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On Nash Theory of Arc Structure of Singularities (*).

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Sunto. – *Si ha l'intenzione d'incominciare lo studio sistematico della teoria abbozzata da John Nash circa venticinque anni fa, dove si associa a una varietà algebrica singolare X spazi i cui ponti corrispondono genericamente a troncature di certi rami analitici parametrizati su X. In questo articolo si presentano accuratamente i fondamenti della teoria. Inoltre, con questi metodi (e nozioni della teoria della equisingolarità per le curve sghembe) s'introducono nuovi invarianti di una singolarità, si spera di studiarli più accuratamente in futuri lavori.*

Introduction.

In the mid-sixties, JOHN NASH wrote a paper entitled «Arc structure of singularities» ([5]), where he introduce some interesting ideas about a possible way to study singularities of algebraic and complex analytic varieties. The basic idea is to consider «parametrized analytic arcs γ» whose origin is in S, the singular set of an algebraic variety X over a field k (and the general point of γ is smooth); if we assume $X \subset A^n$, this arc is given by n power series $\phi_i, \ldots, \phi_n$ in $k[[t]]$. Truncating the series $\phi_i \bmod (t^{n+1})$, and taking the resulting coefficients in a certain order, we get a point in an affine space A^M, $M = M(N)$. Letting the arc γ vary, we get a constructible set in A^M, whose Zariski-closure is the Nash variety $V(X, S, N)$. In [5], some basic properties of these varieties are studied, and a number of examples discussed.

The following are some of the things done in this paper:

1) *A presentation of the foundations of the theory.* The relevant spaces are not introduced simply as algebraic varieties, but also they are endowed with a natural non-reduced structure, which might be useful. For instance, using this, it is possible (under some mild restrictions) to characterize the smoothness of X (and S) in terms of the associated Nash spaces (cf. § 2 and § 3).

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2) *A careful study of the basic defining properties.* One may verify that some of Nash' requirements may be relaxed. For instance, in a suitable sense, the requirement that our «truncated arc γ_N» must be induced by an actual arc, whose general point is smooth, may be replaced by the simpler condition that γ_N can be lifted to $\gamma_{N'}$: Spec $k[\![t]\!]/(t^{N'}) \to X$, N' large enough.

3) *The introduction of new invariants.* In section 5, we verify that, over a dense open set of each irreducible component Σ of the Nash variety $V(X, S, N)$, the points of Σ correspond to truncations of arcs which actually vary in a family, parametrized by an open set of Σ; moreover this family is equisingular, which insures that the relevant invariants of the fibers (Milnor numbers, δ-invariants, etc.) are constant. (cf. specially (5.11)). This allows us to associate to the singularities of X some possibly interesting numerical invariants. We expect to return to a finer study of these in the future.

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0. – Notation and terminology.

In general, we shall follow the conventions of [3], although for us an algebraic variety is not necessary irreducible. We work over a base field k algebraically closed; part of the time we'll assume $k = \boldsymbol{C}$ (the complex numbers).

If X is an algebraic scheme over $\boldsymbol{C}$, X^h denotes its associated analytic space; a metric open of X will be, by definition, an open set of X^h, with its usual topology (which is, as is well known, given by a metric).

All the rings which will appear here are commutative and with an identity, which is preserved by homomorphisms. In general, the ideal of a ring A generated by elements $f_1, \ldots, f_n$ of A will be denoted simply by $(f_1, \ldots, f_n)$. If A is local, $\max(A)$ denotes the maximal ideal of A. If f is a power series, $\operatorname{ord}(f)$ denotes its order.

The germ at X of an analytic space at $x \in X$ is denoted by (X, x), although sometimes, if the center x is clear, we simply talk about the germ X.

The symbol $\subset$ will indicate proper inclusion (i.e., contained but not equal).

The symbols $\boldsymbol{R}$, $\boldsymbol{N}$ denote the real numbers and non negative integers respectively.

1. – The basic constructions.

(1.1) In this section, X denotes an algebraic scheme of finite type over an algebraically closed field k, $\operatorname{Sing}(X)$ denotes the *singular locus of* X, $S \subset X$ is a closed subscheme.

we let: $\mathcal{T} = k[\![t]\!]$ (formal power series in t), $\mathcal{T}_N = k[\![t]\!]/(t^{N+1})$, $T = \operatorname{Spec}(\mathcal{T})$, $T_N = \operatorname{Spec}(\mathcal{T}_N)$.

An *analytic arc* (or, for short, just an arc) on X is a morphism $T \to X$. An *N-trun-*

cated arc on X (or just an *N-arc*) is a morphism $T_N \to X$. An N-arc $\phi_N: T_N \to X$ is *liftable* if there is an arc $\phi: T \to X$ such that $\phi_N = \phi i$, where $i: T_N \to T$ is the natural inclusion. In this case we also say that ϕ_N is obtained from ϕ by «truncation (mod t^{N+1})», or that ϕ_N is the N-truncation of ϕ. In a similar way we define: ϕ_N can be lifted to $\phi_{N'}$, $N > N'$. If ϕ, ϕ' are arcs inducing the same $\phi_N: T_N \to X$ by truncation (mod t^{N+1}), we write $\phi \equiv \phi' \pmod{t^{N+1}}$.

An *S-arc* (or an *arc, relative to S*) is an arc $\phi: T \to N$ such that $\phi(0) \in S$, where 0 is the closed point of T. In a similar fashion one defined the notion of «an *N-arc, relative to S*» (also called «an N-truncated S-arc»). A *general S-arc* (resp. an *N-truncated general S-arc*) is one such that the image of T (resp. of T_n) is not contained in S.

Note that if $\phi_N: T_N \to X$ is an N-truncated S-arc (i.e., $\phi_N(0) \in S$) and $\phi: T \to X$ is a lifting of ϕ_N, then automatically ϕ is an S-arc.

(1.2) We shall need the following:

THEOREM. – Given equations $f_i(x_1, \dots, x_r) = 0$, $i = 1, \dots, m$, with $f_i \in k[x_1, \dots, x_r]$ (polynomial ring) and an integer $N_0 \geq 0$, then there is an integer N (depending on N_0), having the following properties: if $a_i(t) \in k[\![t]\!]$, $i = 1, \dots, r$ are such that $f_i(a_1, \dots, a_r) \equiv 0 \bmod t^{N+1})$, $i = 1, \dots, m$, then there are series $b_i(t) \in k[\![t]\!]$, $i = 1, \dots, r$, satisfying $f_i(b_1, \dots, b_r) = 0$, $i = 1, \dots, m$, and $b_i \equiv a_i \pmod{t^{N_0+1}}$, $i = 1, \dots, r$.

This is true because $k[\![t]\!]$ has the «strict approximation property». cf. [6], Section 1.

A consequence of this is the following result:

(1.3) PROPOSITION. – Let X be a scheme, $S \subset X$ a subscheme (as in (1.1)), N_0 a positive integer. Then, there exists an integer N_1 with the following property: if $\phi: T_{N_0} \to X$ is an N_0-arc (resp. a N_0-truncated S-arc) which can be lifted to an N_1-arc (resp. a N-truncated S-arc), then ϕ can be lifted to an arc (resp., an S-arc) $\phi: T \to X$.

PROOF. – Cover X with affine opens $\mathrm{Spec}(A_i)$, $i = 1, \dots, l$, with A_i of the form $k[x_1, \dots, x_{r(i)}]/(f_1, \dots, f_{m(i)})$; use Theorem (1.2) on each A_i, with our given N_0. If $N(i)$ is the integer of the conclusion of (1.2), then clearly $N = \max(N(1), \dots, N(l))$ works.

(1.4) Next we want to parametrize, in a suitable sense, the N-arcs on a scheme X. Recall the following basic, well-known facts.

If $f: W \to Z$ is a morphism of schemes, then $\mathrm{Im}(f)$ is the closed subscheme of Z defined by the sheaf of ideals $I = \mathrm{Ker}(\mathcal{O}_Z \to f_* \mathcal{O}_W)$. From now on, $\mathrm{Im}(f)$ will always denote the image of f in the sense. If we have a sequence of morphisms $X = X_0 \leftarrow X_1 \dots$ and $f_n: X_n \to X$ is the composite map, then we have inclusions $\mathrm{Im} f_1 \supseteq \mathrm{Im} f_2 \supseteq \dots$, corresponding to the chain of $\mathcal{O}_X$-ideals $I_1 \subseteq I_2 \subseteq \dots$, $I_j = \mathrm{Ker}(\mathcal{O}_X \to f_{j*} \mathcal{O}_{X_j})$. In the noetherian situation of (1.1), the chain $\{I_j\}$ eventually stabilizes, i.e., there is an n_0 such that $\mathrm{Im}(f_n) = \mathrm{Im}(f_{n_0})$, for $n \geq n_0$.

(1.5) Recall that if $\mathcal{H} = \mathrm{Hom}(T_N, X)$ is the functor from the category of algebraic schemes over k to sets defined by $\mathcal{H}(U) = \{f\colon T_N \times U \to X \times U / f$ commutes with the projections on $U\}$, then it is represented by an algebraic scheme $\mathrm{Hom}(T_N, X)$. Concretely, if $X \subset \boldsymbol{A}_k^r$ is the affine scheme defined by $f_1, \dots, f_m$ in $k[X_1, \dots, X_r]$, take truncated series $a_i \in k[\![t]\!]/(t^{N+1})$, let M be the total number of the resulting coefficients $\{a_j^{(i)}\}$. Write

$$f_q(a_1(t), \dots, a_r(t)) \equiv \sum_{s=0}^{N} F_s^{(q)}(\{a_j^{(i)}\})\, t^s \pmod{t^{N+1}},$$

then the subscheme of $\boldsymbol{A}_k^M$ defined by $F_0^{(1)}, \dots, F_N^{(m)}$ is $\mathrm{Hom}(T_N, X)$ in this case. In general, cover X with affines X_i, apply the process just described to each of these, and glue in the usual way.

In a similar fashion, one represents the functor $\mathcal{H}_S$ (defined by $\mathcal{H}_S(U) = \{f \in \mathcal{H}(U) / f(0 \times U) \subset S \times U\}$, 0 being the closed point of T_N) by means of a closed subscheme $\mathrm{Hom}_S(T_N, X)$ of $\mathrm{Hom}(T_N, X)$. In the affine situation ($X = \mathrm{Spec}\, k[X_1, \dots, X_r]/(f_1, \dots, f_m)$) of above, and using the same notation, $\mathrm{Hom}_S(T_N, X)$ is the subscheme of $\boldsymbol{A}^M$ defined by the ideal $(F_0^{(1)}, \dots, F_N^{(m)}, \bar{h}_1, \dots, \bar{h}_s)$, of $k[\{a_j^{(i)}\}]$, $1 \leq i \leq m$, $0 \leq j \leq N$, where $(h_1, \dots, h_s) \subset k[X_1, \dots, X_r]$ define S, and $\bar{h}_j = h_j(a_0^{(i)}, \dots, a_0^{(r)})$, $j = 1, \dots, s$.

From the fact that $T_N \subsetneq T_{N'}$ if $N \leq N'$, one readily obtains «projection maps»:

$$f_{N'N}\colon \mathrm{Hom}_S(T_{N'}, X) \to \mathrm{Hom}_S(T_N, X), \tag{1.5.1}$$

essentially corresponding to forgetting the last $N' - N$ terms of the truncations. Since clearly $\mathrm{Hom}_S(T_0, X) \approx S$, it follows that $\mathrm{Hom}_S(T_0, X)$ is an S-scheme, for each $N \geq 0$.

(1.6) Fix an integer $N \geq 0$. According to (1.4), the sequence of images (cf. (1.5.1.)):

$$\mathrm{Im}\, f_{N+1,N} \supseteq \mathrm{Im}\, f_{N+2,N} \supseteq \cdots$$

eventually stabilizes, i.e., all inclusions become equalities. Let:

$$E_N = \mathrm{Im}\, f_{M,N}, \qquad M \geq M_0 \text{ large enough}. \tag{1.6.1}$$

Intuitively, points of a dense open set of E_N correspond to truncations of liftable S-arcs. In fact, we may view E_N as $\mathrm{Im}\, f_{M,N}$, where $M \geq M_0$ (of (1.6.1)) and $M \geq N_1$ (the number of Proposition (1.3)), points of an open contained in that image correspond to «truncations» of S-arcs $\phi\colon T_M \to X$, these maps lift by (1.3).

(1.7) THE NASH FAMILIES. – The morphisms $f_{M,N}$ (1.5.1) induce morphisms $g_{M,N}\colon E_M \to E_N$. It is easy to verify that $\mathrm{Im}(g_{M,N}) = E_N$.

Let $S_0 = \mathrm{Sing}(X)$, and consider $Z_M = E_M \cap \mathrm{Hom}(T_M, S_0)$ (this takes place in $\mathrm{Hom}(T_M, X)$, of which both are closed subschemes). Consider the image $Z_{M,N} \subsetneq E_N$ of Z_M via $g_{M,N}$. By (1.4), $Z_{M,N} = Z_{M_1,N}$ if $M \geq$ suitable M_1 (which we may assume $\geq M_0$, the number of (1.6.1)). Let $\tilde{V}_N$ be the closure of $E_N - Z_{M_1,N}$. This is called the Nash scheme of liftable truncated N-arcs (and their specializations). We see, from the used construction, that generically points of $\tilde{V}_N$ correspond to N-truncations of arcs $T \to X$, such that $\phi(0) \in S$ and $\phi(T) \not\subset \operatorname{Sing}(X)$ (the scheme $\tilde{V}_N$ might have a non-reduced structure).

Let $V_N = (\tilde{V}_N)_{\mathrm{red}}$, this is called the *Nash variety of N-truncated arcs.*

Sometimes we shall use the more precise notation $E_N = E(X, S, N)$, $\tilde{V}_N = \tilde{V}(X, S; N)$, etc.

(1.8) In general, E_N and $\tilde{V}_N$ won't be reduced. For instance, take the plane affine curve X defined by $y^2 - x^3 = 0$, let $S = \{(0,0)\}$. Then it is easy to compute the equations defining E_N (in a suitable $\boldsymbol{A}^M$). To get them, one considers expressions $x = \sum a_i t^i$, $y = \sum b_j t^j$, which must satisfy $y^2 = x^3$; moreover the condition «$\phi(0) \in S$» means $a_0 = b_0 = 0$. So we get, among others, the equation $b_1^2 = 0$. Clearly, in the affine ring of E_N we get $b_1^2 = 0$, but not $b_1 = 0$.. In section (3.2) we'll see that in this example $E_N = \tilde{V}_N$, hence $\tilde{V}_N$ could be non-reduced too.

It is easy to construct many similar examples (e.g., $y^2 - x^3 = 0$ but in 3-space, with S the origin or the z-axis; $z^2 - x^3 - y^3 = 0$, with the origin as S, etc.).

We have the following result, which easily follows from the definitions.

(1.9) Let X be an algebraic variety, $S, S_1, \ldots, S_m$ closed subvarieties such that $S = \bigcup_{i=1}^{m} S_i$. Then, $V(X, S; N) = \bigcup_{i=1}^{m} V(X, S_i; N)$, for all N (this happens in $\operatorname{Hom}(T_N, X)$, so the equality makes sense).

(1.10) Finally, we present some comments on the complex-analytic case.

If X is an algebraic scheme over $\boldsymbol{C}$, X^h its associated analytic space and ϕ: $T \to X$ is an arc (cf. (1.1)), then, given any N, there is a morphism $\boldsymbol{D}_\varepsilon \to X^h$ ($\boldsymbol{D}_\varepsilon = \{z \in \boldsymbol{C} / |z| < \varepsilon\}$) inducing (in an obvious sense) the same N-arc $T_N \to X$. This is an immediate consequence of the Analytic Approximation Lemma (cf. [1]). Consequently, in this situation one may develop the theory using convergent analytic arcs rather than formal ones, if one prefers.

More generally, the theory can be developed in the context of Analytic Geometry. We shall use, as an auxiliary tool, very basic facts only (the «local situation», i.e. an analytic set X is an open U of $\boldsymbol{C}^m$, defined as the zeroes of functions holomorphic in U, specially when X is non-singular); since we are primarily interested in the algebraic case we shall omit the details of the general constructions in the analytic context.

2. – Finer properties.

(2.1) In this section, the base field will be the complex numbers, $\boldsymbol{C}$. We present some results about the singularities of $E_N = E(X, S; N)$ and $\tilde{V} = \tilde{V}(X, S; N)$.

First we assume X is a smooth algebraic variety. Note that in this case clearly $E_N = \tilde{V}_N$ for all N. We have:

(2.2) PROPOSITION. – Let X be a smooth algebraic variety, $S \subset X$ a closed algebraic subset, then $E(X, S; N) = \tilde{V}(X, S; N)$ is smooth if and only if S is smooth.

PROOF. – E_N will be smooth as a scheme if and only if it is so as an analytic space. Regarded as an analytic space, E_N is covered by opens $\mathcal{V}$ obtained as follows: take a coordinate neighborhood U of X, i.e., U can be identified to a ball in $\boldsymbol{C}^d$, $d = \dim X$, where S is defined by equations $g_j(u_1, \ldots, u_d) = 0$, $j = 1, \ldots, s$; then $\mathcal{V} \approx \mathrm{Hom}_S(T_n, U)$. But a point of $\mathrm{Hom}_S(T_N, U)$ is given by the coefficients of d N-truncations of power series, $u_j = \sum_{i=0}^{N} a_i^{(j)} t^i$ such that $a_0^{(1)}, \ldots, a_0^{(d)}) \in S$, i.e., $\mathcal{V}$ is identified to the analytic subset of $U \times \boldsymbol{C}^N$ defined by $g_j(a_0^{(1)}, \ldots, a_0^{(d)}) = 0$, $j = 1, \ldots, s$, i.e. to $(U \cap S) \times \boldsymbol{C}^N$. It becomes clear that E_N is smooth if and only if S is smooth.

Next we turn to the singular case.

(2.3) LEMMA. – Let X, S be as in (1.1), assume no irreducible component of X is contained in $\mathrm{Sing}(X)$. Let N be an integer $\geqslant 0$, $P \in S$ and $\phi_P \colon T_N \to X$ the constant morphism, equal to P. Then, $\phi_P \in \tilde{V}(X, S; N)$.

PROOF. – We may assume that, locally near P, X is embedded in some $\boldsymbol{A}^n$, with P corresponding to the origin. We can get a curve C containing P, with a branch $\mathcal{B}$ at P not contained in $S_0 = \mathrm{Sing}(X)$ (take a section of X with a general plane through P). Parametrize $\mathcal{B}$: $x_j = \sum_{i=1}^{\infty} a_i^{(j)} t^j$, $j = 1, \ldots, n$. By changing the parameter, (e.g., via $t = u^{N+1}$) if necessary, we may assume $a_i^{(j)} = 0$, $i \leqslant N$ all j. This proves that ϕ_p lifts to an arc $T \to X$, generically not in S_0, hence $\phi_P \in \tilde{V}(X, S; N)$.

(2.4) PROPOSITION. – Let X, S be as in (2.3). Assume there is a smooth point P of S which is in $\mathrm{Sing}(X)$. Then, for N large enough, $\phi_P \colon T_N \to X$ (the constant morphism equal to P, which by (2.3) is in E_N) is a singular point of E_N.

PROOF. – We may assume (by considering a suitable affine neighborhood of P in X) that X is contained in $\boldsymbol{C}^r$, defined by equations $f_1, \ldots, f_m$, while S is defined by $h_1, \ldots, h_s$ (as explained in (1.5), whose notation and terminology we shall follows). We also assume that P corresponds to the origin and that r is the embedding dimension of X at P. Then it is a well-known basic fact that $\mathrm{ord}(f_i) \geqslant 2$, all i. We'll denote by Y^h the analytic space associated to the scheme Y. In local coordinate $z_1, \ldots, z_n$ (near 0, in $\boldsymbol{C}^r$) we may assume that S is defined by $h_i = z_i$, $i = 1, \ldots, s$; hence the local ring of $\mathrm{Hom}_S(T_M, X)$ at ϕ_P is $\boldsymbol{C}\{A(M)\}/\mathfrak{J}(M)$, where $A(N)$ is the set of variables $a_j^{(i)}$ $i = 1, \ldots, r$, $0 < j \leqslant N$ if $i = 1, \ldots, s$ and $0 \leqslant j \leqslant N$ otherwise, and generators $F_j^{(l)}$ of $\mathfrak{J}(M)$

are determined by the identities:

$$(2.4.1)\qquad f_l(a^{(1)}(t), \ldots, a^{(r)}(t)) = \sum_j F_j^{(l)}(A(M))\, t^j \bmod (t^{N+1}), \qquad l = 1, \ldots, m,$$

where $a^{(i)}(t) = \sum_j a_j^{(i)} t^j$ are series with formal coefficients. On the other hand, the analytic local ring of $\mathrm{Im}(f_{M,N})$ at ϕ_p is defined by $J_{MN} = \{g \in C\{A(N)\}\} / i_{MN}(g) \in \mathfrak{J}(M)$, where i_{MN} is the inclusion $C\{A(N) \subset C\{A(M)\}$. Now, from the assumption that $\mathrm{ord} f_l \geq 2$, all l, it follows that each generator $F_j^{(l)}$ of $\mathfrak{J}(M)$ has order ≥ 2; this easily implies that any element of J_{MN} has order ≥ 2. This implies that $C\{A(N)\}/J_{MN}$ (the analytic local ring of $\mathrm{Im}(f_{MN})$ at «the origin» ϕ_P) is singular, provided $J_{MN} \neq 0$. But this will be certainly the case if N is large enough (a sufficiently large truncation of an arc on X through P will do). If we take M large enough with respect to N, then $\mathrm{Im}(f_{MN})$ is precisely E_N (cf. (1.6)), thus E_N (as an analytic space, or as a scheme) is singular at ϕ_P).

(2.5) REMARKS. – (*a*) Obvious cases where the hypothesis of (2.4) are satisfied, are: X is an algebraic variety, and either $S = \mathrm{Sing}(X)$ or S is smooth and $S \cap \mathrm{Sing}(X) \neq \emptyset$.

(*b*) In (3.2) we'll show that, in certain cases, $E_n = \tilde{V}_N$.

3. – Relationship of $\tilde{V}_N$ and E_N.

(3.1) In this section we prove that, in the case where $S = \mathrm{Sing}(X)$ (we are using the notation of (1.1)), «generically» E_N and $\tilde{V}_N$ are the same, i.e., generically the condition «the general point of an arc must not be in $\mathrm{Sing}(X)$» is superfluous. Precisely, we have:

(3.2) THEOREM. – Let X be a complex algebraic variety, $S \subset X$ a closed subvariety, assume X equidimensional. Consider, for $N \geq 0$, the diagram

$$\begin{tikzcd} V^{\sim}(X,S;N) \arrow[rr, "i"] \arrow[dr, "q"'] & & E(X,S;N) \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

Then there is an open dense set $\mathcal{U} \subset S$ such that i induces the identity $q^{-1}(\mathcal{U}) = p^{-1}(\mathcal{U})$. Moreover, if N is large enough, the section $s\colon \mathcal{U} \to q^{-1}(\mathcal{U})$ (geometrically defined by sending $P \in \mathcal{U}$ to the truncation of the constant arc equal to P) is such that $s(P)$ is a singular point of $\tilde{V}(X,S,N)$ (or of $E(X,S,N)$), for each $P \in \mathcal{U}$.

PROOF. – The last part of the conclusion follows from the first one and (2.4).

For the first part, clearly we may assume that $\mathrm{Sing}(X) \subseteq S$, and we start with:

Case (i). The codimension of S in X is 1. According to the theory of equisingularity (cf. e.g., [8]) in this case there is an open set $\mathcal{U}_1 \subset S$ such that on it locally there is a simultaneous parametrization. This means: for each $P \in \mathcal{U}_1$, locally near P, X is embeddable in C^n, and (assuming $X \subset C^n$) there is a metric neighborhood $\mathcal{G}$ of P in C^n, with coordinates $x_1, \dots, x_{d-1}, y_d, \dots, y_n$, such that: (*a*) $\mathcal{G} \cap S$ is defined by $y_i = 0$, $i = d, \dots, n$; (*b*) the closure (in $\mathcal{G}$) of the connected components of $(\mathcal{G} \cap X) - S$ are all the analytic branches $\mathcal{B}_1, \dots, \mathcal{B}_\rho$ of S at P; and there are ρ polydisks $\mathcal{P}_j$; in C^d, with coordinates

$$(x_1, , \dots, x_{d-1}, u_j), \qquad j = 1, \dots, \rho$$

and morphisms:

$$\phi_j \colon \mathcal{P}_j \to \mathcal{B}_j, \qquad \phi_j(x_1, , \dots, x_{d-1}, u_j) = (x_1, \dots, x_{d-1}, \alpha_d(x, u_j), \dots, \alpha_n(x, u_j)),$$
$$x = (x_1, \dots, x_{d-1}), \; j = 1, \dots, \rho,$$

which are homeomorphisms, where $\alpha_i(x, 0) = 0$ for all j. (The disjoint union of the $\mathcal{P}_j$ is the normalization of $\mathcal{G} \cap X$).

In the sequel, let $\tilde{V}_N = \tilde{V}(X, S; N)$ and $E_N = E(X, S; N)$. Consider a general point of E_N given by an N-arc $\gamma_N \colon T_N \to X$, $\gamma_N(0) = P$, liftable to $\gamma \colon T \to X$. If $\operatorname{Im}(\gamma) \not\subseteq S$, then $\gamma_N \in V_N$. So assume $\operatorname{Im}(\gamma) \subset S$. According to (1.10), we may assume that (using the coordinates on $\mathcal{G}$ above) γ is defined by convergent power series $\psi_1(t), \dots, \psi_n(t))$, where $\psi_j \equiv 0$ for $j \geq d$; moreover (after re-numbering if necessary), $\operatorname{Im}(\gamma) \subseteq \mathcal{B}_1$. Then consider the arc $\gamma_1 \colon T \to \mathcal{P}_1$ given by $x_i = \psi_i(t)$, $i = 1, \dots, d-1$, $u_1 = t^{N+1}$. Then clearly the arc $\gamma' = \phi_1 \gamma_1$ satisfies: $\operatorname{Im}(\gamma') \not\subseteq S$ and its N-truncation is γ_N. This proves that $\gamma_N \in \tilde{V}_N$; and from this the conclusion of Case (i) is clear.

Case (ii). The codimension of S is arbitrary. In this case, take the blowing-up $\pi \colon X' \to X$ of X with center S, let E be the exceptional divisor. Using the theorem of generic smoothness and the Case (i), we get open dense sets $\mathcal{U} \subset S$, $\mathcal{V} \subset E$ such that the conclusion holds for points of $\mathcal{V}$, and π induces a smooth morphism $g \colon \mathcal{V} \to \mathcal{U}$. Now given $\gamma_N \colon T_N \to X$, liftable to $\gamma \colon T \to S \subset X$, by smoothness we may lift γ to $\delta \colon T \to E$, by the Case (i) we get $\delta_1 \colon T \to X$, with $\operatorname{Im} S_1 \not\subseteq E$ and $\delta_1 \equiv \delta (\operatorname{mod} t^{N+1})$. Then clearly $\pi\delta_2$ is an arc where image is not in S, and its N-truncation is γ_N. This proves the theorem.

4. – Irreducible components of V_N.

(4.1) This is mainly an expository section where we present some results of Nash on the irreducible components of V_N. This is taken from [5], although our presentation is rather different, specially our proof of (4.5), which is more algebraic than Nash's.

Let X, S be a pair algebraic variety-closed subvariety, as in (1.1), $S_0 = \operatorname{Sing}(X)$.

We'll see that for any N, the number $C(N)$ of irreducible components of $V(X, S; N)$ is bounded by a number independent of N, and for N large enough $C(N)$ becomes constant. This is done by comparing $C(N)$ to the corresponding number for a desingularization of X, so we study the smooth case first. In this section we work again over $k = \boldsymbol{C}$, we denote $T = \operatorname{Spec} \boldsymbol{C}[\![t]\!]$ and $T_N = \operatorname{Spec} \boldsymbol{C}[\![t]\!]/(t^{N+1})$.

(4.2) PROPOSITION. – Let Z be a smooth algebraic variety, $S \subset Z$ a closed subvariety, with irreducible components $S_1, \ldots, S_s$. Then, $V(Z, S_i; N)$, $i = 1, \ldots, s$ are the irreducible components of $v(Z, S; N)$.

PROOF. – First of all, by (1.9) $V(Z, S; N) = \bigcup_{i=1}^{s} V(Z, S_i; N)$. Next we shall see that:

(a) $V_i = V(Z, S_i; N)$ is irreducible, for each i, and

(b) $V_i \not\subset V_j$ if $j \neq j$, this will conclude the proof.

For this, we shall verify that the induced projection $\pi_i \colon V_i \to S_i$ is smooth, with all the closed fibers isomorphic to the same affine space; from this (a) and (b) are immediate. For the smoothness, it suffices to show that each point $P \in S_i$ has an open metric neighborhood $\mathcal{U}$ such that $\mathcal{V} := V(\mathcal{U}, S_i \cap \mathcal{U}; N) \approx (S_i \cap \mathcal{U}) \times \boldsymbol{A}^M$ (for some M), so that the first projection agrees with the map induced by π_i. But if we take a coordinate neighborhood $\mathcal{U}$ of P in the manifold Z, with coordinates $u_1, \ldots, u_m$, where S_i is defined by the vanishing of functions $f_1, \ldots, f_r$, then an element of $\operatorname{Hom}_{S_i}(T_N, \mathcal{U})$ is given n elements of $\boldsymbol{C}[\![t]\!]/(t^{N+1})$, $\mu_i = \sum_{i=0}^{N} a_j^{(i)} t^j$, subject to the condition $f_j(a_0^{(1)}, \ldots, a_0^{(n)}) = 0$, $j = 1, \ldots, r$ only. Clearly, this means: $\operatorname{Hom}_{S_i \cap \mathcal{U}}(T_N, \mathcal{U}) \approx (S_i \cap \mathcal{U}) \times \boldsymbol{A}^M$, $M = Nn$. By the smoothness of Z, all these truncations are liftable to arcs $T \to V$, and so $\operatorname{Hom}_{S_i \cap \mathcal{U}}(T_N, \mathcal{U}) = V(\mathcal{U}, S_i \cap \mathcal{U}, N)$. The requirement on π_i is clearly satisfied, and this concludes the proof.

(4.3) THEOREM. – Let X, S be as in (4.1), $c(N)$ = number of irreducible components of $V(X, S; N)$. Let $\pi \colon X' \to X$ be any desingularization of X which induces an isomorphism

$$X' - \pi^{-1}(S_0) \xrightarrow{\sim} X - S_0, \tag{4.3.1}$$

where $S_0 = \operatorname{Sing}(X)_1$, $S' = \pi^{-1}(S)$, with irreducible components $S_1', \ldots, S_s'$. Then, $c(N) \leq s$.

PROOF. – Let $W_i = V(X', S_i'; N) - \operatorname{Hom}_{S_0'}(T_N, X')$, with $S_0' = \pi^{-1}(S_0)$, $T_n = \operatorname{Spec} k[t]/(t^{N+1})$. Then it is readily checked (e.g., using the description of $V(X', S_i', N) = V_i'$ given in the proof of (4.2)) that W_i is dense in V_i'. Now, if $\pi' \colon \bigcup_{i=1}^{s} W_i \to V(X, S; N) = V_N$ is the morphism induced by π, then the image of π' is dense in V_N. In fact, an open dense of V_N is composed of points corresponding to truncations of arcs $T \to X$, sending the origin to S and whose image is not contained in S_0. Using the isomorphism (4.3.1) and the fact that π is proper, we may lift such an arc (defining a general point $P \in V_N$) to an element of $\mathrm{Hom}_S(T, X')$, inducing a point $Q \in \bigcup_{i=1}^{s} W_i$, such that $\pi(Q) = P$. So $\mathrm{Im}(f)$ is dense in V_N, hence for each component V_i of V_N there is some j, $1 \leqslant j \leqslant s$ such that $V_i = \overline{\pi(W_j)}$, which proves the theorem.

(4.4) PROPOSITION. – We keep the notation and assumptions of (4.3). Then, for N large enough, the number $c(N)$ becomes constant.

PROOF. – If $N' \geqslant N$, then there is a natural projection

$$p_{N'N}: V(X, S; N') \to V(X, S; N)$$

induced by $f_{N'N}$ of (1.5). It is easily seen (as in the proof of (4.3)) that these are dominating maps, i.e., with dense image. Thus the number of irreducible components of $V(X, S; N')$ is no less than that of $V(X, S; N)$. Since this number is bounded, it must become constant.

Next we prove the following:

(4.5) THEOREM. – We keep the assumptions and notation of (4.3). Then, there is an integer N_0 such that for $N \geqslant N_0$, for each irreducible component V_i of $V(X, S; N)$, there is a unique irreducible component $V'_{j(i)}$ of $V(X', S'; N)$ such that the induced morphism $V'_{j(i)} \to V_i$ is dominant.

As we know, such a $V'_{j(i)}$ is of the form $V(X', S'_{j(i)}; N)$, where $S'_{j(i)}$ is an irreducible component of $S' = \pi^{-1}(S)$. These irreducible components of S' are called the essential components of $S' = \pi^{-1}(S)$. If $S - \mathrm{Sing}(X)$, they are simply called essential components of the resolution π; it can be proved that they are the components of the exceptional locus which appear (in a suitable sense) in any resolution of the singularities of X (cf. [5]).

To prove Theorem (4.5), we begin by proving a lemma.

(4.6) LEMMA. – The notation and assumptions are as in (4.3). Then, there is an integer M, with the following property. If $N > M$, there is an open dense set $\mathcal{V}_N$ in $V(X, S; N)$ such that if α, β are S-arcs inducing elements of $\mathcal{V}_N$ and satisfying $\alpha \equiv \beta (\mathrm{mod}\ t^{N+1})$, $N > M$, then if α', β': $T \to X'$ are liftings of α, β respectively, we have $\alpha' \equiv \beta' \bmod t^{N-M+1}$.

PROOF. – Clearly, the situation is local in X, so we may assume $X \approx \mathrm{Spec}\, A$, a reduced finitely generated k-algebra. Also we may assume that π: $X' \to X$ is the blowing-up of X with center an ideal $I = (h_0, \ldots, h_m) \subset A$. ([3], p. 166). From the usual local theory of the blowing-up (cf. [3]) we get: after re-indexing, if necessary, the local theory lifting α' of α to X' has image in an affine open U_0 of X' of the form $\mathrm{Spec}\, A[h_1/h_0, \ldots, h_m/h_0] \subset \boldsymbol{A}^m_A$, and if $\mathrm{Spec}\, A \subset \boldsymbol{A}^d$ (i.e., A is a quotient of

$k[Z_1, \ldots, Z_d]$) and $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j \in h[\![t]\!]$ for all j, then α' is given by

$$(4.6.1) \qquad (\alpha_1(t), \ldots, \alpha_d(t), \bar{h}_1(t)/\bar{h}_0(t), \ldots, \bar{h}_m(t)/\bar{h}_0(t)),$$

where $\bar{h}_j(t) = H_j(\alpha_1(t), \ldots, \alpha_d(t))$, $j = 1, \ldots, m$, H_j being any polynomial in $k[Z_1, \ldots, Z_d]$ inducing h_j. Now consider $H_0(Z_1, \ldots, Z_d)$, and let

$$\delta_0 = \min\{\mathrm{ord}\, H(\gamma_1(t), \ldots, \gamma_d(t))/\gamma = (\gamma_1, \ldots, \gamma_d) \text{ is an } S\text{-arc}\}$$

(clearly this number is finite, e.g., take a section of S with a sufficiently general plane). If $N > \delta$, let $\mathcal{V}_N^{(0)} = \{\gamma/\gamma$ is an N-arc relative to S and $\mathrm{ord}(H(\gamma_1, \ldots, \gamma_d) = \delta_0\}$ ($\mathrm{ord} H(\gamma_1, \ldots, \gamma_d)$ is defined in the obvious way). This is an open dense of $V(X, S; N)$.

Now let $N > \delta$ and α, β be S-arcs inducing elements of $\mathcal{V}_N^{(0)}$, with $\alpha \equiv \beta \bmod t^{N+1}$, then, their lifting α', β' are parametrized as in (4.6.1) (with the same U_0, because $\alpha \equiv \beta (\bmod\, t)$). Now, from (4.6.1) and our assumption that $H_0(\alpha(t))$ and $H_0(\beta(t))$ have the same order, it easily follows that α' and β' are congruent $\bmod\, t^{N-\delta_0+1}$. With a similar procedure, we obtain numbers δ_j, $j = 0, \ldots, m$, valid for the other affine opens U_i which cover X', as well as open dense sets $\mathcal{V}_N^{(j)} \subset V(X, S; N)$, for each $N > M$. It is clear that $M = \max\{\delta_0, \ldots, \delta_m\}$, together with $\mathcal{V}_N = \bigcup_{j=0} \mathcal{V}_N^{(j)}$, for $N > M$, are the objects we needed.

(4.7) PROOF OF THEOREM (4.5). – Let $N_0 = M + 1$ (M is the number of Lemma (4.6)), let $N \geq N_0$ and assume by contradiction that two irreducible components V_1' and V_2' of $V(X', S'; N) = V$ have dense image in the same component V_0 of $V(X, S; N) = V'$, via the canonical morphism $\pi_N : V \to V'$. We know that $V_i' = V(X', S_i'; N)$, $i = 1, 2$, for suitable irreducible components S_1', S_2' of S'. Let $p_i : V_i' \to S_i'$, $i = 1, 2$, be the canonical projections; $G_1' = S_1' - S_2'$, $G_2' = S_2' - S_1'$. Clearly, we may find open dense sets $G_i \subseteq V_1'$, $i = 1, 2$, such that $G_i \subseteq p_i^{-1}(G_i') \cap \pi_N^{-1}(\mathcal{V}_N)$ where $\mathcal{V}_N$ is as in Lemma (4.6). By a Theorem of Chevalley, $\pi_N(G_1) \cap \pi_N(G_2)$ constains an open dense G of V_0. Let $\alpha_0 \in G$, $\alpha_0 = \pi_N(\beta_i)$, $i = 1, 2$, where $\beta_i \in G_i$ comes from an S'-arc $\tilde{\beta}_i$, and hence α_0 comes from $\tilde{\alpha} = \pi \cdot \tilde{\beta}_i : T \to X$. Applying Lemma 4.6, with $\alpha = \beta = \tilde{\alpha}$, we see that $\beta_1 \equiv \beta_2 \ (\bmod\, t)$, hence they have the same origin, i.e. $p_1(\beta_1) = p_2(\beta_2)$. This contradicts the choice of the G_i's.

5. – Families of parametrized arcs.

(5.1) Throughout this section, X denotes an algebraic variety, $S \subset X$ a Zariski closed set, $S_0 = \mathrm{Sing}\, X$. For simplicity the base field will be $\boldsymbol{C}$.

We want to show that, for N large enough, if Σ is an irreducible component of $V(X, S; N)$ then points of an open dense of Σ correspond to truncations of parametrized arcs which vary in a family, moreover we may demand that they vary in a nice, «equisingular» fashion. This allows us to introduce certain numerical invariants. To make this more precise, we need:

(5.2) DEFINITION. – Given analytic varieties U, V, a family of parametrized arcs of V, with parameter space U, is a pair $(\mathcal{D}, \beta)$ where $\mathcal{D} \subset U \times \boldsymbol{C}$ is an open neigborhood of $U \times \{0\}$ and $\beta: \mathcal{D} \to V$ is a holomorphic mapping, such that for each $u \in U$ there is some $\varepsilon > 0$ such that β induces a homeomorphism of $D_u(\varepsilon) := \{u\} \times \{t \in \boldsymbol{D} / |t| < \varepsilon\}$ with its image.

Thus, if $C_u = \beta(D_u(\varepsilon))$, then $(C_u, \beta(u, 0))$ is a germ of an analytic branch and β induces an irreducible parametrization of this («irreducible» means that we cannot reparametrize by means of a substitution $t^r = \tau$, $r > 1$, cf. [9], p. 94).

(5.3) DEFINITION. – Given analytic varieties U, V, a family of curves on V, parametrized by U, is a flat morphism $\pi: \mathcal{C} \to U$, where $\mathcal{C}$ is a closed subspace of $U \times V$; such that $\pi^{-1}(u)$ is a purely one-dimensional subspace of $\{u\} \times V$ for each $u \in U$.

These concepts are related as follows:

(5.3) LEMMA. – Given analytic varieties U, V and a family $(\mathcal{D}, \beta)$ of parametrized arcs of V, parametrized by U, then there are open sets U_i in U (resp. V_i in V); $i \in I$ (a suitable index set) such that the union of the U_i is dense in U, and for each i there is a family of reduced curves on V_i, $\pi_i: \mathcal{C}_i \to U_i$, $i \in I$, together with sections $s_i: U_i \to \mathcal{C}_i$, such that for each i, $u \in U_i$, the germ $(\pi^{-1}(u), s_i(u))$ is irreducible, and β induces a parametrization of this germ.

PROOF. – Let $u_0 \in U$. We'll see that for some (possibly deleted) neighborhood of u_0 in U, say U' and on open V' in V there is a family $\mathcal{C}' \to U'$, of curves on V' and a section with the required properties. This clearly proves the lemma. To see this, clearly we may find an open U_1 in U (resp. V_1 in V), with $u_0 \in U_1$, such that for each $u \in U_1$, $C_u := \beta((\{u\} \times \boldsymbol{C}) \cap \mathcal{D})$ is a closed one dimensional subspace of V_1. Consider the morphism $\gamma: \mathcal{D} \cap (U_1 \times \boldsymbol{C}) \to U_1 \times V$ given by $\gamma(u, t) = (u, \beta(u, t))$; let $\mathcal{D}_1 = \gamma^{-1}(U \times V_1)$. Then, it is easily seen that $\gamma(\mathcal{D}_1)$ is closed in $U_1 \times V_1$, and the induced map $\gamma_1: \mathcal{D}_1 \to U_1 \times V_1$ is proper. Hence, $\mathcal{C}_1 = \gamma_1(\mathcal{D}_1)$ is a closed analytic subset of $U_1 \times V_1$. Consider the projection $\pi_1: \mathcal{C}_1 \to U_1$, we have a section $s_1: U_1 \to \mathcal{C}_1$ given by $s_1(u) = \beta(u, 0)$. For a dense open set $U' \subset U_1$, the pull back $\pi': \mathcal{C}' \to U'$ of π_1 will be flat, with reduced fibers ($\mathcal{C}_1$ is reduced), and this family (with the section s' induced by s_1) clearly satisfies all the requirements.

Returning to our basic situation of (5.1), we have:

(5.4) PROPOSITION. – Let X, S and S_0 be as in (5.1). Then, there is an integer N_0 such that for $N \geqslant N_0$, for any irreducible component Σ of $V(X, S; N)$, there is an open dense $\Sigma^{(0)} \subset \Sigma$ such that $\Sigma^{(0)}$ is covered by metric open sets U, with the property that for each U there is a family of parametrized arcs of X, with parameter space U, satisfying: for each $u \in U$, the point of $V(X, S; N)$ corresponding to $\beta_u: D_u \to X$, by truncating mod N, is u. (More precisely: β_u induces a morphism

$T_N \to X$ (cf. (1.1)), we claim that this point of $\mathrm{Hom}(T_N, X)$ is in $V(X, S; N)$ and is precisely u).

PROOF. – We fix a desingularization $f: X' \to X$ of X, which induces an isomorphism over the complement of $S_0 = \mathrm{Sing}(X)$. If N is large enough (say, $N \geq N_1$), the number of irreducible components of $V(X, S; N)$ is constant and if Σ is such a component, then there is a unique irreducible component Σ' of $V(X', S; N)$ where image by the morphism $\delta: \Sigma' \to V(X, S; N)$ induced by f is dense in Σ; we necessarily have $\Sigma' = V(X', E; N)$, where E is a suitable irreducible component of $f^{-1}(S)$ (cf. § 4).

Now, I make the following:

CLAIM. – There are metric open sets $\mathcal{U}_i$ ($i \in I$, an index set) of Σ' such that there families of parametrized arcs on X, parametrized by $\mathcal{U}_i$, with the property:

(5.4.1) The N-truncation of the arc corresponding to any $u \in \mathcal{U}_i$ is $\delta(u) \in \Sigma$, moreover the union $\mathcal{U}$ of these $\mathcal{U}_i$ is dense in Σ'.

To see this, view X' as a complex analytic manifold, and take a coordinate neighborhood $\mathcal{N}$ of a point P of $\mathcal{E}$ (the set of smooth points of E), with an isomorphism $\beta: \mathcal{N} \to \mathcal{P}$, where $\mathcal{P}$ is a poly-disk in $\boldsymbol{C}^n$ ($n = \dim X'$) and $\mathcal{P}' = (\mathcal{N} \cap \mathcal{E})$ is defined by $z_{m+1} = \ldots = z_n = 0$ ($z_1, \ldots, z_n$ coordinates in $\boldsymbol{C}^m$; of course if $S \subseteq S_0$, then $m = n-1$).

Then, as was explained in the proof of (4.2), there are isomorphisms:

$$V(\mathcal{N}, \mathcal{N} \cap \mathcal{E}; N) \approx V(\mathcal{P}, \mathcal{P}'; N) \approx \mathcal{P}' \times \boldsymbol{A}^M, \tag{5.4.2}$$

for a suitable M (the points of $\boldsymbol{A}^M$ correspond of truncated series:

$$z_j = \sum_{i=0}^{N} a_i^{(j)} t^j, \tag{5.4.3}$$

where $z_1, \ldots, z_n$ are coordinates of $\mathcal{P}$ and $a_0^{(j)} = 0$ if $j = m+1, \ldots, n$).

If $\mathcal{A}'$ is an open set in $\mathcal{P}' \times \boldsymbol{A}^m$ and $\rho: \mathcal{A}' \to \boldsymbol{R}$ is a continuous, positive valued function, let $(\mathcal{A}' \times \boldsymbol{C})_\rho := \{(a, t) / |t| < \rho(a)\}$. It is easy to see that we may choose $\mathcal{A}'$ and ρ in such a way that the equations (5.4.3) define a family of parametrized arcs $\beta': (\mathcal{A}' \times \boldsymbol{C})_\rho \to X'$. If $\mathcal{A}$ is the open of Σ' corresponding to $\mathcal{A}'$ by means of the isomorphisms (5.4.2) (and the natural identification of $V(\mathcal{N}, \mathcal{N} \cap \mathcal{E}; N)$ with an open of Σ'), the β' induces a family of arcs of X' parametrized by $\mathcal{A}; \mathcal{D}_1 \to X'$. Composing this with $f: X' \to X$, we get a family $\mathcal{D}_1 \to X$. It is clear from the construction that the property (5.4.1) holds for this family. Letting P vary in $\mathcal{E}$, the families $(\mathcal{U}_i, \mathcal{D}_i)$ constructed in this way are the ones we need to prove the claim.

Now, shrinking $\mathcal{U}$ if necessary, by using the theorem on generic smoothness and some elementary considerations, we find that the canonical diagram

$$\begin{array}{ccc} V(X', f^{-1}(S); N) & \longrightarrow & V(X, S; N) \\ \downarrow & & \downarrow \\ f^{-1}(S) & \longrightarrow & S \end{array}$$

induces a commutative square

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{p} & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \longrightarrow & \mathcal{T} \end{array}$$

where $\mathcal{U}$, $\mathcal{V}$, $\mathcal{S}$, $\mathcal{T}$ are suitable dense open sets of Σ', Σ, $\mathcal{S}_1$ and $f(E)$ respectively, and all the morphisms are smooth.

Now let Q_0 be any point of $\mathcal{V}$, let $P_0 \in p^{-1}(Q_0)$, assume $p_0 \in \mathcal{U}_i$. By the smoothness of p, there are open neighborhoods U of P_0, U' of Q_0 and a section $s\colon U \to U'$ of the map $U' \to U$ induced by p. The pull back (by s) of the restriction to U' of the family of parametrized arcs $(\mathcal{U}_i, \mathcal{D}_i)$ yields a family of arcs on X, parametrized by U, with the property required in the statement of the Proposition. This proves (5.4).

To refine Proposition (5.4) we need some results on truncations. There are similar results in the literature (cf. [4], p. 155), but I couldn't find it in the form I need.

(5.5) Given an analytic arc in $\boldsymbol{C}^m$, $\gamma\colon \boldsymbol{D}_\varepsilon \to \boldsymbol{C}^m$, defined by power series $(\phi_1(t), \dots, \phi_m(t))$, its associated branch is, by definition, the germ corresponding to the analytic algebra $\boldsymbol{C}\{\phi_1, \dots, \phi_m\} \subset \boldsymbol{C}\{t\}$; it will be denoted by $\operatorname{Im}(\gamma)$.

Recall: the δ-invariant of a germ of a curve (C, x) is $\delta(C) = \operatorname{length}(\overline{A}/A)$, where $A = \mathcal{O}_{C,x}$, $\overline{A}$ the integral closure of A in its total ring of fractions. The conductor of A in $\overline{A}$ will be denoted by $\mathcal{C}(A)$; if A is a domain $c(A)$ will denote its degree (i.e., $\min\{r/t^n \in \mathcal{C}(A) \text{ for } n \geq r\}$, where $\overline{A} \approx \boldsymbol{C}\{t\}$).

(5.6) LEMMA. – Let C be a branch in $\boldsymbol{C}^m$, with local ring A, $\overline{A} = \boldsymbol{C}\{t\}$, $\delta = \delta(C)$, $\nu =$ multiplicity of A, $\mathcal{M} = \max(A)$. Then

$$t^{(N+1)\nu + c - 1} \in \mathcal{M}^N, \qquad \text{for all } N \geq 0.$$

PROOF. – Let C be parametrized by $\phi_1, \dots, \phi_m$, i.e., $A = \boldsymbol{C}\{\phi_1, \dots, \phi_m\}$. We may assume $\operatorname{ord}(\phi_1) \leq \operatorname{ord}(\phi_2) \leq \dots \leq \operatorname{ord}(\phi_m)$, thus $\nu = \operatorname{ord}(\phi_1)$. Consider the following two filtrations on A: $\{\mathcal{M}^r\}$ ($\mathcal{M} = \max A$) and $\{I_r\}$, $r = 0, 1, \dots$, where $I_r = (t^{\nu r}) \cap A$. It is easily seen that $\mathcal{M}^r \subset I_r$, for all r. I claim: If r_0 is any integer $\geq c/\nu$, with $c = c(A)$, then $\mathcal{M} I_r = I_{r+1}$ for each $r \geq r_0$. In fact, if $\alpha \in I_{r+1}$, then we can write (in $\boldsymbol{C}\{t\}$): $\alpha = t^\nu[a_1 t^{\nu r} + a_2 t^{\nu r+1} + \dots]$. Writing $\phi_1 = t^\nu \beta$ (where necessarily β is a unit), then $\alpha = \phi_1 \gamma$, with $\gamma = \beta^{-1}[a_1 t^{\nu r} + \dots]$ of order $\geq \nu r$. So, if $r \geqslant r_0$, $\operatorname{ord}\gamma \geqslant c$ and so $\gamma \in A$, so $\alpha = \phi_1 \gamma$, with $\phi_1 \in \mathfrak{M}$ and $\gamma \in (t^{\nu r}) \cap A$, i.e., $\alpha \in \mathfrak{M} I_r$. The other inclusion being immediate, the claimed equality follows.

From this it easily follows: $\mathfrak{M}^N I_{r_0} = I_{r_0+N}$, whence $I_{r_0+N} \subseteq \mathfrak{M}^N$, for all $N \geqslant 0$, i.e., $(t^{N\nu + r_0\nu}) \cap A = (t^{N\nu + r_0\nu}) \subset \mathfrak{M}^N$ (the equality because $N\nu + r_{0\nu} \geqslant c$). If r_0 is the smallest integer $\geqslant c/\nu$, then $c + \nu - 1 \geqslant r_0 \nu$, thus $t^{(N+1)\nu + c - 1} \in \mathfrak{M}^N$, as we wanted to show.

(5.7) LEMMA. – Notation as in (5.6). Let C_0 be a general plane projection of C, $\delta_0 = \delta(C_0)$. Then, if $\delta_0 > 0$, $8\delta_0 \geqslant 3\nu + c$. Always, $t^{8\delta_0 + 2} \in \mathfrak{M}^2$.

PROOF. – The inequality follows from the following well known facts: (i) $\delta_0 = \delta$, (ii) $2\delta \geqslant c$ ([7], p. 80), (iii) $2\delta_0 + 2 > \beta_g$ (the last characteristic exponent of C_0) and $\beta_g \geqslant \nu + 1$ ([10], ch. II, § 3). The second assertion follows from (5.6) and the inequality just gotten.

(5.8) PROPOSITION. – Let C be a parametrized branch in $\boldsymbol{C}^m$, with irreducible parametrization $(\phi_1(t), \ldots, \phi_m(t))$, let C_0 be a general plane projection of C, $\delta_0 = \delta(C_0)$, and $N_0 = 8\delta_0 + 2$. Then, N_0 has the property that if D is another branch in $\boldsymbol{C}^m$, with a parametrization $(\psi_1, \ldots, \psi_m)$ satisfying:

$$\psi_j(t) \equiv \phi_j(t) (\operatorname{mod} t^N), \qquad N \geqslant N_0, \; j = 1, \ldots, m \tag{5.8.1}$$

then C and D are isomorphic.

PROOF. – In the sequel, $\mathcal{O}_E$ denotes the local ring of a branch E.

(i) We shall see that if D_0 is a general plane projection of D, then $\delta_0 = \delta(D_0)$. We shall use certain facts from the theory of equisingularity for plane curves, as explained in [11] or [12]. To begin with, since equisingularity is an open condition, if E_0 and E_1 are general plane projections of a branch E, then E_0 and E_1 will be equisingular, hence $\delta(E_0) = \delta(E_1)$. So, we may assume that we take a common (linear) projection for both C and D, i.e., a general linear change of coordinates in $\boldsymbol{C}^m$, followed by the projection $(z_1, \ldots, z_m) \to (z_1, z_2)$, to get C_0 and D_0 respectively. It is clear that the assumption (5.8.1) implies: we may parametrize C_0 (resp. D_0) by $x = \tau^\nu$, $y = \alpha(\tau)$, $\operatorname{ord}\alpha \geqslant \nu$ (resp. $x = \tau^{\nu'}$, $y = \beta(\tau)$) in such a way that $\alpha \equiv \beta (\operatorname{mod} \tau^{8\delta_0 + 2})$. This, using the inequality «$2\delta_0 + 2 > \beta_g$ = last characteristic exponent of C_0» ([10], ch. II, § 3) implies $\nu' = \nu$ and all the characteristic exponents of D_0 agree with those of C_0. Hence C_0 and D_0 are equisingular, and $\delta_0 = \delta(C_0) = \delta(D_0)$.

(ii) By the assumption (5.8.1), we may write:

$$\phi_j(t) = \alpha_j(t) + \gamma_j(t), \qquad j = 1, \ldots, m,$$

$$\psi_j(t) = \alpha_j(t) + \delta_j(t), \qquad j = 1, \ldots, m,$$

where $\alpha_j(t)$ is a polynomial of degree $< N_0$ and γ_j, δ_j are series of order $\geqslant N_0$ each. Since $N_0 = 8\delta_0 + 2$, where (by (i)) δ_0 is the δ-invariant of a general plane projection both of C and D, by Lemma (5.7), γ_j (resp. δ_j) is in $(\max \mathcal{O}_C)^2$ (resp. $(\max \mathcal{O}_D)^2$), hence these are power series in m variables f_j (resp. g_j), $j=1,\dots,m$ of order ≥ 2, such that $\gamma_j = f_j(\phi_1,\dots,\phi_m)$ (resp. $\delta_j = g_j(\psi_1,\dots,\psi_m)$.).

Consider the change of variables in C^m $z_j = x_j - f_j(x_1,\dots,x_m)$ (resp. $z_j = x_j - g_j(x_1,\dots,x_m)$). Then, in the new variables, the branch C (resp. D) has a parametrization $z_j = \alpha_j(t)$, $j=1,\dots,m$ (resp. $z_j = \alpha_j(t)$, $j=1,\dots,m$). Now it is clear that C and D (having in suitable coordinates the same parametrization) are isomorphic.

(5.9) Finally, for the refinement of Proposition (5.4) that will be presented, we need to recall some facts about equisingularity for space curves (cf. [2]). In this case, the situation is more complex than in the planar case, since we loose the equivalence of almost any conceivable reasonable definition that we have in the plane; instead one gets a rather complex chart of implications (cf. [2], p. 4). We shall say that a family of curves, with a section (cf. (5.3)), parametrized by a manifold U, is $\mathcal{S}$-equisingular along the section if it is equisaturated. This is the strongest of the standard definitions of equisingularities for space curves ([2], p. 4), it is an open condition and it implies: the numbers $\delta(C_u)$ and $l(C_u) := \delta(C'_u) - \delta(C_u)$, where C'_u is a general plane projection of C_u, are constant (and actually, $l(C_u)$ and the Milnor number $\mu(C_u)$ are constant (along the section) if and only if $\mathcal{S}$-equisingularity holds).

(5.10) We shall say that two germs (C, x_0) and (D, x_1) have the same $\mathcal{S}$-type if there are families (with sections) of curves $(p_i\colon \mathcal{C}_i \to U_i, s_i)$, $i=1,\dots,r$, which are equisingular along the sections, points y_i, z_i in U_i, $i=1,\dots,r$, such that $p_1^{-1}(z_i) \approx p_{i+1}^{-1}(y_{i+1})$, $j=1,\dots,r$, and $C \approx p_1^{-1}(y_1)$, $D \approx p_r^{-1}(z_r)$ (we mean, of course, isomorphisms of germs, we drop the base points to simplify the notation).

If C and D have the same $\mathcal{S}$-type, then they have the same Milnor number, δ-invariant, number of irreducible components, etc. (cf. [2]).

Now we may state the main result of this section.

(5.11) THEOREM. – Let X, S, $S_0 = \operatorname{Sing}(X)$ be as in (5.1) (or in Proposition (5.4)). Then, there is an integer $K \geq N_0$ (the number of (5.4)), such that for $N \geq K$, it holds:

(*a*) for each irreducible component Σ of $V(X', S; N)$ there is an open dense set $\Sigma^{(1)} \subset \Sigma$ which is covered by metric open sets U_j, j in an index set I, such that for each j we have a commutative diagram:

$$\begin{tikzcd} \mathcal{D}_j \arrow[r, "\gamma_j"] \arrow[dr, "q_j"'] & \mathcal{C}_j \arrow[d, "p_j"] \\ & U_j \arrow[u, bend right, "s_j"'] \end{tikzcd}$$

where: (i) p_j defines a family of branches on a suitable open V_j of X (in particular, $\mathcal{C}_j \subset U_j \times V_j$) and s_j is a section of p_j for each $u \in U_j$; (ii) the family $\mathcal{C}_j \to U_j$ is equisingular along s_j; (iii) $\mathcal{O}_j \subset U_j \times \boldsymbol{C}$ is a neighborhood of $U_j \times \{0\}$ and for each $u \in U_j$, induces a parametrization of the germ $(p_j^{-1}(u), s_j(u))$, moreover the resulting induced morphism $T_N = \operatorname{Spec} \boldsymbol{C}[t]/(t^{N+1}) \to X$ is precisely u (note $\Sigma \subset \operatorname{Hom}_S(T_N, X)$, naturally).

(*b*) If $u \in \Sigma^{(1)}$, γ and γ' are arcs (i.e., map germs $(\boldsymbol{D}, 0) \to X$) both inducing (via N-truncation) the point u, then the germs $\operatorname{Im}(\gamma)$ and $\operatorname{Im}(\gamma')$ (cf. (5.5)) are isomorphic.

(*c*) If u, v are points of $\Sigma_N^{(1)}$, γ and γ' are arcs whose N-truncations are u, v respectively, then the $\mathcal{S}$-type of the germ $\operatorname{Im}(\gamma)$ is the same as that of $\operatorname{Im}(\gamma')$.

PROOF. – To get (*a*) use the method of the proof of Proposition (5.4) (with the same N_0) to get opens U' with the property of the U's of (5.4). Then, using Lemma (5.3) we may cover a dense open set of each U' with metric opens U'' parametrizing families of branches (with sections), with the properties listed in (5.3). Using the openness of $\mathcal{S}$-equisingularity, we find inside each U'' a dense open $\tilde{U}$ such that induced family is $\mathcal{S}$-equisingular. These form a collection $\{U_j\}$ of opens with the desired properties, we let $\Sigma^{(1)} := \bigcup_{j \in I} U_j$.

Concerning (*b*), let us note that if we fix an index j, letting $C_u = p_j^{-1}(u)$, $u \in U_j$ (where $p_j \colon \mathcal{C}_j \to U_j$ is the family of part (*a*)) and C'_u being a general plane projection of C_u, then $\delta(C'_u)$ is independent of $u \in U_j$ (this is because the family is $\mathcal{S}$-equisingular, cf. (5.9)). Let δ_j be this common number. Now let $K_j := \max(8\delta_j + 3, N_0)$. If $u \in U_j$ and γ is any arc whose N-truncation is u, with $N \geqslant K_j$, then by Proposition (5.8) it follows that $\operatorname{Im}\gamma$ and $(C_j, s; (u))$ are isomorphic germs.

Now assume U_j is another of our opens (with family $p_j \colon \mathcal{C}_j \to U_i$, with section s_j) and $U_i \cap U_j \neq \emptyset$. Let $u \in U_i \cap U_j$. As before, we get a value $\delta_i = \delta(C''_v)$, C''_v a general plane projection of $p_i^{-1}(v)$, $v \in U_i$. As we have shown, since $p_i^{-1}(C_u)$ is the image of an arc whose truncation is u, $C_j \approx p_i^{-1}(C_u)$. It follows that $\delta_j = \delta_i$. Since the union of the U_i's is connected (an open dense in the irreducible Σ) we see: δ_i is the same for all i, say $= \delta_0$, so the numbers K_j defined above are all equal, say to K. Now (*b*) (with this value of K) becomes clear.

Statement (*c*) is an immediate consequence of (*b*) and the definitions.

(5.12) For instance, as is explained in [5], for X defined in $\boldsymbol{C}^3$ by $z^{n+1} = xy$, $S = \{0\}$ (the A_N-rational double point), if N is large enough, $V(X, S; N)$ has n irreducible components. The points of the j-th one will correspond, generically, to N-truncations of arcs $x = \alpha t^j + \dots$, $y = \beta t^{n+1-j} + \dots$, $z = ct + \dots$, with $c^{n+1} = ab \neq 0$; i.e. a non-singular branch. Note that for $n = 1$, i.e. the cone $z^2 = xy$, $V(X, S; N)$ is irreducible, its general point corresponding to a smooth arc; i.e., the same situation as in the case «X is smooth» (cf. (2.4), (2.5)).

In the case X: $z^2 + x^3 + y^6 = 0$ in $\boldsymbol{C}^3$), $S = \{0\}$, again for N large $V(X, S; N)$ is irreducible, and generically its points correspond to truncation of smooth arcs $x = at^2 + \dots$, $y = bt + \dots$, $z = ct^3 + \dots$, $a, b, c \neq 0$. But here all these have a common tangent line, namely the line $x = z = 0$.

Note. After this article was completed, I received a copy of «Courbes tracées sur un germe d'hypersurface», by M. LEJEUNE-JALABERT (I thank her for sending this preprint). This paper also discussed the foundations of Nash's theory, in the case of an algebroid variety, following an approach similar to the one used in Section 1. It also contains interesting results of the equations defining Nash varieties, in the case of a hypersurface.

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