u\rangle_\Sigma. \tag{7.92b}$$

Term (1). We rewrite (7.88) as

$$(1)=\operatorname{Re}\left([P,\chi^{\mathcal{B}_r^-}]D_t^{r-2}D_t^{q-1}u,\ D_t^{3-2r}D_t^{q}\chi^{\mathcal{B}_r^-}u\right)_Q. \tag{7.93}$$

We now invoke (5.43) of Theorem 5.3 and (7.4) of Proposition 7.2 on the left hand side term of the inner product in (7.93); while we invoke assumption (7.1), $3-2r<0$ and (5.25*a*) on the right hand side term. We obtain

$$|(1)|\leqslant C_r|g|_\Sigma^2,\qquad C_r \text{ independent of } q,\ 1/2\leqslant q\leqslant r-1. \tag{7.94}$$

Term (2). We rewrite (7.89) as

$$(2)=\operatorname{Re}\left(D_t^{-r}D_1\chi^{\mathcal{B}_r^-}D_t^q u,\ D_t^q\chi^{\mathcal{B}_r^-}u\right)_Q \tag{7.95}$$

where recalling (5.91*a*) of Theorem 5.16, using $D_t^{-r}\in OPS_{1,0}^{-r}$ [T.1, Proposition 1.3, (1.7), p. 37] and the product theorem [T.1, Theorem 4.4, p. 46 with $\varrho=\min(\varrho',\varrho'')=$ $=r-1$, $\delta=\max(\delta',\delta'')=2-r$] we obtain that

$$D_t^{-r}D_1\chi^{\mathcal{B}_r^-}\in OPS^0_{r-1,2-r}(R^n_{ty}),\qquad \text{uniformly in } x\in R^1_{x^+}. \tag{7.96a}$$

Thus, from Lemma 3.1 *a*) with $s=0$ we have

$$D_t^{-r}D_1\chi^{\mathcal{B}_r^-}:\quad \text{continuous } L_2(Q)\to L_2(Q). \tag{7.96b}$$

Using now (7.96*b*), assumption (7.1) and (5.25*a*) in (7.95), we conclude that

$$|(2)| \leqslant C_r |g|_\Sigma^2 , \quad C_r \text{ independent of } q , \ 1/2 \leqslant q \leqslant r-1 . \tag{7.97}$$

Term 3. By (5.25*a*) and assumption (7.1) we obtain

$$|(3)| \leqslant C_r |g|_\Sigma^2 , \quad C_r \text{ independent of } q , \ 1/2 \leqslant q \leqslant r-1 . \tag{7.98}$$

Term 4. As to (7.92*a*) since $q \leqslant r-1 < r/2$ by assumption, we obtain

$$|(4_1)| \leqslant C_r |g|_\Sigma , \quad C_r \text{ independent of } q \tag{7.99}$$

by use also of the trace regularity of (7.3). As to (7.92*b*) we rewrite it as

$$(4_2) = \operatorname{Im} \langle [B, \boldsymbol{\chi}^{\mathcal{B}_r^-}] D_t^{r/2} u|_{x=0}, D_t^{2(q-r+1)} \boldsymbol{\chi}^{\mathcal{B}_r^-} u|_{x=0} \rangle . \tag{7.100}$$

Invoking now (5.87) of Theorem 5.15 with $s = 0$ and $k = r-2$ on the left term of the inner product in (7.100), along with $q - r + 1 \leqslant 0$ (by assumption) and the trace regularity (7.3) on the right term, we conclude that

$$|(4_2)| \leqslant C_r |g|_\Sigma^2 , \quad C_r \text{ independent of } q , \ 1/2 \leqslant q \leqslant r-1 . \tag{7.101}$$

Then (7.87) combined with (7.93), (7.95), (7.98), (7.91), (7.99) and (7.100) yields the desired conclusion (7.85). Lemma 7.14 is proved. □

COROLLARY 7.15. – Under the assumption of Theorem 7.13 we have $\Lambda^{q-r/2} \tilde{D}_x \boldsymbol{\chi}^{\mathcal{B}_r^-} u \in \in L_2(Q)$; more precisely

$$\| \Lambda^{q-r/2} \tilde{D}_x \boldsymbol{\chi}^{\mathcal{B}_r^-} u \|_Q \leqslant C_r |g|_\Sigma \tag{7.102}$$

with constant C_r independent of q, $1/2 \leqslant 0 \leqslant r-1$, where the operator $\Lambda_{t,y} = \Lambda$ is defined by (4.1*a*). □

PROOF OF COROLLARY 7.15. – We use (7.85) of Lemma 7.14 along with the fact that the operators $D_t^\theta \tilde{D}_x \boldsymbol{\chi}^{\mathcal{B}_r^-}$ and $\Lambda_t^\theta \tilde{D}_x \boldsymbol{\chi}^{\mathcal{B}_r^-}$ belong to the same class with $\sigma \sim |\eta|$ in

$$\mathcal{B}_r^- \cup \mathcal{B}_{\mathrm{tr},r} \cup \mathcal{G}_{\mathrm{tr}}^{\mathrm{II}} = \\ = \operatorname{supp} \chi^{\mathcal{B}_r^-} \supset \operatorname{supp} [\text{symbol of } D_t^\theta \tilde{D}_x \boldsymbol{\chi}^{\mathcal{B}_r^-}], \operatorname{supp} [\text{symbol of } \Lambda^\theta \tilde{D}_x \boldsymbol{\chi}^{\mathcal{B}_r^-}] . \qquad \square$$

STEP 2. – We now use identity (4.3) of the trace Theorem 4.1 with v there replaced by $\boldsymbol{\chi}^{\mathcal{B}_r^-} u$ now. We obtain

$$|\boldsymbol{\chi}^{\mathcal{B}_r^-} u|_{H^\theta(\Sigma)}^2 = [1] + [2] , \tag{7.103}$$

$$[1] = 2 \operatorname{Im} (\Lambda^\theta \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^\theta \boldsymbol{\chi}^{\mathcal{B}_r^-} u)_Q , \tag{7.104}$$

$$[2] = \mathcal{O}(\| \Lambda^\theta \boldsymbol{\chi}^{\mathcal{B}_r^-} u \|)_Q^2 . \tag{7.105}$$

Since Λ is self-adjoint on $L_2(R^n_{ty})$, we rewrite (7.104) as

$$(7.106)\qquad [1] = 2\,\mathrm{Im}\int_{R^1_{x+}} (\Lambda^\theta \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^\theta \chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx = $$
$$= 2\,\mathrm{Im}\int_{R^1_{x+}} (\Lambda^{q-r/2} \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^{r/2-q+2\theta} \chi^{\mathcal{B}_r^-} u)_{L_2(R^n_{ty})}\, dx =$$
$$= 2\,\mathrm{Im}(\Lambda^{q-r/2} \tilde{D}_x \chi^{\mathcal{B}_r^-} u, \Lambda^{r/2-2q+2\theta}\Lambda^q \chi^{\mathcal{B}_r^-} u)_Q\,.$$

Recalling now (7.102) of Corollary 7.15 and the left term of the inner product in in (7.106) and using $\Lambda^q \chi^{\mathcal{B}_r^-} u \in L_2(Q)$ (which follows from assumption (7.1) as in the proof of Lemma 7.3), we obtain from (7.106):

$$(7.107a)\qquad |[1]| \leqslant C_r |g|^2_\Sigma\,, \quad C_r \text{ independent of } q\,,\ 1/2 \leqslant q \leqslant r-1$$

provided $r/2 - 2q + 2\theta \leqslant 0$; i.e. *provided*

$$(7.107b)\qquad \theta \leqslant q - r/4\,.$$

Similarly, we obtain from (7.105) that

$$(7.108a)\qquad |[2]| \leqslant C_r |g|^2_\Sigma\,, \quad C_r \text{ independent of } q\,,\ 1/2 \leqslant q \leqslant r-1$$

provided

$$(7.108b)\qquad \theta \leqslant q\,.$$

We thus conclude from (7.103), (7.107*a*)-(107*b*) and (7.108*a*)-(108*b*) that

$$(7.109)\qquad |\chi^{\mathcal{B}_r^-} u|_{H^\theta(\Sigma)} \leqslant C_r |g|_\Sigma\,, \quad C_r \text{ independent of } q\,,\ 1/2 \leqslant q \leqslant r$$

provided $\theta \leqslant q - r/4$. Selecting $\theta = q - r/4$ yields (7.84) as desired. The proof of Theorem 7.13 is complete. □

7.1.5. *Completion of the proof of Theorem* 7.1.

Theorems 7.10, 7.12 and 7.13 provide the regularity of the various components $\chi^{\mathcal{G}} u|_\Sigma \in H^q(\Sigma)$; $\chi^{\mathcal{B}_r^-} u|_\Sigma \in H^{q-(2-r)}(\Sigma)$, $1/2 \leqslant q \leqslant r/2$; and $\chi^{\mathcal{B}_r^-} u|_\Sigma \in H^{q-r/4}(\Sigma)$, $1/2 \leqslant q \leqslant r-1$ of the partition of unity decomposition in (6.1) for the trace of the solution. Intersecting the segments $\{q - (2-r), 3/2 < r < 2\}$ (with $1/2 \leqslant q \leqslant r/2$) and $\{q - r/4, 3/2 < r < 2\}$ (with $1/2 \leqslant q \leqslant r-1$), we obtain (as already announced)

$$(7.110)\qquad r = 8/5$$

(as in Section 6 in the case $f \in L_2(Q)$, $g = 0$), in which case $q - (2 - 8/5) = q - 8/5 \cdot 1/4 = q - 2/5$ is the optimal value which provides the highest regularity to

$$\chi^{\mathcal{G}} u|_\Sigma \,, \quad \chi^{\mathcal{B}_r^+} u|_\Sigma \,, \quad \text{and} \quad \chi^{\mathcal{B}_r^-} u|_\Sigma$$

simultaneouly; i.e.

$$\text{for } r = 8/5 \to \chi^{\mathcal{G}} u|_\Sigma,\ \chi^{\mathcal{B}_r^+} u|_\Sigma,\ \chi^{\mathcal{B}_r^-} u|_\Sigma \in H^{q-2/5}(\Sigma)\,.$$

The proof of Theorem 7.1 is complete. □

7.2. *From trace regularity* $u|_\Sigma \in H^{q-2/5}(\Sigma)$ *back to interior regularity* $u \in H^{q/2+3/10}(Q)$. *Theorem* 7.16: *an improvement of* $[3/10 - q/2]$ *over the a-priori information* $u \in H^q(Q)$.

The main result of the present section is the following.

THEOREM 7.16. – As in the statement of Theorem 7.1, assume hypothesis (7.1) for the corresponding solution of problem (1.6) with $u_0 = u_1 = f = 0$. Then, in fact, $u \in H^{q/2+3/10}(Q)$; more precisely

$$\|u\|_{H^{q/2+3/10}(Q)} \leqslant C|g|_\Sigma \tag{7.11}$$

with C independent of q, $1/2 \leqslant q \leqslant 3/5$. □

REMARK 7.3. – Conclusion (7.111) represents as improvement of $3/10 - q/2 = (q/2 + 3/10) - q$ in all variables over the original a-priori information (7.1) (only in the t-variable alone). This fact will give rise to a «boost-strap» argument in Section 7.3. □

PROOF OF THEOREM 7.16.

STEP 1. – We begin by improving the regularity of u in the t variable as needed in (7.111).

LEMMA 7.17. – Under the assumption (7.1) of Theorem 7.16 we have that $D_t^{q/2-7/10} u \in H^1(Q)$; more precisely

$$\|D_t^{q/2-7/10} u\|_{H^1(Q)} \leqslant C|g|_\Sigma\,, \quad \text{independent of } q\,,\ 1/2 \leqslant q \leqslant 3/5\,. \tag{7.112}$$

PROOF OF LEMMA 7.17. – We shall invoke identity (4.12a) for problem (1.6) with $Pu = f = 0$ (rewritten as (7.46) in the proof of Lemma 7.7) as it applies to the localized problem

$$\text{(7.13)}\qquad \begin{cases} P(D_t^\theta u) = 0 & \text{in } \Omega\,, \\ B(D_t^\theta u) = D_t^\theta g & \text{in } \Gamma\,, \end{cases} \qquad -+<t<\infty\,,$$

where θ is a negative real number, $\theta < 0$ to be determined below. The version of (4.12*a*) or (7.46) which corresponds to problem (7.113) is

$$\text{(7.114)}\qquad \gamma C_0 \|D_t^\theta u\|^2_{H^1(Q)} \leqslant (1) + (2)\,,$$

$$\text{(7.115)}\qquad (1) = -2\,\mathrm{Re}\,\langle D_t^\theta g, D_2 D_t^\theta u\rangle_\Sigma\,,$$

$$\text{(7.116)}\qquad (2) = 2\gamma\,\mathrm{Im}\,\langle D_t^\theta g, aD_t^\theta u\rangle_\Sigma\,.$$

We shall now exploit the new information on the trace $u|_\Sigma$ provided by (7.2) of Theorem 7.1.

Term (1). We rewrite (7.115) as

$$\text{(7.117)}\qquad (1) = -2\,\mathrm{Re}\,\langle g, D_2 D_t^{-1} D_t^{2\theta+1+2/5-q} D_t^{q-2/5} u\rangle_\Sigma\,.$$

By (3.12) and [T.1, Proposition 1.3, (1.7), p. 37, and Theorem 4.4, p. 46] we have $D_2 D_t^{-1}|_{x=0} \in OPS^0_{1,0}$ and hence

$$\text{(7.118)}\qquad D_2 D_t^{-1}|_{x=0}\colon \text{ continuous } H^s(\Sigma) \to H^s(\Sigma)\,.$$

Thus, by (7.118) with $s = 0$ and (7.2) of Theorem 7.1 applied on (7.117) we obtain

$$\text{(7.119)}\qquad |(1)| \leqslant C|g|^2_\Sigma\,, \quad \text{provided } 2\theta \leqslant q - 1 - 2/5 = q - 7/5$$

with C independent of q, $1/2 \leqslant q \leqslant 3/5$.

Term (2). Similarly for the lower order term (7.116)

$$\text{(7.120)}\qquad |(2)| \leqslant 2\gamma|\langle g, aD_t^{2\theta-q+2/5} D_t^{q-2/5} u\rangle_\Sigma| \leqslant C\gamma|g|^2_\Sigma\,, \quad \text{provided } 2\theta \leqslant q - 2/5\,,$$

Using (7.119)-(7.120) in (7.114) we obtain as desired

$$\text{(7.121)}\qquad \gamma C_0 \|D_t^\theta u\|^2_{H^1(Q)} \leqslant C(1+\gamma^2)|g|^2_\Sigma\,, \quad \text{provided } 2\theta \leqslant q - 7/5 < 0\,.$$

Choosing the best value $2\theta = q - 7/5 < 0$ in (7.121) results in (7.112) as desired. Lemma 7.17 is proved. □

STEP 2. – We next improve the regularity in the t and y variables for $\chi^{\mathcal{B}} u$.

COROLLARY 7.18 (to Lemma 7.17). – Under the assumption of (7.1) of Theorem 7.16 we have:

(i)

$$\left.\begin{array}{l}\|D_t^{q/2-7/10}\chi^{\mathcal{B}}u\|_{H^1(Q)}\\ \|D_t^{q/2-7/10}\chi^{\mathcal{G}}u\|_{H^1(Q)}\end{array}\right\}\leq C|g|_\Sigma\,, \tag{7.122}$$

C independent of q, $1/2\leq q\leq 3/5$.

(ii) Moreover

$$\|\Lambda^{q/2-7/10}\chi^{\mathcal{B}}u\|_{H^1(Q)}\leq C|g|_\Sigma \tag{7.123}$$

C independent of q, $1/2\leq q\leq 3/5$, where the operator $\Lambda=\Lambda_{ty}$ is defined by (4.1). □

PROOF OF COROLLARY 7.18. – Part (i) is a consequence of (7.112). As to part (ii), we use as usual that $D_t\chi^{\mathcal{B}}$ and $D_y\chi^{\mathcal{B}}$ belong to the same operator class since $\sigma\sim|\eta|$ in $\mathcal{B}\cup\mathcal{G}_{tr}^{I}\cup\mathcal{G}_{tr}^{II}=\operatorname{supp}\chi^{\mathcal{B}}$. □

STEP 3. – We now find the desired regularity for $\chi^{\mathcal{B}}u$.

LEMMA 7.19. – Under the assumption (7.1) of Theorem 7.16 we have $\chi^{\mathcal{B}}u\in H^{q/2+3/10}(Q)$; more precisely

$$\|\chi^{\mathcal{B}}u\|_{H^{q/2+3/10}(Q)}\leq C|g|_\Sigma\,. \tag{7.124}$$

C independent of q, $1/2\leq q\leq 3/5$.

PROOF OF LEMMA 7.19. – From (7.123) we obtain (for the same reason as in the proof of Corollary 7.18)

$$\Lambda\Lambda^{q/2-7/10}\chi^{\mathcal{B}}u=\Lambda^{q/2+3/10}\chi^{\mathcal{B}}u\in L_2(Q)\,, \tag{7.125}$$

$$D_x\Lambda^{q/2-7/10}\chi^{\mathcal{B}}u\in L_2(Q)\,, \tag{7.126}$$

continuously in $g\in L_2(\Sigma)$. But (7.125)-(7.126) imply

$$\chi^{\mathcal{B}}u\in L_2(R^1_{x+};\,H^{q/2+3/10}(R^n_{ty}))\,, \tag{7.127}$$

$$D_2\chi^{\mathcal{B}}u\in L_2(R^1_{x+};\,H^{q/2-7/10}(R^n_{ty}))\,, \tag{7.128}$$

continuously in $g\in L_2(\Sigma)$ where to obtain (7.128) we use that the constant coefficient operator Λ commutes with D_x. We next interpolate between (7.127) and (7.128) to obtain [L-M.1]

$$D_x^\varrho\chi^{\mathcal{B}}u\in L_2(R^1_{x+};\,H^{q/2+3/10-\varrho}(R^n_{ty}))\,,\qquad 0\leq\varrho\leq 1\,. \tag{7.129}$$

If we now specialize to $0 < \varrho = q/2 + 3/10 < 1$, we obtain from (7.129)

$$D_x^{q/2+3/10}\chi^{\mathcal{B}} u \in L_2(Q) \tag{7.130}$$

continuously in $g \in L_2(\Sigma)$, which along with (7.125) yields (7.124), as desired.
The proof of Lemma 7.19 is complete. □

STEP 4. – We now find the desired regularity for $\chi^{\mathcal{G}} u$.

LEMMA 7.20. – Under the assumption (7.1) of Theorem 7.16 we have $\chi^{\mathcal{G}} u \in H^{q/2+3/10}(Q)$; more precisely

$$\|\chi^{\mathcal{G}} u\|_{H^{q/2+3/10}(Q)} \leqslant C|g|_\Sigma , \tag{7.131}$$

C independent of q, $1/2 \leqslant q \leqslant 3/5$.

PROOF OF LEMMA 7.20. – From (7.122*b*) we have a fortiori:

$$D_t D_t^{q/2-7/10}\chi^{\mathcal{G}} u = D_t^{q/2+3/10}\chi^{\mathcal{G}} u \in L_2(Q) , \tag{7.132}$$

$$D_y D_t^{q/2-7/10}\chi^{\mathcal{G}} u = D_t^{q/2-7/10} D_y \chi^{\mathcal{G}} u \in L_2(Q) , \tag{7,133}$$

re-written as

$$\chi^{\mathcal{G}} u \in L_2(R^n_{x+y}; H^{q/2+3/10}(R^1_t)) , \tag{7.134}$$

$$D_y \chi^{\mathcal{G}} u \in L_2(R^n_{x+y}; H^{q/2-7/10}(R^1_t)) , \tag{7.135}$$

continuously in $g \in L_2(\Sigma)$. By interpolating between (7.134) and (7.135), we obtain [L-M.1]:

$$D_y^\theta \chi^{\mathcal{G}} u \in L_2(R^n_{x+y}, H^{q/2+3/10-\theta}(R^1_t)) , \quad 0 \leqslant \theta \leqslant 1 . \tag{7.136}$$

If we now specialize to $0 < \theta = q/2 + 3/10 < 1$, we obtain from (7.136)

$$D_y^{q/2+3/10}\chi^{\mathcal{G}} u \in L_2(Q) \tag{7.137}$$

continuously in $g \in L_2(\Sigma)$.
A similar argument yields

$$D_x^{q/2+3/10}\chi^{\mathcal{G}} u \in L_2(Q) \tag{7.138}$$

continuously in $g \in L_2(\Sigma)$, if we replace (7.133) with

$$D_x D_t^{q/2-7/10} u = D_t^{q/2-7/10} D_x \chi^{\mathcal{G}} u \in L_2(Q) . \tag{7.139}$$

Thus, (7.132), (7.137) and (7.138) combined yield (7.131) as desired, and Lemma 7.20 is proved. □

Finally, (7.124) of Lemma 7.19 and (7.131) of Lemma 7.20 together yield (7.111). Theorem 7.16 is proved. □

7.3. *Final step in the proof of Theorem* 1.3: *the « boost-strap » argument.*

Starting with the a-priori information on the regularity of problem (1.6) with $u_0 = u_1 = f = 0$ and $g \in L_2(\Sigma)$:

interior: $u \in H^q(Q)$ (indeed, only $D_t^q u \in L_2(Q)$ was needed),

boundary: $u|_\Sigma \in H^0(\Sigma)$,

we have obtained improvements on the regularity of u expressed by Theorem 7.1 and Theorem 7.16, according to the following scheme

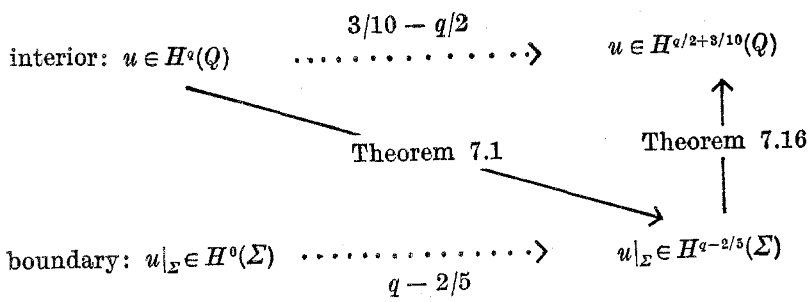

Figure 7.1. First Step.

We now repeat the step taking the regularity on the right column of Fig. 7.1 as new a-priori information. We obtain since

$$3/10 - \frac{q/2 + 3/10}{2} = 3/10 - q/4:$$

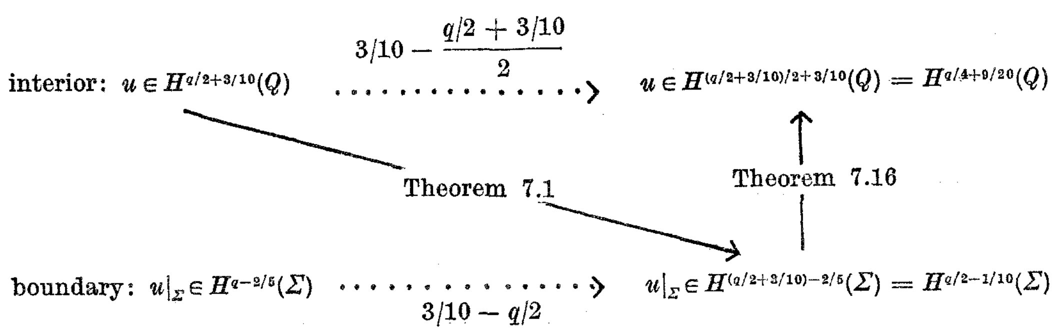

Figure 7.2. Second Step.

Taking the regularity of the right column of Fig. 7.2 as new information we have at the third step

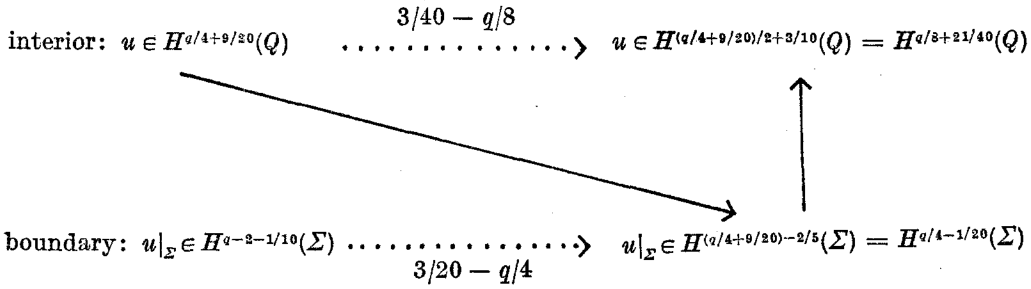

Figure 7.3. Third Step.

and so on. But at the outset we know that $q = q_0 = 1/2$. Hence the subsequent improvement on the *interior regularity* are

$$\frac{3}{10} - \frac{q_0}{2} = \frac{1}{20}; \quad \frac{3}{20} - \frac{q_0}{4} = \frac{1}{40}; \quad \frac{3}{40} - \frac{q_0}{8} = \frac{1}{80}; \quad \text{etc.}. \tag{7.139}$$

Thus since

$$\frac{1}{2} + \frac{1}{20} + \frac{1}{40} + \frac{1}{80} + \ldots = \frac{1}{2} + \frac{1}{20}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right] = \tag{7.140}$$
$$= \frac{1}{2} + \frac{1}{20}\frac{1}{1-\frac{1}{2}} = \frac{1}{2} + \frac{1}{10} = \frac{3}{5}$$

we can repeat the steps an arbitrary number of finitely many times, we conclude that the *interior regularity* of u is

$$u \in H^{3/5-\varepsilon}(Q), \quad \forall \varepsilon > 0 \tag{7.141}$$

continuously in $g \in L_2(\Sigma)$, as desired. Similarly, starting with $q = q_0 = 1/2$, the subsequent improvement of the *trace regularity* are

$$q_0 - \frac{2}{5} = \frac{1}{10}; \quad \frac{3}{10} - \frac{q_0}{2} = \frac{1}{20}; \quad \frac{3}{20} - \frac{q_0}{4} = \frac{1}{40}; \quad \text{etc.}. \tag{7.142}$$

Thus, similarly, since

$$\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots = \frac{1}{10} + \frac{1}{20}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right] = \tag{7.143}$$
$$= \frac{1}{10} + \frac{1}{20}\frac{1}{1-\frac{1}{2}} = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$$

we conclude that the *trace regularity* of u is

$$u|_{\Sigma} \in H^{3/5-\varepsilon}(\Sigma)\,, \qquad \forall \varepsilon > 0\,. \tag{7.144}$$

continuously in $g \in L_2(\Sigma)$.

Conclusions (7.141) and (7.144) prove Theorem 1.3. □

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